CHAPTER 5

The Hypergeometric Function and Its Generalizations

5.1 The Lauricella Functions $F_D$

The Gaussian hypergeometric function $_2F_1$ is intimately associated with the Laplace and wave equations in four-dimensional space and their complexifications. The $_2F_1$ arises from these equations via separation of variables, and the conformal symmetry groups account for many of the properties of this function. Rather than pursue this differential equations approach, we shall instead treat the hypergeometric function directly, together with some of its generalizations that have been studied in the past 150 years.

The Lauricella functions $F_A, F_B, (B.21)-(B.24)$, were defined by Lauricella [71] and studied in detail by Appell and Kampe de Feriet [4]. As can easily be seen from the power series definitions, these functions are generalizations of $_2F_1$ to $n$ complex variables: When $n = 1$, each of $F_A, F_B$ reduces to $_2F_1$. Although the Lauricella functions were not originally obtained by separation of variables, we shall see that each of these functions can be obtained by a partial separation of variables in a system of $n$ second-order partial differential equations. The $F_D$ are most interesting generalizations of the $_2F_1$ from a group-theoretic view, and we shall devote much of our attention to these functions, obtaining results for $F_A$ by setting $n = 1$. As can be seen from (B.24),

$$F_D[a; b_1, \ldots, b_n; c; z_1, \ldots, z_n]$$

depends on $n+2$ complex parameters $a, b, c$ and $n$ complex variables $z_1, \ldots, z_n$. For $n = 1$ we have

$$F_D[a; b; c; z] \equiv _2F_1\left(\begin{array}{c}a, b \\ c\end{array}\mid z\right).$$

(1.1)
We can use (B.24) to derive differential recurrence relations obeyed by the functions \( F_D \) and then construct a Lie algebra from the recurrence relations. The details are similar to computations presented earlier, so here we list only the results. We define a family of functions

\[
\Psi_c^{a_1, \ldots, a_n}(s, u_1, \ldots, u_n, t, z_1, \ldots, z_n) = \Psi_c^{a, b}(s, u_j, t, z_j)
\]

\[
= \frac{\Gamma(c-a)}{\Gamma(c)} F_D(a; b_1, \ldots, b_n; c, z_1, \ldots, z_n) s^a u_1^{b_1} \cdots u_n^{b_n} t^c
\]  

(1.2)

where \( c \neq 0, -1, -2, \ldots \), and \( s, u_j, t \) are complex variables. Furthermore, we define operators:

\[
E^a = s \left( \sum_j z_j \partial_{z_j} + s \partial_t \right), \quad E^{a \beta, \gamma} = sz_k u_k t \partial_{z_k},
\]

\[
E^\beta_k = u_k (z_k \partial_{z_k} + u_k \partial_{u_k}), \quad E_\gamma = t^{-1} \left( \sum_j z_j \partial_{z_j} + t \partial_t - 1 \right),
\]

\[
E^{a \gamma} = st \left( \sum_j (1 - z_j) \partial_{z_j} - s \partial_s \right), \quad E^\gamma = t \left( \sum_j (1 - z_j) \partial_{z_j} + t \partial_t - s \partial_s - \sum_j u_j \partial_{u_j} \right),
\]

\[
E_\alpha = s^{-1} \left( \sum_j z_j (1 - z_j) \partial_{z_j} + t \partial_t - s \partial_s - \sum_j z_j u_j \partial_{u_j} \right),
\]

\[
E^\beta_k = u_k^{-1} \left( z_k (1 - z_k) \partial_{z_k} + \sum_{j \neq k} (1 - z_j) \partial_{z_j} + t \partial_t - z_k s \partial_s - \sum_j u_j \partial_{u_j} \right),
\]

\[
E^{\beta, \gamma} = u_k t \left( (z_k - 1) \partial_{z_k} + u_k \partial_{u_k} \right),
\]

\[
E_{\alpha \gamma} = s^{-1} t^{-1} \left( \sum_j z_j (1 - z_j) \partial_{z_j} - \sum_j z_j u_j \partial_{u_j} + t \partial_t - 1 \right)
\]

\[
E_{\alpha \beta_k} = s^{-1} u_k^{-1} t^{-1} \left( \sum_j z_j (z_j - 1) \partial_{z_j} - t \partial_t + z_k s \partial_s + \sum_j z_j u_j \partial_{u_j} - z_k + 1 \right),
\]

\[
E^{\beta, \gamma} = u_k^{-1} t^{-1} \left( z_k (z_k - 1) \partial_{z_k} + \sum_{j \neq k} (z_k - 1) z_j \partial_{z_j} + z_k s \partial_s - t \partial_t + 1 \right),
\]

\[
E^{\beta, \gamma} = u_k^{-1} (z_k - z_p) \partial_{z_k} + u_k \partial_{u_k}, \quad J_\alpha = s \partial_s - \frac{1}{2} t \partial_t,
\]

\[
J_{\beta_k} = u_k \partial_{u_k} - \frac{1}{2} t \partial_t + \frac{1}{2} \sum_{j \neq k} u_j \partial_{u_j},
\]

\[
J_\gamma = t \partial_t - \frac{1}{2} (s \partial_s + \sum_j u_j \partial_{u_j} + 1), \quad k, p = 1, 2, \ldots, n.
\]

(1.3)
Unless otherwise indicated, \( j \) is summed from 1 to \( n \). The action of the foregoing operators on the basis (1.2) is

\[
E_{a}^{b} \Psi_{c}^{a,b} = (c - a - 1) \Psi_{c+1}^{a+1,b},
\]

\[
E_{b}^{a} \Psi_{c}^{a,b} = b_{k} \Psi_{c}^{a,b},
\]

\[
E_{c}^{a} \Psi_{c}^{a,b} = \left( \sum_{j} b_{j} - c \right) \Psi_{c+1}^{a+1,b},
\]

\[
E_{a} \Psi_{c}^{a,b} = (a - 1) \Psi_{c-1}^{a-1,b},
\]

\[
E_{b} \Psi_{c}^{a,b} = b_{k} \Psi_{c}^{a,b},
\]

\[
E_{c} \Psi_{c}^{a,b} = \left( c - \sum_{j} b_{j} \right) \Psi_{c+1}^{a,b},
\]

\[
E_{a} \Psi_{c}^{a,b} = \left( c - \sum_{j} b_{j} \right) \Psi_{c+1}^{a,b},
\]

\[
E_{b} \Psi_{c}^{a,b} = b_{k} \Psi_{c}^{a,b},
\]

\[
E_{c} \Psi_{c}^{a,b} = \left( c - \sum_{j} b_{j} \right) \Psi_{c+1}^{a,b},
\]

\[
E_{a} \Psi_{c}^{a,b} = (a - 1) \Psi_{c-1}^{a-1,b},
\]

\[
E_{b} \Psi_{c}^{a,b} = (a - c + 1) \Psi_{c-1}^{a,b},
\]

\[
E_{c} \Psi_{c}^{a,b} = (a - c / 2) \Psi_{c}^{a,b},
\]

\[
J_{a} \Psi_{c}^{a,b} = (b_{k} - \frac{1}{2} c + \frac{1}{2} \sum_{j \neq k} b_{j}) \Psi_{c}^{a,b},
\]

\[
J_{b} \Psi_{c}^{a,b} = \left[ c - \frac{1}{2} \left( a + \sum_{j} b_{j} + 1 \right) \right] \Psi_{c}^{a,b},
\]

\[
J_{c} \Psi_{c}^{a,b} = \left[ c - \frac{1}{2} \left( a + \sum_{j} b_{j} + 1 \right) \right] \Psi_{c}^{a,b},
\]

\[k, p = 1, \ldots, n. \quad (1.4)\]

The symbols \( \bar{b}_{k} \) and \( \bar{b}_{k} \) are defined by

\[
\bar{b}_{k} = b_{1}, \ldots, b_{k-1}, b_{k}, b_{k+1}, \ldots, b_{n},
\]

\[
\bar{b}_{k} = b_{1}, \ldots, b_{k-1}, b_{k}, b_{k+1}, \ldots, b_{n}. \quad (1.5)\]

The differential recurrence relations for the \( F_{D} \) are obtained by factoring the dependence on \( s, y, \) and \( t \) from both sides of the expressions (1.4). Moreover, for \( n = 1 \) these relations reduce exactly to the recurrence formulas (B.5) for the \( 2F_{1} \).

Relations (1.4) can be verified by routine computation. Furthermore, it is straightforward to show that the operators (1.3) form a basis for the Lie algebra \( sl(n+3, \mathbb{C}) \) of dimension \( (n+3)^{2} - 1 \). Recall that \( SL(n+3, \mathbb{C}) \) is the group of all \( (n+3) \times (n+3) \) complex matrices \( A \) such that \( \det A = 1 \). The Lie algebra \( sl(n+3, \mathbb{C}) \) of \( SL(n+3, \mathbb{C}) \) consists of all \( (n+3) \times (n+3) \) complex matrices \( \mathcal{C} \) such that \( \text{tr} \mathcal{C} = 0 \) [85]. Denoting by \( \delta_{ij} \) the matrix with a one in row \( i \), column \( j \), and zeros everywhere else (see (6.4), Section 3.6), we see that the matrices \( \delta_{ij}, i \neq j, \) and \( \delta_{ii} - \delta_{33}, 1 \leq i, j \leq n+3, \) form a
basis for $sl(n + 3, \mathbb{C})$. The commutation relations can be obtained from the general formula

\[ [\mathcal{E}_{ji}, \mathcal{E}_{kl}] = \delta_{jk} \mathcal{E}_{li} - \delta_{il} \mathcal{E}_{kj}. \]  

We can check that the appropriate commutation relations are satisfied if we make the identifications

\[
\begin{align*}
E^a &= \mathcal{E}_{12}, \\
E_a &= \mathcal{E}_{21}, \\
E^\beta &= \mathcal{E}_{k+3,3}, \\
E_\beta &= \mathcal{E}_{k+3,p+3}, \\
E_\gamma &= \mathcal{E}_{31}, \\
E^\alpha_\gamma &= \mathcal{E}_{32}, \\
E_\alpha_\gamma &= \mathcal{E}_{23}, \\
E_\alpha\beta_\gamma &= -\mathcal{E}_{k+3,2}, \\
J_\alpha &= \frac{1}{2}(\mathcal{E}_{11} - \mathcal{E}_{22}), \\
J_\beta &= \frac{1}{2}(\mathcal{E}_{k+3,k+3} - \mathcal{E}_{33}), \\
J_\gamma &= \frac{1}{2}(\mathcal{E}_{33} - \mathcal{E}_{11}), \\
&1 \leq k, p \leq n, k \neq p.
\end{align*}
\]  

Let

\[ C_k = E^a E_\beta - E^\alpha_\beta_\gamma E_\gamma, \quad 1 \leq k \leq n. \]  

It is straightforward to check that the solution $f$ of the simultaneous equations

\[
\begin{align*}
C_k f &= 0, \\
J_\alpha f &= (a - c/2) f, \\
J_\beta f &= \left( b_k - \frac{1}{2} c + \frac{1}{2} \sum_{j \neq k} b_j \right) f, \\
J_\gamma f &= \left[ c - \frac{1}{2} \left( a + \sum_{j} b_j + 1 \right) \right] f, \\
&k = 1, \ldots, n,
\end{align*}
\]  

analytic in a neighborhood of $z_1 = \ldots = z_n = 0$ is

\[ f = F_D(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n) s^n u_1^b \cdots u_n^b t^c, \]  

unique to within a multiplicative constant. In fact, the last $n + 2$ equations imply

\[ f = F(z_1, \ldots, z_n) s^n u_1^b \cdots u_n^b t^c \]

and the first $n$ imply

\[
\left\{ \left( \sum_{j=1}^n z_j \partial_{z_j} + a \right) (z_k \partial_{z_k} + b_k) - \partial_{z_k} \left( \sum_{j=1}^n z_j \partial_{z_j} + c - 1 \right) \right\} F = 0,
\]  

\[ k = 1, \ldots, n, \]
which are the partial differential equations for \(F_D\). The operators \(C_k\) do not commute with all the elements of \(sl(n+3, \mathcal{C})\), but each element leaves the solution space of the system of equations invariant. It follows from these remarks that if \(\Psi(s, u_z, t, z)\) is a solution of \(C_k \Psi = 0, k = 1, \ldots, n\), which has a Laurent expansion

\[
\Psi = \sum_{a, b_1, \ldots, b_n, c} g_{ab_1 \ldots b_n c} \Psi^1 \Psi^2 \cdots \Psi^n \Psi^c,
\]

and if \(\Psi\) is analytic at \(z_1 = \cdots = z_n = 0\), then

\[
g_{ab_1 \ldots b_n c} = k(ab_1 \ldots b_n c) F_D(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n)
\]

where \(k\) is a constant. Moreover, the elements of \(sl(n+3, \mathcal{C})\) map a solution of \(C_k \Psi = 0, 1 \leq k \leq n\), into another solution.

We see now that the \(F_D\) arise as separable solutions of the system of \(n\) second-order partial differential equations \(C_k \Psi = 0, 1 \leq k \leq n\), in the coordinates \(s, u_z, t, z_j\). To simplify this system we perform the \(R\)-transformation \(\Phi = t^{-1} \Psi\) to remove the multiplier term from \(E_\gamma\). Then we transform to new variables \(v, v_j, w, w_j\) such that

\[
E^a = \partial_v, \quad E^b_z = \partial_{v_z}, \quad E^{a_1 \ldots a_n} = \partial_{w_k}, \quad E_\gamma = \partial_w.
\]

Explicitly,

\[
s = -1/v, \quad u_j = -1/v_j, \quad t = w, \quad z_j = w v_j / v_j, \quad 1 \leq j \leq n,
\]

and the equations \(C_k \Psi = 0\) become

\[
(\partial_v \partial_{v_z} - \partial_w \partial_{w_k}) \Phi = 0, \quad 1 \leq k \leq n.
\]

From (1.2) it follows that the equations (1.15) have solutions

\[
\Phi^{a, b} = t^{-1} \Psi^{a, b} = \frac{\Gamma(c - a) \Gamma(a)}{\Gamma(c)} F_D(a; b_1, \ldots, b_n; c; \frac{ww_j}{vv_j}, \ldots, \frac{ww_n}{vv_n})
\times \left(-\frac{1}{v}\right)^{a-b_1} \cdots \left(-\frac{1}{v_n}\right)^{b_n} w_v^{-1} t^{b-c-1}
\]

In the special case where \(n = 1\) we can set \(v = (z + t)/2, v_1 = (z - t)/2, w = \)
\[(ix+y)/2, \omega = (ix-y)/2, \text{ and transform (1.15) to the complex wave equation}\]
\[
(\partial_t - \partial_{xx} - \partial_{yy} - \partial_{zz}) \Phi(t, x, y, z) = 0
\]

for which (1.16) yields the \( F_1 \) as separated solutions. It is straightforward to show that the symmetry algebra of this equation is \( \mathfrak{o}(6, \mathcal{C}) = \mathfrak{sl}(4, \mathcal{C}) \).

Returning now to the operators (1.3), we will determine the group action of \( SL(n+3, \mathcal{C}) \) induced by these operators. Rather than determine the global group action, we note that each of the triplets
\[
\{ J^+, J^-, J^0 \} \equiv \{ E^\alpha, E_\alpha, J_\alpha \}, \{ E^p, E_p, J_p \}, \{ E^\alpha \gamma, E_\alpha \gamma, J_\alpha + J_\gamma \},
\{ E^{\alpha \gamma}, E^{\alpha \gamma}, J_\alpha + J_\gamma \}, \{ E^{\gamma \alpha}, E^{\gamma \alpha}, J_\gamma + J_\alpha \},
\{ E^{p \gamma}, E^{p \gamma}, J_p + J_\gamma \}, 1 \leq k \leq n, 1 \leq j < p \leq n, \]

satisfies the commutation relations
\[
[J^0, J^\pm] = \pm J^\pm, [J^+, J^-] = 2J^0
\]
and forms a basis for a subalgebra of \( \mathfrak{sl}(n+3, \mathcal{C}) \) isomorphic to \( \mathfrak{sl}(2, \mathcal{C}) \). Furthermore, each triplet generates a local Lie subgroup of \( SL(n+3, \mathcal{C}) \) isomorphic to \( SL(2, \mathcal{C}) \) and the subgroups so obtained suffice to generate the full group action of \( SL(n+3, \mathcal{C}) \).

We pass from the Lie algebra action generated by \( \{ J^+, J^-, J^0 \} \) to the group action via the relation
\[
T(A) = \exp(-bd^{-1}J^+) \exp(-cdJ^-) \exp(\tau J^0), \quad \exp(\tau/2) = d^{-1},
\]

where
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathcal{C})
\]
(see (4.14), Section 2.4). We find that the triplet \( \{ E^\alpha, E_\alpha, J_\alpha \} \) generates the group action
\[
T_1(A) \Psi(s, u_j, t, z_j) = \Psi \left[ \frac{as + c}{d + bs} \cdot \frac{u_j(as + c)}{as + c(1-z_j)} \cdot \frac{ts}{as + c} \cdot \frac{z_j s}{(d + bs)(as - cz_j + c)} \right]
\]
and the triplet \( \{ E^\beta_k, E_{\beta_k}, J_{\beta_k} \} \) generates

\[
T_{2,k}(A)\Psi(s, u_j, u_k, t, z_j, z_k) = \Psi \left( \frac{s(au_k + c)}{au_k + c(1 - z_k)}, \frac{u_j}{u_k} \frac{(au_j + c)}{d + bu_k}, \frac{u_k t}{au_k + c}, \frac{au_k z_j + c(z_j - z_k)}{au_k + c(1 - z_k)}, \frac{z_k u_k}{(d + bu_k)(au_k - cz_k + c)} \right),
\]

\( k = 1, \ldots, n. \)

In (1.20) \( j \) runs from 1 to \( n \) excluding \( k \). The triplet \( \{ E^\gamma, E_{\gamma}, J_{\gamma} \} \) generates

\[
T_3(A)\Psi(s, u_j, t, z_j) = \left( a + \frac{c}{t} \right)^{-1} \Psi \left( \frac{s(d + bt)}{d + bt}, \frac{u_j(d + bt)}{d + bt} \right),
\]

\[
\frac{at + c}{d + bt}, \left[ \frac{dz_j - br(1 - z_j)}{(a + \frac{c}{t})} \right],
\]

the triplet \( \{ E^{\alpha_k \gamma}, E_{\alpha_k \gamma}, J_{\alpha} + J_{\beta_k} + J_{\gamma} \} \) generates

\[
T_{4,k}(A)\Psi(s, u_j, u_k, t, z_j, z_k) = \left( a + \frac{c(1 - z_k)}{u_k ts} \right)^{-1} \Psi \left( \frac{as - cz_k}{u_k t}, \frac{u_j}{u_k} \frac{as u_k t - cz_k}{as u_k t + cz_j - cz_k}, \frac{cz_k}{st}, t \frac{as u_k t + c(1 - z_k)}{as u_k t - cz_k}, \frac{z_k d - bs u_k t}{as u_k t + c(z_j - z_k)} \right), \]

the triplet \( \{ E^{\alpha \gamma}, E_{\alpha \gamma}, J_{\alpha} + J_{\gamma} \} \) generates

\[
T_5(A)\Psi(s, u_j, t, z_j) = \left( a - \frac{c}{st} \right)^{-1} \Psi \left( \frac{s}{d - bst}, \frac{u_j st}{ast - cz_j}, \frac{at - c}{s} \frac{(dz_j - bst)(ast - c)}{(ast - cz_j)(d - bst)} \right),
\]

(1.23)
the triplet \( \{ E^{\alpha \beta \gamma}, E^{\beta \gamma}, J^{\beta \gamma}, J^{\gamma} \} \) generates

\[
T_{6,k}(A) \Psi(s, u_j, u_k, t, z_j, z_k) = \left( a + \frac{c}{u_k t} \right)^{-1} \Psi \left( \frac{su_k t}{a u_k t + cz_k} \cdot u_j, \frac{u_k t}{d + bu_k t} \cdot at + \frac{c}{u_k t}, \frac{z_j (au_k t + c)}{a u_k t + cz_k}, \frac{dz_k + bu_k t (au_k t + c)}{(d + bu_k t) (au_k t + cz_k)} \right).
\]  
(1.24)

and the triplet \( \{ E^{\beta \gamma}, E^{\gamma}, J^{\beta \gamma} - J^{\gamma} \} \) generates

\[
T_{7,k,p}(A) \Psi(s, u_j, u_k, u_p, t, z_j, z_k, z_p) = \Psi \left( s, u_j, \frac{u_k u_p}{du_p + bu_k}, \frac{u_p u_k}{au_k + cu_p}, t, z_j, \frac{dz_k u_p + bz_p u_k}{du_p + bu_k}, \frac{az_k u_k + cz_k u_p}{au_k + cu_p} \right), \quad 1 \leq k < p \leq n.
\]  
(1.25)

Each of the operators \( T_i(A) \) maps a solution \( \Psi \) of the system \( C_k \Psi = 0, \ 1 \leq k \leq n \), into another solution.

To compute the matrix elements of the group operators \( T_i(A) \) with respect to the basis \( \{ \Psi_{c}^{a,b} \} \) it is useful to construct a simpler model of relations (1.4). Such a model is provided by the functions

\[
J_{c,a,b}^{a,b}(s, u_j, t) = s^a u_1^{b_1} \cdots u_n^{b_n} t^c
\]

of \( n+2 \) complex variables and the operators

\[
E^a = s(t \partial_t - s \partial_s - 1), \quad E^{a \beta \gamma} = su_k^2 t \partial_{u_k}, \quad E^{\beta \gamma} = u_k^2 \partial_{u_k},
\]

\[
E_\gamma = t^{-1}(t \partial_t - s \partial_s - 1), \quad E_\alpha = s^{-1}(s \partial_s - 1),
\]

\[
E^{\beta \gamma} = u_k^{-1} \left( t \partial_t - \sum_j u_j \partial_{u_j} \right), \quad 1 \leq k \leq n,
\]  
(1.26)

which generate \( sl(n+3, \mathbb{C}) \). For some examples of matrix elements computed in this way and associated generating functions for the \( F_D \) and the \( \frac{1}{2} F_1 \) see [90] and [82, Chapter 5].
5.2 Transformation Formulas and Generating Functions for the $F_D$

We now show that the transformation formulas for the $F_D$ are consequences of the $SL(n + 3, \mathbb{C})$ symmetry. Let

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{C}).$$

Expressions (1.2) and (1.19) imply

$$T_1(I) \Psi_c^{a,b} = (-1)^{a+c} \frac{\Gamma(c-a) \Gamma(a)}{\Gamma(c)} F_D \left( a; b_j; c; -\frac{z_j}{1-z_j} \right) \times (1-z_1)^{-b_1} \cdots (1-z_n)^{-b_n} s_c^{-a} u_1^{b_1} \cdots u_n^{b_n} t^c$$

(2.2)

However, $T_1(I) \Psi_c^{a,b}$ is a simultaneous eigenfunction of $J_\alpha, J_\beta, J_\gamma$ analytic at $z_1 = \cdots = z_n = 0$. Thus,

$$T_1(I) \Psi_c^{a,b} = k F_D (c-a; b_j; c; z_j) s_c^{-a} u_1^{b_1} \cdots u_n^{b_n} t^c.$$  

(2.3)

Setting $z_1 = \cdots = z_n = 0$ in (2.2), (2.3), we can evaluate the constant $k$ and obtain the transformation formula

$$F_D \left( a; b_j; c; \frac{z_j}{z_j-1} \right) = F_D (c-a; b_j; c; z_j)$$

(2.4)

(see [4, Chapter VII]). Similarly, $T_{2,k}(I) \Psi_c^{a,b}$ yields the formulas

$$F_D \left( a; b_j, b_k; c; \frac{z_k-z_j}{z_k-1}, \frac{z_k}{z_k-1} \right)$$

$$= F_D \left( a; b_j, c - \sum b_l; c; z_j, z_k \right), \quad k=1, \ldots, n.$$  

(2.5)

The remaining transformation formulas for the $F_D$ can be obtained by composition from (2.4) and (2.5). The transformation formulas for $F_D$ follow from (2.4) for $n = 1$.

Computing $T_3(I) \Psi_c^{a,b}$, we find that

$$F_D \left( a; b_j; a + \sum b_l - c + 1; 1-z_j \right)$$

(2.6)

is a solution of equations (1.11), analytic at $z_1 = \cdots = z_n = 1$. Computing
\[ T_{s}(J)\Psi_{c}^{a,b} \text{, we see that} \]
\[ z_{1}^{-b_{1}} \cdots z_{n}^{-b_{n}} F_{D} \left( \sum_{l} b_{l} - c + 1; b_{j}; \sum_{l} b_{l} - a + 1; z_{j}^{-1} \right) \quad (2.7) \]

is another solution of (1.11). Similarly, \( T_{6,k}(J)\Psi_{c}^{a,b} \) yields the solution
\[ z_{k}^{-a} F_{D} \left( a; b_{j}, a - c + 1; a - b_{k} + 1; \frac{z_{j}}{z_{k}}, \frac{1}{z_{k}} \right). \quad (2.8) \]

For \( A \) close to the identity in \( SL(2,\mathbb{C}) \) the expressions \( T_{j}(A)\Psi_{c}^{a,b} \) can be expanded by use of matrix elements computed from the recurrence relations (1.4). However, for \( A \) far from the identity (e.g., \( A = I \)), these expansions are no longer valid. For example,
\[ \exp(\lambda E_{c}^{a}) \Psi(s,u_{j},t,z_{j}) = \Psi \left( \frac{s}{1+\lambda st}, u_{j}, t, \frac{z_{j} + \lambda st}{1+\lambda st} \right). \quad (2.9) \]

For \( |\lambda| \) small we find
\[ \exp(\lambda E_{c}^{a}) \Psi_{c}^{a,b} = \sum_{h=0}^{\infty} \left( \sum_{h} b_{l} - c \right)_{h} \Psi_{c+h}^{a+h} \lambda^{h}, \]
that is,
\[ (1+\lambda)^{-a} F_{D} \left( a; b_{j}; c; \frac{z_{j} + \lambda}{1+\lambda} \right) \]
\[ = \sum_{h=0}^{\infty} \left( \sum_{h} b_{l} - c \right)_{h} \frac{(a)_{h}}{(c)_{h}} F_{D}(a+h;b_{j};c+h;z_{j}) \lambda^{h}, \quad (2.10) \]

\[ |\lambda| < 1. \]

If \( \lambda = 1 \) and \( |\tau| < 1 \) where \( \tau = s^{-1}t^{-1} \), then \( \exp(E_{c}^{a})\Psi_{c}^{a,b} \) is not analytic at \( z_{1} = \cdots = z_{n} = \tau = 0 \). However, we can apply \( \exp(E_{c}^{a}) \) to the solution (2.6) and use (1.12), (1.13) to obtain
\[ (1+\tau)^{-a} F_{D} \left( a; b_{j}; a + \sum_{l} b_{l} - c + 1; \frac{\tau(1-z_{j})}{1+\tau} \right) \]
\[ = \sum_{h=0}^{\infty} B_{h} F_{D} (-h;b_{j};c-a-h;z_{j}) \tau^{h}. \quad (2.11) \]
To evaluate the constants \( B_h \) we set \( z_1 = \cdots = z_n = 0 \):

\[
(1+\tau)^{-a} F_D \left( a; b_j; a + \sum_l b_l - c + 1; \tau/(1+\tau) \right)
= (1+\tau)^{-a} {}_2F_1 \left( a, \sum_l b_l; a + \sum_l b_l - c + 1; \tau/(1+\tau) \right) = \sum_{h=0}^{\infty} B_h \tau^h.
\]

Thus,

\[
B_h = \binom{-a}{h} {}_2F_1 \left[ \begin{array}{c} -h, \sum_l b_l \\ a + \sum_l b_l - c + 1 \end{array} \right] = \binom{-a}{h} \frac{(a-c+1)_h}{(a + \sum_l b_l - c + 1)_h} (2.12)
\]

from [82, p. 211], and Vandermonde's theorem [120, p. 28].

Expanding \( T_l(A) \psi^{a,\beta}_\gamma \) as a power series in \( \tau = s^{-1} \), we obtain

\[
a^{-\gamma} b^{-\gamma} \left( 1 + \frac{\alpha}{a} \right)^{a - \sum \beta_j - \gamma} \left( 1 + \frac{\beta_j}{b} \right)^{-\beta_j} \left( 1 + \frac{\alpha (1 - z_l)}{a} \right)^{-\alpha} \\
\cdots \left( 1 + \frac{\alpha (1 - z_n)}{a} \right)^{-\alpha} F_D \left( \alpha; \beta_j; \gamma; \frac{z_j \tau}{(b + \alpha \tau)(a + \alpha \tau(1 - z_j))} \right)
= \sum_{h=0}^{\infty} B_h F_D (-h; \beta_j; \gamma; z_j) \tau^h. (2.13)
\]

Setting \( z_1 = \cdots = z_n = 0 \) and using identity (5.124) [82, p. 206], we find

\[
B_h = \binom{a}{b} a^{-\gamma} \binom{-\gamma}{h} {}_2F_1 \left( \begin{array}{c} -h, \alpha \\ -1/bc \end{array} \right), \quad ad - bc = 1. (2.14)
\]

If \( a = b = d = 1 \) and \( c = 0 \), the identity becomes

\[
(1+\tau)^{-a} F_D \left( \alpha; \beta_j; \gamma; \frac{z_j \tau}{1+\tau} \right) = \sum_{h=0}^{\infty} \binom{-\alpha}{h} F_D (-h; \beta_j; \gamma; z_j) \tau^h, (2.15)
\]

\(|\tau| < 1\),
and, if \( a = c = 1, b = -w^{-1} \) it reduces to

\[
(1 + \tau)^{a + \sum \beta_i - 1}[1 + (1 - w)\tau]^{-a} \prod_{l=1}^{n} [1 + (1 - z_l)\tau]^{-\beta_l} \\
\times F_D\left(\alpha; \beta_j; \gamma; \frac{-z_j\tau w}{[1 + (1 - w)\tau][1 + (1 - z_j)\tau]}\right) \\
= \sum_{h=0}^{\infty} \binom{-\gamma}{h} \binom{-h, \alpha}{\gamma} F_D(-h; \beta_j; \gamma; z_j)\tau^k,
\]

(2.16)

\(|\tau| < \min(1, |1 - z_j|^{-1}, |1 - w|^{-1}).\)

More generally, we can derive generating functions for the \( F_D \) through the characterization of a solution \( \Psi \) of \( C_k \Psi = 0, 1 \leq k \leq n \), by the requirement that \( \Psi \) is a simultaneous eigenfunction of \( n + 2 \) commuting (or almost commuting) operators constructed from the enveloping algebra of \( \mathfrak{sl}(n + 3, \mathbb{C}) \). Such a characterization of \( \Psi^{a,b} \) is given by (1.9).

As an example we compute the solution \( \Psi \) of the simultaneous equations

\[
E^a \Psi = \Psi, \quad J_{\beta_k} \Psi = \left( \beta_k + \frac{1}{2} \sum_{j \neq k} \beta_j - \frac{1}{2} \gamma \right) \Psi,
\]

\[
(J_{\gamma} + \frac{1}{2} J_{a}) \Psi = \left( \frac{3}{4} \gamma - \frac{1}{2} \sum_{j} \beta_j - \frac{1}{2} \right) \Psi, \quad C_k \Psi = 0, \quad k = 1, \ldots, n,
\]

(2.17)

which is analytic at \( z_1 = \cdots = z_n = 0 \). The first \( n + 2 \) equations have the general solution

\[
\Psi = f\left( \frac{z_j}{s} \right) \exp(-s^{-1}) u_1^{\beta_1} \cdots u_n^{\beta_n} \gamma
\]

where \( f \) is arbitrary. Substitution of this expression into \( C_k \Psi = 0, 1 \leq k \leq n \), yields

\[
f(x_j) = \Phi(\beta_j; \gamma; x_j)
\]

\[
= \sum_{m_j=0}^{\infty} \frac{(\beta_1)_m \cdots (\beta_n)_m x_1^{m_1} \cdots x_n^{m_n}}{(\gamma)_{m_1 + \cdots + m_n} m_1! \cdots m_n!}
\]

\[
= \lim_{\alpha \to \infty} F_D\left( \alpha; \beta_j; \gamma; \frac{z_j}{\alpha} \right).
\]

(2.18)
Expanding $T_1(A)\Psi$ as a power series in $\tau = s^{-1}$, we obtain

$$
\exp \left[-\left(\frac{dt + b}{a + c\tau}\right)\right] (a + c\tau)^{\sum_{l=1}^{n} \beta_l - \gamma} \prod_{l=1}^{n} \left[ a + c\tau(1 - z_l) \right]^{-\beta_l} \\
\times \Phi \left( \frac{z_j\tau}{(a + c\tau)[a + c\tau(1 - z_j)]} \right) = \sum_{h=0}^{\infty} B_h F_D (-h; \beta_j; \gamma; z_j) \tau^h, \quad ad - bc = 1.
$$

(2.19)

Setting $z_1 = \cdots = z_n = 0$ and using the generating function (4.11), Section 2.4, for Laguerre polynomials, we find

$$
B_h = a^{-\gamma}e^{-b/a}(c/a)^h L^{(\gamma - 1)}((ac)^{-1}),
$$

(2.20)

where $L^{(\alpha)}_n(z)$ is a generalized Laguerre polynomial. If $b = c = 0, a = d = 1$, the identity simplifies to

$$
\exp(-\tau)\Phi(\beta_j; \gamma; z_j\tau) = \sum_{h=0}^{\infty} F_D (-h; \beta_j; \gamma; z_j)(-\tau)^h/h!.
$$

(2.21)

If $a = c = d^{-1} = w^{-1/2}, b = 0$, we find

$$
\exp \left[-\frac{w\tau}{(1 + \tau)}\right] (1 + \tau)^{\sum_{l=1}^{n} \beta_l - \gamma}[1 + \tau(1 - z_l)]^{-\beta_l} \\
\times \prod_{l=1}^{n} [1 + \tau(1 - z_n)]^{-\beta_l} \Phi \left( \frac{\beta_j}{\gamma} \frac{z_j\tau}{(1 + \tau)[1 + \tau(1 - z_j)]} \right) = \sum_{h=0}^{\infty} L^{(\gamma - 1)}(w) F_D (-h; \beta_j; \gamma; z_j) \tau^h, \quad |\tau| < \min(1, |z_j - 1|^{-1}).
$$

(2.22)

If $b = -c = 1, a = d = 0$, then $T_1(A)\Psi$ becomes

$$
e^s(1 - z_1)^{-\beta_1}(1 - z_n)^{-\beta_n} \Phi \left( \frac{z_j\gamma}{1 - z_j} \right) u_1^\beta_1 \cdots u_n^\beta_n t^\gamma.
$$

Expanding this function in powers of $s$, we obtain

$$
e^s(1 - z_1)^{-\beta_1}(1 - z_n)^{-\beta_n} \Phi \left( \frac{z_j\gamma}{1 - z_j} \right) = \sum_{h=0}^{\infty} F_D (\gamma + h; \beta_j; z_j) s^h/h!.
$$

(2.23)
Note that $\Phi$, (2.18), is a confluent form of $F_D$. For $n = 1$ we have

$$\Phi\left(\frac{\beta}{\gamma} \middle| z\right) = _1F_1\left(\frac{\beta}{\gamma} \middle| z\right).$$

Note also that we have demonstrated that the system of equations (1.15) admits a partial $R$-separation of variables in terms of the coordinates $z_j/s, s, u, t$ and the partially separated solutions are characterized by operator equations of the form (2.17). An exhaustive classification of generating functions for the $F_D$ awaits the classification of partially separable coordinate systems for the equations (1.15).

We can use the differential recurrence relations obeyed by the functions $\Phi$ and other confluent forms of the $F_D$ to obtain a Lie algebraic theory of these functions. The corresponding Lie algebras can also be derived as contractions of the symmetry algebra of the $F_D$ [90].

The Lauricella functions $F_{A, B, C}$ are other $n$-variable generalizations of the $F_1$ which can be treated by similar Lie algebraic methods (see [91, 92]). However, not all of the recurrence relations obeyed by the $F_1$ can be extended to these functions, which therefore seem somewhat less interesting than the $F_D$. Similarly, the generalized hypergeometric functions $\rho F_q$ can be treated by Lie algebraic methods [88].

**Exercises**

1. Show that the $2(p + q) + 1$ operators

$$E^a = t_x (z \partial_z + t_y \partial_y), \quad E_{\beta_k} = u^{-1}_k (z \partial_z + u_k \partial_u_k - 1),$$

$$E^{a_1 \cdots a_p} = t_{a_1} \cdots t_{a_p} u_{a_1} \cdots u_{a_p} \partial_z,$$

$$T_l = t_l \partial_{u_l}, \quad U_k = u_k \partial_{u_k},$$

$$l = 1, \ldots, p, \quad k = 1, \ldots, q,$$

form a basis for a Lie algebra $\mathfrak{g}_{p, q}$ by working out the commutation relations.

2. Making use of the differential recurrence formulas (B.20) for the generalized hypergeometric functions $\rho F_q$, determine the action of $\mathfrak{g}_{p, q}$ on the basis functions

$$\Psi_{\beta_j} (t_x, u_j, z) = _p F_q\left(\begin{array}{c} a_1^q \\ b_j^p \end{array} \middle| t_1^{a_1} \cdots t_p^{a_p} u_1^{b_1} \cdots u_q^{b_q} \right).$$

3. Show that the differential equation (B.19) for $\rho F_q$ is equivalent to $L_{p, q} \Psi_{\beta_j} = 0$ where

$$L_{p, q} = E^{a_1} \cdots E^{a_p} - E^{a_1} \cdots \beta_j E_{\beta_1} \cdots E_{\beta_q}.$$
4. Show that the $\Psi_{b_j}^{a}$ can be characterized as the solutions $\Psi(t, u_j, z)$ of the equations

$$L_{p, q} \Psi = 0, \quad T_l \Psi = a_l \Psi, \quad U_k \Psi = b_k \Psi, \quad 1 \leq l \leq p, \quad 1 \leq k \leq q,$$

which are analytic at $t = 0$. (This proves that the $p F_q$ arise from the partial differential equation $L_{p, q} \Psi = 0$ by a separation of variables.)

5. Prove that $\mathfrak{s}_{p, q}$ is a symmetry algebra of the equation $L_{p, q} \Psi = 0$.

6. Determine the special function identities for the $p F_q$ which are associated with the expressions

$$\exp(c E^{\alpha}) \Psi_{b_j}^{a}, \quad \exp(c E^{\beta}) \Psi_{b_j}^{a}, \quad \text{and} \quad \exp(c E^{\alpha_1, \ldots, \alpha_s}) \Psi_{b_j}^{a}.$$

7. Use Weisner's principle and the expression $\exp(1 E^{\alpha}) \Psi_{b_j}^{a}$ to derive the identity

$$(1 - \tau)^{-\sigma} \rho F_q \left( \begin{array}{c} \sigma, a_i \\ b_j \end{array} \right) = \sum_{n=0}^{\infty} \frac{\Gamma(\sigma + n)}{\Gamma(\sigma)n!} \rho F_q \left( \begin{array}{c} -n, a_i \\ b_j \end{array} \right) \tau^n, \quad |\tau| < 1.$$

8. Show that a solution $\Psi$ of $L_{p, q} \Psi = 0$ such that

$$E^{\alpha} \Psi = \Psi, \quad T_l \Psi = a_l \Psi, \quad 1 \leq l \leq p - 1; \quad U_k \Psi = b_k \Psi, \quad 1 \leq k \leq q;$$

takes the form

$$\Psi = f(z/t_p) \exp(-t_p^{-1}) i_t^{a_i} \ldots i_{t_{p-1}}^{a_{t-1}} u_{t_1}^{b_1} \ldots u_{t_p}^{b_p};$$

that is, $\Psi$ is $R$-separable in the coordinates $t_1, \ldots, t_{p-1}, u_1, \ldots, u_p, z/t_p$. Show that if $\Psi$ is analytic at $z = 0$, then to within a constant multiple,

$$f(x) = \rho F_q \left( \begin{array}{c} a_i \\ b_j \end{array} \right)^x.$$

Apply Weisner's principle to derive the identity

$$e^{\tau \rho^{-1} F_q \left( \begin{array}{c} a_i \\ b_j \end{array} \right)} = \sum_{n=0}^{\infty} \rho F_q \left( \begin{array}{c} a_i - n \\ b_j \end{array} \right) \frac{\tau^n}{n!}.$$

(For further identities for the $p F_q$ derived by group-theoretic methods, see [88].)

9. Use the differential recurrence relations (B.5) for the $2 F_1$ to verify relations (1.4) in the special case where $n = 1$.

10. Work out the identities (2.21), (2.22), and (2.23) when $n = 1$, in which case they become generating functions for the $2 F_1$. 