Homework Problem Set #2

Math 5467

March 8, 2004

Exercise 1 *Expand* $f(t) = \pi^2 - 2\pi|t|$ *in a Fourier series on the interval* $-\pi \leq t \leq \pi$. *Plot both* f *and the partial sums*

$$S_k(t) = \frac{a_0}{2} + \sum_{n=0}^{k}(a_n \cos nt + b_n \sin nt)$$

for $k = 1, 2, 5, 7$. *Observe how the partial sums approximate* f.

Exercise 2 *Expand* $f(t) = -t$ *in a Fourier series on the interval* $-\pi \leq t \leq \pi$. *Plot both* f *and the partial sums* S_k *for* $k = 5, 10, 20, 40$. *Observe how the partial sums approximate* f. *What accounts for the slow rate of convergence?*

Exercise 3 *let* $f_1(t) = t$ *and* $f_2(t) = \pi^2 - 3t^2$, $-\pi \leq t \leq \pi$. *Find the Fourier series of* f_1 *and* f_2 *and use them to sum the following series:*

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
4. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

Exercise 4 *Let*

$$\Pi(t) = \begin{cases} 1 & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

be the box function on the real line. We will often have the occasion to express the Fourier transform of $f(at+b)$ in terms of the Fourier transform $\hat{f}(\lambda)$ of $f(t)$, where a, b are real parameters. This exercise will give you practice in correct application of the transform.

1. *Sketch the graphs of* $\Pi(t)$, $\Pi(t-3)$, *and* $\Pi(2t-3) = \Pi(2(t-3/2))$.

2. *Sketch the graphs of* $\Pi(t)$, $\Pi(2t)$, *and* $\Pi(2(t-3))$. *Note: In the first part a right 3-translate is followed by a 2-dilate; but in the second part a 2-dilate is followed by a right 3-translate. The results are not the same.*

3. *Find the Fourier transforms of* $g_1(t) = \Pi(2t-3)$ *and* $g_2(t) = \Pi(2(t-3))$ *from parts 1 and 2.*

4. *Set* $g(t) = \Pi(2t)$ *and check your answers to part 3 by applying the translation rule to*

$$g_1(t) = g(t - \frac{3}{2}), \qquad g_2(t) = g(t-3), \qquad \text{noting} \;\; g_2(t) = g_1(t - \frac{3}{2}).$$

Exercise 5 *Let $f(t)$ have Fourier transform*

$$\hat{f}(\lambda) = \begin{cases} 1 & 0 \leq \lambda < 1 \\ 2 & 1 \leq \lambda < 2 \\ 3 & 2 \leq \lambda < 3 \\ 0 & \text{otherwise,} \end{cases}$$

Sketch the graph of the Fourier transform of the function:

- $f(-t)$
- $2f(2t)$
- $e^{4it} f(t)$
- $f'(t)$
- $(f * f)(t)$
- $f(t) \cdot f(t)$.

Exercise 6 *Sum the Fourier series* $\sum_{n=0}^{\infty} 2^{-n} \cos nt$ *and* $\sum_{n=1}^{\infty} 2^{-n} \sin nt$. *Hint: Take the real and imaginary parts of* $\sum_{n=0}^{\infty} 2^{-n} e^{int}$. *You should be able to sum this last series directly.*

Exercise 7 *Let* $f(t) = e^{iat}$ *for* a *real but not an integer, and* $-\pi \leq t \leq \pi$. *Use Parseval's formula for complex Fourier series, and evaluate the* L^2 *norm* $||f||^2$ *in two ways to deduce that*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a-n)^2} = \frac{\pi^2}{\sin^2 a\pi}.$$

Exercise 8 *(Haar wavelets on [0,1]) Let* $\phi(t)$ *be the Haar scaling function*

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let n *be a positive integer and* $h_k(t) = \sqrt{n}\phi(nt - k), \quad k = 0, 1, \cdots, n-1$

1. *Show that* $\{h_0, \cdots, h_{n-1}\}$ *is an ON set in* $L^2[0,1]$.

2. *Let* $f(t)$ *be a continuous function on* $[0,1]$ *and form the projection* $f_n(t)$ *on the subspace* S_n *of* $L^2[0,1]$ *spanned by* $\{h_0, \cdots, h_{n-1}\}$*:*

$$f_n = \sum_{k=0}^{n-1} (f, h_k) h_k.$$

   Show that $f_n(t) \to f(t)$ *pointwise in* t *as* $n \to \infty$.

3. *For* $f(t) = t$, *compute explicitly the Haar wavelet decomposition for* $n = 4$ *and* $n = 8$. *Plot the results.*