Transformation and Reduction Formulas for Two-Variable Hypergeometric Functions on the Sphere S_2

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We classify the two-variable hypergeometric functions that arise as eigenfunctions of the Laplace-Beltrami operator on S_2 and characterize these functions in terms of elements in the enveloping algebra of so(3). This operator characterization is used to derive transformation and reduction formulas for the functions.

1. Introduction

In a previous paper we have obtained Lie algebraic characterizations for all 34 of the two-variable hypergeometric functions classified by Horn [1]. Here we list those functions which are related to the complex Lie algebra $\mathrm{sl}(2) \cong \mathrm{so}(3)$ and show how our operator characterizations lead simply to transformation and reduction formulas for the Horn functions.

In Sec. 2 we list all Horn functions that are representable in terms of the type-A operator realization of sl(2). (Type-A operators define a multiplier representation on the complex sphere S_2 [2].) Each such function is a simultaneous eigenfunction of the Casimir operator for sl(2) and a second operator in the enveloping algebra of sl(2) that is part first-order, part second-order. If two families of functions have operators which lie on the same orbit under the adjoint action of sl(2), then these families have a transformation formula that relates them. Reduction formulas arise when the second operator equation factors as a product of first-order operators.

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In Sec. 3 we list the Horn functions that can be characterized in terms of angular-momentum operators for so(3), a special case of type-A operators. These functions are all solutions of the Laplace-Beltrami eigenvalue equation on the complex sphere S_2. In addition to transformation and reduction formulas that can be derived by the methods of the preceding section, we find a new type of reduction formula. It is known that every purely second-order operator in the enveloping algebra of so(3) characterizes a separable coordinate system for the Laplace-Beltrami eigenvalue equation on S_2 [3, 4]. Thus for each Horn function on S_2 whose defining operator equation is purely second-order, we can expand this function in terms of purely separable eigenfunctions.

We have not in fact treated all Horn functions that are related to so(3). We have confined our attention here to operator realizations associated with S_2 and omitted the type-B operator realizations for confluent-limit Horn functions, which were listed in Ref. [1]. Furthermore we have omitted all discussion of the Appell function F_4, which is more properly associated with the sphere S_3 and so(4) and will be treated in another publication. (Indeed, the important reduction formula for F_4 as a product of two functions ${}_2F_1$ [5, p. 81] is a consequence of the fact that for certain parameter choices the operator characterization of F_4 corresponds to a separable solution of the Laplace-Beltrami eigenvalue equation on S_3.)

Our operator techniques provide considerable insight into the theory of two-variable hypergeometric functions. In forthcoming papers we shall show how these techniques can be used to obtain generating functions and expansion formulas for hypergeometric functions.

2. Type-A operator realizations of Horn functions

We start by listing the two-variable hypergeometric functions that are associated with the type-A operators:

$$L^+ = -vz\partial_z - v^2\partial_v, \qquad L^0 = v\partial_v - \frac{\mu+1}{2}$$

$$L^- = -\frac{z(1-z)}{v}\partial_z + \partial_v + \frac{z(\mu+\xi+1)}{2v} - \frac{\mu+1}{v}. \tag{2.1}$$

Here, v,z are complex variables and μ,ξ are complex constants. These operators satisfy the commutation relations

$$[L^0,L^\pm] = \pm L^\pm, \qquad [L^+,L^-] = 2L^0; \tag{2.2}$$

hence they define a multiplier representation of the Lie algebra sl(2) [2]. Furthermore, the operators (2.1) are closely associated with the type-A factorizations of Infeld and Hull [2, 6].

The functions $F(u,v)$ in Table 1 all satisfy the Casimir eigenvalue equation

$$(L^+L^- + L^0L^0 - L^0)F = \tfrac{1}{4}(\mu^2-1)F \tag{2.3}$$

Table 1
Type-A Realizations of Horn Functions

Function; related functions		Operator equation; typical solution
(1)	F_1;	$L^+L^0+L^0L^0+\frac{1}{2}(\beta+2\beta'-\alpha-\gamma+1)L^+-\alpha L^0$ $\sim\frac{1}{4}(\mu+1)(\mu-2\alpha+1)$, $\mu=\alpha+\beta-\gamma,\ \xi=-\alpha+\beta+\gamma-1$;
	G_2	$F_1(\alpha,\beta,\beta',\gamma,1-z,v)v^\alpha$
(2)	Φ_1;	$L^0L^0+L^+-\alpha L^0\sim\frac{1}{4}(\mu+1)(\mu-2\alpha+1)$, $\mu=\alpha+\beta-\gamma,\ \xi=-\alpha+\beta+\gamma-1$;
	Φ_2, Γ_1	$\Phi_1(\alpha,\beta,\gamma,1-z,v)v^\alpha$
(3)	F_2;	$L^+[L^0-\alpha+\beta'+(\mu+1)/2]+[L^0-\alpha+(\mu+1)/2]$ $\times[L^0-\alpha+\gamma'+(\mu-1)/2]\sim 0$, $\mu=\gamma-1,\ \xi=2\beta-\gamma$;
	F_3, H_2	$F_2(\alpha,\beta,\beta',\gamma,\gamma',z,v)v^\alpha$
(4)	H_4;	$(L^0-\alpha+\beta+2\gamma-1-\xi/2)(L^0-\alpha+\beta-\xi/2)+4L^+L^+\sim 0$, $\mu=\delta-1,\ \xi=2\beta-\delta$;
	H_7	$H_4(\alpha,\beta,\gamma,\delta,-v^2,z)v^\alpha$
(5)	Ψ_1;	$(L^0-\alpha+\beta+\gamma'-1-\xi/2)(L^0-\alpha+\beta-\xi/2)+L^+\sim 0$, $\mu=\gamma-1,\ \xi=2\beta-\gamma$
	H_{11}	$\Psi_1(\alpha,\beta,\gamma,\gamma',z,v)v^{\alpha-\gamma+1}z^{\gamma-1}$
(6)	Ξ_1;	$L^+(L^0+\gamma-1-\xi/2)-i(L^0+\alpha'+\gamma-1-\xi/2)\sim 0$, $\mu=\alpha-\beta,\ \xi=\alpha+\beta-1$;
	H_2	$\Xi_1(\alpha,\alpha',\beta,\gamma;z^{-1},iv^{-1})z^{-\beta}v^{\beta-\gamma+1}$
(7)	Ξ_2;	$-iL^+(L^0+\gamma-1-\xi/2)\sim 1$, $\mu=\alpha-\beta,\ \xi=\alpha+\beta-1$;
	H_3	$\Xi_2(\alpha,\beta,\gamma,z^{-1},-iv^{-1})z^{-\beta}v^{\beta-\gamma+1}$

in addition to a second operator equation that is partly first-order, partly second-order in the enveloping algebra of sl(2).

These results follow from Table 3 of Ref. [1] and constitute the list of all type-A operator realizations of two-variable hypergeometric functions for which the multiplier is nonzero. Corresponding to each function in the list we also give all related hypergeometric functions. (We regard two hypergeometric functions as related if the standard pair of partial differential equations for one of these functions (as listed in Ref. [7], pp. 224–227) can be transformed into the standard pair of differential equations for the other by a multiplier transformation and change of independent variables and parameters. The explicit transformations can easily be obtained from the "canonical" equations of Ref. [1]. By an operator equation $C\sim\alpha$, $\alpha\in\mathbb{C}$, for a function $F(v,z)$ we mean that F satisfies $CF=\alpha F$ as well as relation (2.3).

The operator equations for F_1 and Φ_1 are of a different nature than the remaining equations on our list. It is well known that the standard equations for

F_1 and Φ_1 admit exactly three linearly independent solutions, whereas those for F_2, H_4, Ψ_1, Ξ_1, and Ξ_2 admit four linearly independent solutions [8, 9]. Now the sl(2) operator equations for $F_2, \ldots, \Xi_2$ in Table 1 correspond precisely to the standard equations for these functions. However, as follows from their derivation in Ref. [1] the operator equations for F_1 and Φ_1 are more general than the standard equations, i.e., each solution of the standard equations satisfies the operator equations, but the converse is not true. Indeed, using standard techniques, [8, pp. 44–49], one can show that the operator equations for F_1 and Φ_1 admit four linearly independent solutions.

Comparing the operator equations for F_1 and F_2, and taking into account the expressions (2.1), we see that these equations are identical provided either

case 1:
$$\begin{aligned} a = c &= \alpha + \beta - \gamma + 1, \\ b &= \beta, \\ b' &= \beta + \beta' - \gamma + 1, \\ c' &= \beta - \gamma + 2, \end{aligned}$$

or

case 2:
$$\begin{aligned} a = \alpha, \qquad b &= \beta, \qquad b' = \beta', \\ c &= \alpha + \beta - \gamma + 1, \\ c' &= -\beta + \gamma, \end{aligned}$$

where a, b, b', c, c' are the parameters of F_2, and $\alpha, \beta, \beta', \gamma$ are the parameters of F_1. It follows that any solution of the F_1-equations can be expressed as a linear combination of independent solutions of the F_2-equations. For example, in case 1 one can see from the list of 60 solutions of the standard F_1-equations [8, p. 62] that

$$v^\alpha z^{\gamma-\alpha-\beta}(1-v)^{-\beta'} F_1\left(\gamma-\alpha, \gamma-\beta-\beta', \beta', \gamma+1-\alpha-\beta, z, \frac{z}{1-v}\right)$$

is a solution of the F_1-operator equations. (These 60 solutions can be derived by group-theoretic arguments; see Ref. [10].) On the other hand it is straightforward to check that the only solution of the F_2-operator equations of the form

$$v^\alpha z^{\gamma-\alpha-\beta} G(z, v),$$

where G is analytic in a neighborhood of (0,0), is

$$v^\alpha z^{\gamma-\alpha-\beta} F_2(\gamma-\beta, \gamma-\alpha, \beta', \gamma-\alpha-\beta+1, \gamma-\beta, z, v);$$

see Ref. [8], p. 50. Thus

$$F_2(\gamma-\beta,\gamma-\alpha,\beta',\gamma-\alpha-\beta+1,\gamma-\beta,z,v)$$
$$=(1-v)^{-\beta'}F_1\left(\gamma-\alpha,\gamma-\beta-\beta',\beta',\gamma-\alpha-\beta+1,z,\frac{z}{1-v}\right). \quad (2.4)$$

Though this particular identity is not new, the method of proof is new and, we believe, very simple. Similar computations allow one to expand F_1- or G_2-functions as linear combinations of F_2-, F_3-, or H_2-functions. Analogous reasoning permits expansion of Φ_1-, Φ_2-, or Γ_1-functions in terms of Ψ_1- or H_{11}-functions.

More generally we can relate two classes of Horn functions if the operator equations defining one class are equivalent to the operator equations of the other class under the adjoint action of sl(2). For example, if $F(v,z)$ is a solution of the F_2-operator equations (3) in Table 1 and we denote the parameters of F_2 by a,b,b',c,c', then we see that $\exp[\ln 4+i\frac{1}{2}\pi L^0]\exp[\frac{1}{2}L^+]F$ satisfies the H_4-operator equations (4) if $a=\alpha$, $b=\beta$, $c=\delta$, and $c'=2b'=2\gamma$. Thus

$$F\left(\frac{4iv}{1+2iv},\frac{z}{1+2iv}\right)$$

can be expanded in a basis of H_4 solutions. A simple example is

$$F_2\left(\alpha,\beta,\gamma,\delta,2\gamma,\frac{z}{1+2v},\frac{4v}{1+2v}\right)(1+2v)^{-\alpha}=H_4(\alpha,\beta,\gamma,\delta,v^2,z). \quad (2.5)$$

Reduction formulas of the second kind can also be easily derived from our operator realizations. To explain the method we consider the case of the Appell functions F_1. The operator equation $C\sim 0$ for these functions as given in Table 1 factors into a product of first-order operators in sl(2) if and only if there exist constants a,b such that

$$C\equiv(L^++L^0+a)(L^0+b)\sim 0 \quad (2.6)$$

or constants c,d such that

$$C\equiv(L^0+c)(L^++L^0+d)\sim 0. \quad (2.7)$$

It is easy to verify that (2.6) is valid precisely when $\beta'=0$ or $\beta'=\gamma-\beta-1$. Restricting our consideration to the first possibility we see that then

$$a=\tfrac{1}{2}(-\alpha-\beta+\gamma-1), \qquad b=\tfrac{1}{2}(-\alpha+\beta-\gamma+1). \quad (2.8)$$

Now any solution of (2.3) and the equation

$$L^0+b\sim 0 \quad \text{(i)}$$

is a solution of (2.6). Except for special values of α,β,γ, a basis for the 2-dimensional solution space of (i) and (2.3) is given by

$$J_1 = {}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \alpha+\beta-\gamma+1\end{matrix}\middle| z\right)v^\alpha, \qquad J_2 = z^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}\gamma-\alpha,\ \gamma-\beta\\ \gamma-\alpha-\beta+1\end{matrix}\middle| z\right)v^\alpha. \tag{2.9}$$

(Indeed—as follows from Ref. [11]—Eq. (2.3) admits separable solutions in only two essentially distinct coordinate systems. In one of the systems, $\{z,v\}$, the separated solutions are characterized as eigenfunctions of L^0, and the z-dependent factor of each separable solution satisfies the Gauss hypergeometric equation. See Refs. [12] and [2, pp. 199–214] for a more complete discussion of these systems.)

If J is a solution of (2.6), then, setting $G=(L^0+b)J$, we see that G is a solution of

$$(L^+ + L^0 + a)G = 0. \tag{2.10}$$

We have already determined those solutions F of (2.3) and (2.6) such that $G\equiv 0$. To find all possible solutions G of (2.3) and (2.10), we first compute all solutions $K(z,v)$ of (2.3) and $(L^0+a)K=0$. Then

$$G = \exp(-L^+)K\left(\frac{z}{1-v},\frac{v}{1-v}\right)$$

satisfies (2.10), where $K=\exp(L^+)G$. A basis for the solution space is

$$\begin{aligned} G_3 &= \left(1-\frac{z}{1-v}\right)^{-\beta}(1-v)^{\gamma-\alpha-\beta-1}v^{\alpha+\beta-\gamma+1}\\ G_4 &= \left(\frac{z}{1-v}\right)^{\gamma-\alpha-\beta}(1-v)^{\gamma-\alpha-\beta-1}\ {}_2F_1\left(\begin{matrix}1,\ \gamma-\alpha\\ \gamma-\alpha-\beta+1\end{matrix}\middle| \frac{z}{1-v}\right)v^{\alpha+\beta-\gamma+1}. \end{aligned} \tag{2.11}$$

except for special values of α,β,γ. (Note that these solutions are separable in the coordinates $\{z/(1-v),v\}$ and are characterized as eigenfunctions of L^0+L^+. In the group-theoretic analysis of variable separation these coordinates are considered as equivalent to $\{z,v\}$, because L^0+L^+ lies on the same orbit as L^0 under the adjoint action of sl(2) [13].) We can find the functions J_i such that $(L^0+b)J_i=G_i$, $i=3,4$, by straightforward integration. For uniqueness we require that the coefficient of v^α be zero in the convergent expansion $J_i=\Sigma_\lambda v^\lambda g^{(i)}(z)$. If $\mathrm{Re}(\beta-\gamma)>-1$, then

$$J_i = v^\alpha\int_0^v u^{-\alpha-1}G_i(z,u)\,du, \qquad i=3,4, \tag{2.12}$$

for (z,v) in a sufficiently small deleted neighborhood of $(0,0)$. The solutions for $\mathrm{Re}(\beta-\gamma)\leqslant -1$ are obtained from (2.12) by analytic continuation in β and γ. Except for some special values of α,β,γ, the functions $\{J_1,\ldots,J_4\}$ form a basis for the solution space of (2.3) and (2.6). Thus for $\beta'=0$, any solution of the F_1-equations (2.3) with $C\sim 0$ can be uniquely expressed as a linear combination of a separable solution in the variables $\{z,v\}$ and the functions (2.12).

As a nontrivial example, consider the solution [8, p. 62]

$$(1-z)^{\beta-\gamma}z^{\gamma-\alpha-\beta}(1-v)^{\beta-1}(1-z-v)^{1-\beta}$$
$$\times F_1\left(1-\beta,\gamma-\beta,1-\alpha,\gamma+1-\alpha-\beta,\frac{vz}{(1-v)(1-z)},\frac{z}{1-v}\right)v^{\alpha}. \quad (2.13)$$

From the behavior of this function in a deleted neighborhood of $z=0$, $v=0$, we see that the solution must be a multiple of J_2. Hence we obtain the identity

$$F_1\left(1-\beta,\gamma-\beta,1-\alpha,\gamma+1-\alpha-\beta,\frac{vz}{(1-v)(1-z)},\frac{z}{1-v}\right)$$
$$=\left(1-\frac{z}{1-v}\right)^{\beta-1}(1-z)^{\gamma-\beta}{}_2F_1\left(\begin{matrix}\gamma-\alpha, & \gamma-\beta\\ \gamma-\alpha-\beta+1\end{matrix}\middle| z\right),$$

or

$$F_1\left(\begin{matrix}1-\beta; & \gamma-\beta, & 1-\alpha\\ & \gamma+1-\alpha-\beta & \end{matrix}\middle| x,y\right)=(1-y)^{\beta-1}{}_2F_1\left(\begin{matrix}1-\beta, & \gamma-\beta\\ \gamma-\alpha-\beta+1\end{matrix}\middle|\frac{y-x}{y-1}\right).$$

Another example is provided by the solution [8, p. 62]

$$v^{\alpha+\beta+1-\gamma}(1-v)^{\gamma-\alpha-1}(1-z-v)^{-\beta}F_1\left(\begin{matrix}1; & \beta, & \alpha+1-\gamma\\ & 2+\beta-\gamma & \end{matrix}\middle|\frac{v}{v+z-1},\frac{v}{v-1}\right),$$

which is easily seen to be $(\beta-\gamma)J_3(z,v)$.

Clearly, remarks similar to the above hold for all values of $\alpha,\beta,\beta',\gamma$ such that either (2.6) or (2.7) is valid. Also, analogous treatments can be given for systems (3), (4), and (6) in Table 1.

3. Angular-momentum operator realizations

Here we study the two-variable hypergeometric functions associated with the angular-momentum operators

$$J_1=x_3\partial_{x_2}-x_2\partial_{x_3},\qquad J_2=x_1\partial_{x_3}-x_3\partial_{x_1},\qquad J_3=x_2\partial_{x_1}-x_1\partial_{x_2},\quad (3.1)$$

where x_1, x_2, x_3 are complex variables. Setting

$$J^{\pm} = iJ_2 \pm J_3, \qquad J^0 = iJ_1, \tag{3.2}$$

we see that the sl(2) commutation relations (2.2) are satisfied. The operators (3.2) correspond essentially to the type-A operators (2.1) with $\xi=0$. Indeed, setting $\xi=0$ in (2.1) and computing the transformed operators $\tilde{L}=\rho^{-1}L\rho$, where the multiplier function $\rho(v,z)=(v/z)^{(\mu+1)/2}$, we find

$$\begin{aligned} \tilde{L}^{+} &= J^{+} = -vz\,\partial_z - v^2\partial_v, \\ \tilde{L}^{-} &= J^{-} = -\frac{z}{v}(1-z)\partial_z + \partial_v, \\ \tilde{L}^{0} &= J^{0} = v\,\partial_v, \end{aligned} \tag{3.3}$$

where

$$x_1 = \frac{r}{z}(2-z), \qquad x_2 = -\frac{r}{z}\left(\frac{z-1}{v}+v\right), \qquad x_3 = -\frac{ir}{z}\left(\frac{z-1}{v}-v\right). \tag{3.4}$$

Note that the operators (3.3) leave invariant the complex sphere $S_2: x_1^2+x_2^2+x_3^2 = 1$ (or $S_2: r=1$).

In one sense the eigenvalue equation for the Laplace-Beltrami operator on S_2,

$$(J^{+}J^{-}+J^{0}J^{0}-J^{0})F = \tfrac{1}{4}(\mu^2-1)F, \tag{3.5}$$

is just a special case of (2.3). However, the solutions of (3.5) have a much richer structure than those of (2.3) for general ξ. Indeed, (2.3) separates in only two coordinate systems, whereas (3.5) separates in five systems [3]. Note in particular that by splitting off an appropriate variable in (2.3) we can obtain general one-variable hypergeometric functions ${}_2F_1$ as solutions. The analogous procedure in (3.5) yields the Gegenbauer functions, which enjoy many transformation properties not shared by the more general ${}_2F_1$.

Setting $\xi=0$ and performing the multiplier transformation $F \to \rho^{-1}F$, we obtain for each function listed in Table 1 an operator characterization of the corresponding hypergeometric solutions of (3.5). The results are listed in Table 2.

The hypergeometric functions appearing in Table 2 all have linear restrictions on their parameters. However, as follows from the results of Ref. [1], there are other hypergeometric solutions of (3.5) on the complex sphere for which all parameters are arbitrary. A complete list of these functions appears in Table 3.

Many interesting identities can be read off rather easily from these tables. For example, denoting the parameters of F_1 by a, b, b' respectively, we see that the operators characterizing entries (1) and (8) agree if $\alpha=a$, $\beta=b'$, $\gamma=a-b+1$. Thus any F_1 solution of the operator equations (1) can be expressed as a linear

Table 2
Special Horn functions on S_2

Function; related functions		Operator Equation; typical solution
(1)	F_1;	$J^+J^0+J^0J^0+(\beta+\beta'-\alpha)J^+-\alpha J^0$ $\sim\frac{1}{4}(\mu+1)(\mu-2\alpha+1)$, $\mu=2\beta-1$;
	G_2	$F_1(\alpha;\beta,\beta',\alpha-\beta+1,1-z,v)v^{\alpha-\beta}z^{\beta}$
(2)	Φ_1;	$J^0J^0+J^+-\alpha J^0\sim\frac{1}{4}(\mu+1)(\mu-2\alpha+1)$, $\mu=2\beta-1$;
	Φ_2,Γ_1	$\Phi_1(\alpha,\beta,\alpha-\beta+1,1-z,v)v^{\alpha-\beta}z^{\beta}$
(3)	F_2;	$J^+(J^0-\alpha+\beta+\beta')+(J^0-\alpha+\beta)(J^0-\alpha+\beta+\gamma'-1)\sim 0$, $\mu=2\beta-1$;
	F_3,H_2	$F_2(\alpha,\beta,\beta',2\beta,\gamma',z,v)v^{\alpha-\beta}z^{\beta}$
(4)	H_4;	$(J^0-\alpha+\beta+2\gamma-1)(J^0-\alpha+\beta)+4J^+J^+\sim 0$, $\mu=2\beta-1$;
	H_7	$H_4(\alpha,\beta,\gamma,2\beta,-v^2,z)v^{\alpha-\beta}z^{\beta}$
(5)	Ψ_1;	$(J^0-\alpha+\beta+\gamma'-1)(J^0-\alpha+\beta)+J^+\sim 0$, $\mu=2\beta-1$;
	H_{11}	$\Psi_1(\alpha,\beta,2\beta,\gamma',z,v)v^{\alpha-3\beta+1}z^{3\beta-1}$
(6)	Ξ_1;	$J^+(J^0+\gamma-1)-i(J^0+\alpha'+\gamma-1)\sim 0$, $\mu=1-2\beta$;
	H_2	$\Xi_1(1-\beta,\alpha',\beta,\gamma,z^{-1},iv^{-1})v^{2\beta-\gamma}z^{1-2\beta}$
(7)	Ξ_2;	$-J^+(J^0+\gamma-1)\sim 1$, $\mu=1-2\beta$;
	H_3	$\Xi_2(1-\beta,\beta,\gamma,z^{-1},-v^{-1})v^{2\beta-\gamma}z^{1-2\beta}$

combination of four basis solutions of the H_3-equations (8). A simple consequence is the identity

$$F_1(a,b,b',a-b+1,w,v)$$

$$=H_3\left(a,b',a-b+1,\frac{w}{(1+w)^2},\frac{v}{1+w}\right)(1+w)^{-a}, \quad (3.6)$$

as can be checked by comparing coefficients of 1, w, v, and wv on each side of this equation. Similarly entry (2) is a special case of (5), and entries (2), (9), and (11) are equivalent. We also have the equivalences (1)$\leftrightarrow$(10), (6)$\leftrightarrow$(12), and (7)$\leftrightarrow$(13).

As follows from the results of Refs. [3] and [4], Eq. (3.5) admits solutions via separation of variables in precisely five coordinate systems. The separable

Table 3
General Horn functions on S_2

Function; related functions		Operator equation; typical solution
(8)	H_3	$J^+(J^0+\beta-\gamma+1)+(J^0-\alpha+\gamma-1)(J^0-\gamma+1)\sim 0$, $\mu=2\alpha-2\gamma+1$;
	G_1, H_6	$H_3\left(\alpha,\beta,\gamma,\dfrac{1-z}{(2-z)^2},\dfrac{v}{2-z}\right)z^{\alpha-\gamma+1}(2-z)^{-\alpha}v^{\gamma-1}$
(9)	H_6	$J^0J^0+J^+-\alpha J^0\sim(1-\gamma)(1-\gamma+\alpha)$, $\mu=2\alpha-2\gamma+1$,
	H_8	$H_6\left(\alpha,\gamma,\dfrac{1-z}{(2-z)^2},\dfrac{v}{2-z}\right)(2-z)^{-\alpha}z^{\alpha-\gamma+1}v^{\gamma-1}$
(10)	H_4	$\left(J^0-\alpha+\delta+\frac{\mu}{2}-\frac{1}{2}\right)(J^0-\alpha+\frac{\mu}{2}+\frac{1}{2})$ $+J^+\left(J^0-\alpha+\beta+\frac{\mu}{2}+\frac{1}{2}\right)\sim 0$, $\mu=2\gamma-2$
	H_7	$H_4\left(\alpha,\beta,\gamma,\delta,\dfrac{z^2}{4(z-2)^2},-\dfrac{2v}{z-2}\right)(z-2)^{-\alpha}z^{\gamma-1/2}v^{\alpha-\gamma+1/2}$
(11)	H_7	$J^++\left(J^0+\alpha+\frac{\mu}{2}+\frac{1}{2}\right)\left(J^0-\alpha+\frac{\mu}{2}-\frac{1}{2}\right)\sim 0$, $\mu=2\gamma-2$;
		$H_7\left(\alpha,\gamma,\delta,\dfrac{z^2}{4(z-2)^2},-\dfrac{2v}{z-2}\right)(z-2)^{-\alpha}z^{\gamma-1/2}v^{\alpha-\gamma+1/2}$
(12)	H_9	$\left(J^0-\alpha-\beta+\frac{\mu}{2}+\frac{1}{2}\right)-J^+\left(J^0-\alpha+\frac{\mu}{2}+\frac{1}{2}\right)\sim 0$, $\mu=2\delta-2$;
		$H_9\left(\alpha,\beta,\delta,\dfrac{z^2}{4(z-2)^2},\dfrac{z-2}{2v}\right)(z-2)^{-\alpha}z^{\delta-1/2}v^{\alpha-\delta+1/2}$
(13)	H_{10}	$J^+\left(J^0-\alpha+\frac{1}{2}\right)\sim 1$, $\mu=2\delta-2$;
		$H_{10}\left(\alpha,\delta,\dfrac{z^2}{4(z-2)^2},\dfrac{2-z}{2iv}\right)(z-2)^{-\alpha}z^{\delta-1/2}v^{\alpha-\delta+1/2}$

coordinates and the corresponding operators K [second-order symmetric operators in the enveloping algebra of so(3)] are listed in Table 4. Here the separated solutions $F_\lambda=F_\lambda^{(1)}(u_1)F_\lambda^{(2)}(u_2)$ in the coordinates $\{u_1,u_2\}$ are characterized by (3.5) and the eigenvalue equation $KF_\lambda=\lambda F_\lambda$, where λ is the separation constant.

Recall that every second-order symmetric operator in the enveloping algebra of so(3) lies on the same orbit [under the adjoint action of so(3)] as exactly one of the operators on Table 4 [3]. Thus by choosing the parameters of the

Table 4
Separable coordinates for S_2

Coordinates	Defining operator
(A) $x_1=\cos u_1$, $x_2=\sin u_1\cos u_2$, $x_3=\sin u_1\sin u_2$	$(J^0)^2$
(B) $x_1=\frac{1}{2}[e^{-iu_1}+(1-u_2^2)e^{iu_1}]$, $x_2=u_2e^{iu_1}$, $x_3=-\frac{1}{2}i[e^{-iu_1}-(1+u_2^2)e^{iu_1}]$	$(J_1+iJ_3)^2$
(C) $x_1=\frac{1}{k'}\mathrm{dn}(u_1,k)\mathrm{dn}(u_2,k)$, $x_2=\frac{ik}{k'}\mathrm{cn}(u_1,k)\mathrm{cn}(u_2,k)$, $x_3=k\,\mathrm{sn}(u_1,k)\mathrm{sn}(u_2,k)$	$J_1^2+k^2J_2^2$
(D) $x_1=\tanh u_1\tanh u_2$, $x_2=i\left[\frac{1}{\cosh u_1\cosh u_2}-\frac{1}{2}\left(\frac{\cosh u_2}{\cosh u_1}+\frac{\cosh u_1}{\cosh u_2}\right)\right]$, $x_3=\frac{1}{2}\left(\frac{\cosh u_2}{\cosh u_1}+\frac{\cosh u_1}{\cosh u_2}\right)$	$-J_1^2-J_2^2+J_3^2-i(J_2J_3+J_3J_2)$
(E) $x_1=-\frac{i}{8u_1u_2}[4+(u_1^2-u_2^2)^2]$, $x_2=\frac{1}{2u_1u_2}[u_1^2+u_2^2]$, $x_3=\frac{1}{8u_1u_2}[4-(u_1^2-u_2^2)^2]$	$(J_3J_2+J_2J_3)+i(J_3J_1+J_1J_3)$

operators in Tables 2 and 3 so that these operators become purely second-order and by identifying the orbits on which the operators lie, we can find all cases in which the operator equations yield solutions via *pure* separation of variables. The possibilities are listed in Table 5.

Note that only two separable systems appear as special cases of the systems in Tables 2 and 3: the type-D coordinates which lead to solutions that are products of Gegenbauer functions, and the type-E coordinates which yield solutions that are products of Bessel functions.

Consider for example the F_2-operator equations in Table 5. If $F(z,v)$ is a solution of these equations, then

$$\exp\left[\left(\ln 2+i\frac{\pi}{2}\right)J^0\right]\exp\left[\tfrac{1}{2}J^+\right]F(z,v) = F\left(\frac{z}{1+iv},\frac{2iv}{1+iv}\right) \tag{3.7}$$

is a solution of (3.5) with $\mu=2\beta-1$ and an eigenfunction of $J^0J^0+J^+J^+$ with eigenvalue $(\alpha-\beta)^2$. It follows that if $F(z,v)$ is any solution of the F_2-equations

Table 5
Systems of Horn functions admitting separation on S_2

Function; separable-coordinate type		Operator equation
(1)	F_1; D	$\frac{1}{2}(J^0J^+ + J^+J^0) + J^0J^0 \sim \beta^2$, $\alpha=0, \beta'=\frac{1}{2}-\beta$
(3)	F_2; D	$\frac{1}{2}(J^0J^+ + J^+J^0) + J^0J^0 \sim (\alpha-\beta)^2$, $\gamma'=2(\alpha-\beta)+1, \beta'=\alpha-\beta+\frac{1}{2}$
(4)	H_4; D	$J^0J^0 + 4J^+J^+ \sim (\alpha-\beta)^2$, $\gamma=\alpha-\beta+\frac{1}{2}$
(7)	Ξ_2; E	$J^0J^+ + J^+J^0 \sim -2$, $\gamma=\frac{3}{2}$
(8)	G_1; D	$\frac{1}{2}(J^0J^+ + J^+J^0) + J^0J^0 \sim \alpha^2$, $\beta=-\alpha, \beta'=\frac{1}{2}$
(10)	H_4; D	$\frac{1}{2}(J^0J^+ + J^+J^0) + J^0J^0 \sim (\beta-\frac{1}{2})^2$, $\gamma=\alpha-\beta+1, \delta=2\beta$
(13)	H_{10}; E	$J^0J^+ + J^+J^0 \sim 2$, $\alpha=0$

on Table 5, then (3.7) can be expressed as a linear combination of four separable solutions. The first factor of each separable solution is either of the Legendre functions $P^{\beta-\alpha}_{\beta-1}(u_j)$, $Q^{\beta-\alpha}_{\beta-1}(u_j)$ for $j=1$, and the second factor is the same except $j=2$. Thus, for $\alpha=0,-1,-2,\ldots$, we have

$$F_2\left(\alpha,\beta,\alpha-\beta+\tfrac{1}{2},2\beta,2(\alpha-\beta)+1,X,X\sqrt{(u_1^2-1)(u_2^2-1)}\,\right)X^{\alpha}$$

$$= c\,{}_2F_1\left(\begin{matrix}\alpha,\ \alpha+1-2\beta\\ \alpha-\beta+1\end{matrix}\middle|\frac{1-u_1}{2}\right){}_2F_1\left(\begin{matrix}\alpha,\ \alpha+1-2\beta\\ \alpha-\beta+1\end{matrix}\middle|\frac{1-u_2}{2}\right), \quad (3.8)$$

$$X=2\left[\sqrt{(u_1^2-1)(u_2^2-1)}+u_1u_2+1\right]^{-1}.$$

Comparing the two sides of the equation for $u_2=1$, we find

$$c=\frac{\Gamma(2\beta)\Gamma(\beta-\alpha)}{\Gamma(2\beta-\alpha)\Gamma(\beta)}.$$

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