Intrinsic Characterization of Variable separation for the Partial Differential Equations of Mechanics

by

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0. – Introduction

In this paper we present a unified treatment of variable separation for both the linear and nonlinear equations of classical and quantum mechanics. Further, for certain equations of special physical interest (Hamilton-Jacobi, Helmholtz, Schrödinger, wave and Laplace equations) we show how geometric and symmetry group methods can be used to characterize the separation.

We believe that the basic results of Section 1, viz. a general definition of variable separation for partial differential equations and the relationship between separation equations and separation constants, are new. (Our definition is an extension of that of Levi-Civita for Hamilton-Jacobi equations [1]). Particularly notable is the generality of our definition and our ability to distinguish and classify several types of separability in the mathematical literature, some stated explicitly, others implicit. (See for example, Koornwinder's paper [2] in which he discusses several earlier definitions and adds one of his own. All of these definitions are special cases of ours). The authors in the past have adopted technical (nonintuitive) definitions of variable separation for ease of computation, e.g., [3], [4]. On the other hand several of the intuitive definitions found in the literature are virtually useless for computational purposes. The definition offered here has the advantage that although it is intuitive it leads immediately to integrability conditions which are convenient for computations.

In Section 2 we apply our separation criteria to the Hamilton-Jacobi, Helmholtz, Schrödinger, Laplace and wave equations and obtain the technical conditions for separation of various types. Each of these technical conditions has appeared in the literature before but their concise derivation from a single general principle is new.

In Section 3 we show how each of the separable systems introduced in Section 2 can be characterized intrinsically in terms of symmetries of the associated partial differential equation. Such a characterization permits the use of group representation theory and functional analysis to obtain properties of the separated solutions, e.g., [5]. At this point symmetry methods have been applied only for linear and first order partial differential equations. The intrinsic characterization of variable separation for general nonlinear differential equations remains unclear.

Unfortunately, space does not permit us to discuss the broader concept of partial separation of variables for differential equations, a subject which is less developed than total separation. See, however, [6] - [9].

1. – The General Concept of Variable Separation

We begin by motivating our definition of additive separability for a partial differential equation

$$\mathscr{H}(x_i, u, u_i, u_{ij}, u_{ijk}, \ldots) = E \tag{1.1}$$

in the coordinates $x_1, \ldots, x_n$. Here, u is the dependent variable, $u_i = \partial_{x_i} u$, $u_{ij} = \partial_{x_i} \partial_{x_j} u$, etc., where $1 \leqslant i, j, k, \ldots \leqslant n$, and E is a parameter. We assume (for convenience) that $\mathscr{H}$ is a polynomial in the variables $u_i, u_{ij}, \ldots$ with coefficients which are real analytic functions of the variables x_i, u, all defined in a common domain $D \times J$, $D \subseteq \mathbb{R}^n$ with $(0, \ldots, 0) \in D$, and J an open interval on the real line.

A *solution* of (1.1) is a function $u = S(x, E)$ defined and analytic for x in a nonzero domain $D' \subseteq D$ and E in an open interval $I \subseteq \mathbb{R}$, such that substitution of this function into (1.1) renders (1.1) an identity for all $(x, E) \in D' \times I$. A *separable solution* is a solution of the form $u = \sum_{j=1}^{n} S^{(j)}(x_j, E)$. We will derive necessary and sufficient conditions on $\mathscr{H}$ and $\{x_i\}$ for the existence of (additively) separable solutions and, for a given coordinate system, will determine the multiplicity of such solutions.

Since for a separable solution $u_{ij} = 0$ for $i \neq j$, without loss of generality we can set all mixed partial derivatives identically equal to zero in (1.1) and obtain the simple equation

$$H(x_i, u, u_i, u_{ii}, \ldots) = E. \tag{1.2}$$

For convenience we set $u_{i,1} \equiv u_i$, $u_{i,j+1} = \partial_{x_i} u_{i,j} = \partial_{x_i} u_{i,j}$; $j = 1, 2, \ldots$ and define m_i to be largest number ℓ such that $\partial_{u_{i,\ell}} H = H_{u_{i,\ell}} \equiv 0$. To avoid discussion of degenerate cases we require $m_i > 0$ for $i = 1, \ldots, n$. Let D_i denote the total differentiation operators

$$D_i = \partial_{x_i} + u_{i,1}\partial_u + u_{i,2}\partial_{u_{i,1}} + \ldots + u_{i,m_i+1}\partial u_{i,m_i} + \ldots \tag{1.3}$$

If u is a separable solution of (1.2) such $H_{u_{j,m_j}} \neq 0$ in some domain for all j, then $D_i H(x, u) = 0$, or

$$u_{i,m_i+1} = -\frac{\tilde{D}_i H}{H_{u_{i,m_i}}}, \quad i = 1, \ldots, n \tag{1.4}$$

where

$$\tilde{D}_i = \partial_{x_i} + u_{i,1}\partial_u + \ldots + u_{i,m_i}\partial_{u_{i,m_i-1}}. \tag{1.5}$$

Clearly, y satisfies the *integrability conditions* $D_j u_{i,m_i+1} = 0, j \neq i$, or

$$\begin{aligned} H_{u_{i,m_i}} H_{u_{j,m_j}} (\tilde{D}_i \tilde{D}_j H) + H_{u_{i,m_i}u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H) = \\ = H_{u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H_{u_{i,m_i}}) + H_{u_{i,m_i}} (\tilde{D}_j H)(\tilde{D}_i H_{u_{j,m_j}}). \end{aligned} \tag{1.6}$$

Note that is expression is a polynomial in the variables $u_{k,\ell}$. In general, (1.6) is a restriction both on the coefficients of H and the form of the particular separable solution u. However, there is an important special case where (1.6) is an identity in the dependent variable u, $u_{k,\ell}$. (Indeed, this case will occur if (1.2) admits so many separable solutions that for each $x^0 \in D$ and each set of real constants $u^0, u_i^0, u_{ii}^0, \ldots, i = 1, \ldots, n$ satisfying $H(x^0, u^0, u_i^0, u_{ii}^0, \ldots) = E$, there is a separable solution $u(x)$ such that $u(x^0) = u^0, u_i(x^0) = u_i^0, \ldots$). Then conditions (1.6) reduce to restrictions on the coefficients of H which are independent of the choice of separable solution. If (1.6) is an identity we say that $\{x_i\}$ is a *regular separable coordinate system* (for the equation $\mathscr{H} = E$).

Suppose $\{x_i\}$ is a regular separable coordinate system and consider the equations

$$\begin{aligned} D_i v &= v_{i,1} \\ D_i v_{j,1} &= \delta_{ij}\, v_{j,2} \\ D_i v_{j,m_j-1} &= \delta_{ij}\, v_{j,m_j} \\ D_i v_{j,m_j} &= -\delta_{ij} \frac{\tilde{D}_j H(x, v)}{H_{u_{j,m_j}}(x, v)}, \quad 1 \leqslant i, \ j \leqslant n. \end{aligned} \tag{1.7}$$

The integrability condition for this system of equations, $D_k D_i v_{j,m_j} = D_i D_k v_{j,m_j}$ is equivalent to (1.6). Since (1.6) is satisfied identically for $x^0 \in D$ and each set of constants $v_{i,j}^0$, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant m_i$, such that $H_{u_{j,m_j}}(x^v, v^0) \neq 0$, there is a unique solution v of the system (1.7) such that $v(x^0) = v^0$, $D_i v(x^0) = v_{i,1}^0$,

$D_i v_{i,j}(x^0) = v^0_{i,j+1}$ ([10], chapter 1). Choose E such that $E = H(x^0, v^0)$. Then $u = v(x)$ is a separable solution of (1.2) such that $u(x^0) = v^0$ and $u_{i,j}(x^0) = v^0_{i,j}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$. Indeed $D_{x_i} H(x, v) = 0$ so $H(x, v) = E$.

THEOREM 1. *If $\{x_i\}$ is a regular separable system for the equation $\mathscr{H} = E$, i.e., if equations (1.6) are satisfied identically, then for every set of $m_1 + m_2 + \ldots + m_n + 1$ constants $\{v^0, v^0_{i,j}\}$ with $H(x0, v^0) = E$ and $H_{u_{j,m_j}}(x^0, v^0) \neq 0$, there is a unique separable solution u of $H(x, u) = E$ such that $u(x^0) = v_0$, $u_{i,j}(x^0) = v^0_{i,j}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_j$.*

If equations (1.6) are not satisfied identically, separable solutions still may exist, but they will depend on fewer than $\sum_i m_i + 1$ parameters. This type of separation is *nonregular*.

EXAMPLE 1. $H = (x_1 + x_2)(u_{11} + u_{22}) - 2(u_1 + u_2)$. Equations (1.6) are satisfied identically so $\{x_1, x_2\}$ is a regular separable system. The general separable solution depends on 5 parameters and is given by

$$u = \left(\alpha x_1^3 + \beta x_1^2 + \gamma x_1 - \frac{E}{2} x_1\right) + (-\alpha x_2^3 + \beta x_2^2 - \gamma x_2 + \delta). \tag{1.8}$$

EXAMPLE 2. $H = u_{11}^2 + u_1 + u_{22}$.

Here we have $u_{111} = -1/2$ (provided $u_{11} \neq 0$) and $u_{222} = 0$ so equations (1.6) are satisfied identically and $\{x_1, x_2\}$ is a regular separable system. The general separable solution depends on 5 parameters:

$$u = \left(-\frac{x_1^3}{12} + \alpha x_1^2 + \beta x_1\right) + \left[\frac{1}{2}(E - 4\alpha^2 - \beta) x_2^2 + \gamma x_2 + \delta\right] \tag{1.9}$$

EXAMPLE 3. $H = x_2 u_{11} + x_1 u_{22} + u_1 + u_2$.

Equations (1.6) reduce to the requirement $u_{11} + u_{22} = 0$. The general separable solution depends on 4 parameters:

$$u = (\alpha x_1^2 + \beta x_1) + [-\alpha x_2^2 + (E - \beta) x_2 + \gamma]. \tag{1.10}$$

This is a nonregular separable system.

EXAMPLE 4. $H = \dfrac{u_{11} + u_{22}}{u}$.

Equations (1.6) are satisfied identically for $u \neq 0$. The general separable solution depends on 5 parameters:

$$u = \alpha \exp(x_1 E^{1/2}) + \beta \exp(-x_1 E^{1/2}) + \gamma \exp(x_2 E^{1/2}) + \delta \exp(-x_2 E^{1/2}), \tag{1.11}$$

for $E > 0$, with obvious modifications for $E \leqslant 0$.

Associated with any separable coordinate system, regular or not, there is a system of separation equations. Let $u = v(x) = \Sigma_j v^j(x_j)$ be a particular separable solution of the equation $H(x, u) = E$, uniquely determined by its initial conditions $u = v^0$, $u_{j,k} = v^0_{j,k}$ at the point $x = x^0$. Now fix i and set $x = (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$ where we consider x_i as a variable and y_j as a parameter, $j \neq i$. Substituting $u = v$ into $H(x, u) = E$ we obtain an ordinary differential equation

$$H^{(i)}(x_i, u^i, v^j(y_j), y_j) = E, \quad i = 1, \ldots, n, j \neq i, \tag{1.12}$$

for the function $u^i = v^i(x_i)$, an equation that depends on the parameters y_j, $v^j(y_j)$. Each such expression (1.12) is a *separation equation* for u^i. It is important to observe that if $u^i = v^i(x_i)$ is any solution of the i-th separation equation (1.12), valid for all values of the y_j, then

$$\hat{u}(x) = \sum_{j \neq i} v^j(x_j) + \hat{v}^i(x_i) \tag{1.13}$$

is a separable solution of (1.1).

To write the separation equations in a normal form we solve for the highest derivative term u_{i,m_i} in (1.12) and obtain

$$u_{i,m_i} = F^{(i)}(x_i, u_{i,\ell}, v^j(y_j), y_j, E) \qquad 0 \leqslant \ell < m_i, j \neq i. \tag{1.14}$$

(Since H is a polynomial in the derivatives of u, there may be several distinct solutions (1.14). We choose that solution which corresponds to v_{i,m_i}).

We say that the separation equations corresponding to the separable coordinates $\{x_i\}$ are *normal* if each of the functions $F^{(i)}$, $i = 1, \ldots, n$ in (1.14) is independent of y_j, $j \neq i$, for each fixed choice of v^j. If the separation equations are normal there is a single equation for each unknown function $u^{(i)}$. The equations then take the form

$$u_{i,m_i} = \mathscr{F}^{(i)}(x_i, u_{i,\ell}, \lambda_1, \ldots, \lambda_q), \quad i = 1, \ldots, n, \tag{1.15}$$

where $1 \leqslant q \leqslant \sum_i m_i + 1$, $\lambda_k = \lambda_k(u_{j,s})$, $1 \leqslant k \leqslant q$, and the parameters λ_k are functionally independent as functions of the $\sum_i m_i + 1$ parameters $u^0_{i,\ell}$. We choose q to be minimal: the q n-vectors $\left(\dfrac{\partial \mathscr{F}^{(i)}}{\partial \lambda_k}(x_i, \lambda_j)\right)$ $k = 1, \ldots, q$ are linearly independent over the field of functions $f(\lambda_1, \ldots, \lambda_q)$.

THEOREM 2. *If the separation equations are normal then $q = n$ and for each set of solutions $u^{(i)}(x_i)$, $i = 1, \ldots, n$, of the ordinary differential equations (1.15) the function $u = \Sigma u^{(i)}(x_i)$ is a separable solution of (1.2). All separable solutions of (1.2) arise in this manner.*

Proof. Suppose the separation equations for the coordinates $\{x_j\}$ are normal and that they take the form (1.15). Since $u^0_{i,m_i} = \mathscr{F}^{(i)}(x^0, u^0_{i,\ell}, \lambda_1, \ldots, \lambda_q)$ it follows immediately that $q \geqslant n$.

From equations (1.7) we can (locally) consider each λ_i as a function of x and $u_{j,k}(x_j)$, $j = 1, \ldots, n$; $k = 0, \ldots, m_j : \lambda_i = S_i(x, u_{j,k})$. (Here $D_\ell S_i = 0$ if u is a separable solution of (1.2)). From (1.15) and the fact $q \geqslant n$ it follows that n of these functions, say $S_1, \ldots, S_n$ have the property $\det\left(\dfrac{\partial S_j}{\partial u_{i,m_i}}\right) \neq 0$. Thus there exist functions T_i such that

$$u_{i,m_i} = T_i(x, \lambda_1, \ldots, \lambda_n, u_{j,\ell}),\ 0 \leqslant \ell < m_j,\ 1 \leqslant i \leqslant n. \tag{1.16}$$

Furthermore,

$$\lambda_s = Q_s(x, \lambda_1, \ldots, \lambda_n, u_{j,\xi}),\ s = n+1, \ldots, q, 0 \leqslant \ell < m_j \tag{1.17}$$

Substituting (1.17) into (1.15) we obtain the separation equations in the form

$$u_{i,m_i}(x_i) = \mathscr{G}^{(i)}(x, u_{i,p}(x_i), \lambda_1, \ldots, \lambda_n, u_{j,\ell}(x_j)), \qquad i = 1, \ldots, n, \tag{1.18}$$

where $1 \leqslant j \leqslant n$, $j \neq i$, $0 \leqslant p < m_i$, $0 \leqslant p < m_i$, $0 \leqslant \ell < m_j$, and $\lambda_q = S_q(x, u_{kr}(x_k))$, $1 \leqslant q \leqslant n$. Normality implies that the right hand side of (1.18) is identically the same in the variables x_i, $u_{i,p}$ as the parameters x_j sweeps over a range of values:

$$\begin{aligned} D_j \mathscr{G}^{(i)} &= D_j \partial_{x_i} \mathscr{G}^{(i)} = D_j \partial_{u_{i,p}} \mathscr{G}^{(i)} = \\ &= D_j \partial^a_{x_i} \partial^{b_1}_{u_{i,p_1}} \ldots \partial^{b_\ell}_{u_{i,p_k}} \mathscr{G}^{(i)} = 0, \quad j \neq 1. \end{aligned} \tag{1.19}$$

LEMMA 1. *Let $1 \leqslant \alpha < \beta \leqslant n$. Then either $\partial_{x_\alpha} \mathscr{G}^{(\beta)} = \partial_{u_{\alpha,\ell}} \mathscr{G}^{(\beta)} = 0, 0 \leqslant \ell < m_\alpha$ or $\partial_{x_\beta} \mathscr{G}^{(\alpha)} = \partial_{x_{\beta,j}} \mathscr{G}^{(\alpha)} = 0, 0 \leqslant j < m_\beta$.*

Proof. For clarity we consider the example $n = m_\alpha = m_\beta = 2$. (The general case follows easily from this). Thus

$$\begin{aligned} u_{11} &= \mathscr{G}(x_1, x_2, u^{(1)}, u^{(2)}, u_1, u_2, \lambda_1, \lambda_2) \\ u_{22} &= G(x_1, x_2, u^{(1)}, u^{(2)}, u_1, u_2, \lambda_1, \lambda_2) \end{aligned} \tag{1.20}$$

The conditions $D_2\mathcal{I} = D_2\partial_{x_1}\mathcal{I} = D_2\partial_{u^{(1)}}\mathcal{I} = D_2\partial_{u_1}\mathcal{I} = 0$ imply (since $D_i\lambda_j = 0$)

$$(1.21)\qquad \begin{aligned} &\mathcal{I}_{x_2} + u_2\,\mathcal{I}_{u^{(2)}} + G\,\mathcal{I}_{u_2} = 0\\ &\mathcal{I}_{x_1x_2} + u_2\,\mathcal{I}_{x_1u^{(2)}} + G\,\mathcal{I}_{x_1u_2} = 0\\ &\mathcal{I}_{u^{(1)}x_2} + u_2\,\mathcal{I}_{u^{(1)}u^{(2)}} + G\,\mathcal{I}_{u^{(1)}u_2} = 0\\ &\mathcal{I}_{u_1u_2} + u_2\,\mathcal{I}_{u_1u^{(2)}} + G\,\mathcal{I}_{u_1u_2} = 0. \end{aligned}$$

If $G_{u_2} = 0$, it follows immediately from the first of equations (1.21) that $\mathcal{I}_{u^{(2)}} = \mathcal{I}_{u_2} = 0$; thus $\mathcal{I}$ is independent of x_2, $u^{(2)}$, u_2. If $\mathcal{I}_{u_2} = 0$ we apply ∂_{x_1} to both sides of the first equation (1.2) and use the second equation to obtain $G_{x_1} = 0$. Similarly the third and fourth equations yield $G_{u^{(1)}} = G_{u_1} = 0$. Thus G is independent of x_1, $u^{(1)}$, u_1. (Q.E.D.)

Returning to the proof of the theorem, we choose two distinct integers α, β, $(1 \leqslant \alpha, \beta \leqslant n)$, and consider the separation equations

$$(1.22a)\qquad u_{\alpha,m_\alpha}(x_\alpha) = \mathcal{I}^{(\alpha)}(x_\alpha, x_\beta, u_{\alpha,p}; \lambda_1, \ldots, \lambda_n, u_{\beta,\ell})$$

$$(1.22b)\qquad u_{\beta,m_\beta}(x_\beta) = \mathcal{I}^{(\beta)}(x_\alpha, x_\beta, u_{\beta,\ell}, \lambda_1, \ldots, \lambda_n, u_{\alpha,p}).$$

(We suppress the dependence of $\mathcal{I}^{(\alpha)}$, $\mathcal{I}^{(\beta)}$ on the variables x_j, $u_{j,p}$, $j \neq \alpha, \beta$). According to the lemma, by interchanging the labels α and β if necessary, we can ensure that $\mathcal{I}^{(\beta)}$ is independent of x_α and $u_{\alpha,p}$:

$$(1.23)\qquad u_{\beta,m_\beta}(x_\beta) = \mathcal{I}^{(\beta)}(x_\beta; u_{\beta,\ell}, \lambda_1, \ldots, \lambda_n).$$

Let $u = \sum_{j=1}^{n} u^{(j)}(x_j)$ be a separable solution of (1.2) corresponding to the parameters $\lambda_1, \ldots, \lambda_n$. Now let $u^{*(\alpha)}(x_\alpha)$ be any solution of (1.22a) corresponding to parameters $\lambda_1, \ldots, \lambda_n$ and the «background» $u^{(k)}(x_k)$, $k \neq$ q. Then $u^* = u^{*(\alpha)}(x_\alpha) + \sum_{k\neq\alpha} u^{(k)}(x_k)$ is a separable solution of (1.2). Since the separation equation (1.22b) is independent of $u^{*(\alpha)}(x_\alpha)$, for any solution $u^{*(\beta)}(x_\beta)$ of (1.22b) corresponding to parameters $\lambda_1, \ldots, \lambda_n$ and «background», $u^{(h)}(x_h)$, $h \neq \alpha, \beta$, we have a separable solution u^{**} of (1.2): $u^{**} = u^{*(\alpha)}(x_\alpha) + u^{*(\beta)}(x_\beta) + \sum_{h\neq\alpha,\beta} u^{(h)}(x_h)$. Now $u^{(\alpha)} = u^{*(\alpha)}$ is *any* solution of (1.22a) but also satisfies

$$u_{\alpha,m_\alpha}(x_\alpha) = \mathcal{I}^{(\alpha)}(x_\alpha, x_\beta, u_{\alpha,p}, \lambda_1, \ldots, \lambda_n, u^*_{\beta,\ell})$$

where $u^{*(\beta)}$ is *any* solution of (1.23). This is possible only if $\mathcal{I}^{(\alpha)}$ is independent of x_β, and $u_{\beta,e}$. Repeating this argument for each pair of indices α, β we find that the separation equations assume the simple form

$$u_{i,m_i} = \mathcal{I}^{(i)}(x_i, u_{i,\ell}, \lambda_1, \ldots, \lambda_n), \quad i = 1, \ldots, n \qquad \text{(Q.E.D.)}$$

COROLLARY 1. *If $\{x_i\}$ is a separable system with normal separation equations then it is a regular separable system.*

Note that Examples 2 and 4 have normal separation equations, while Examples 1 and 3 are nonnormal.

There is a similar theory of additive separation for a partial differential equation of the form (1.1) with $E = 0$, i.e., an equation not depending on a parameter. We make the same assumption on $\mathscr{H}$ as before and take the equation in the form (1.2) with $E = 0$:

$$H_1(x_i, u, u_i, u_{ii}) = 0. \tag{1.24}$$

Then a separable solution u of (1.24) must satisfy the integrability conditions (1.6). In case the integrability conditions are identities in the sense that there exist functions $P_{i,j}(x_k, u, u_{k,\ell})$, polynomials in $u_{k,\ell}$, such that

$$\begin{aligned}\mathscr{F}_{ij} &\equiv H_{u_i,m_i} H_{u_j,m_j} (\tilde{D}_i \tilde{D}_j H) + H_{u_i,m_i,u_j,m_j} (\tilde{D}_i H)(\tilde{D}_j H) = \\ &- H_{u_j,m_j} (\tilde{D}_i H)(\tilde{D}_j H_{u_i,m_i}) - H_{u_i,m_i} (\tilde{D}_j H)(\tilde{D}_i H_{u_j,m_j}) = \\ &= P_{i,j} H, \qquad i \neq j,\end{aligned} \tag{1.25}$$

we say that $\{x_k\}$ is a *regular separable coordinate system* for the equation $\mathscr{H} = 0$.

THEOREM 3. *If $\{x_k\}$ is a regular separable system for $H = 0$ then for every set of $m_1 + m_2 + \ldots + m_n + 1$ constants $\{v^0, v^0_{ij}\}$ with $H(x^0, v^0) = 0$ and $H_{u_j,m_j}(x^0, v^0) \neq 0$, there is a unique separable solution u of $H(x, u) = 0$ such that $u(x^0) = v^0$, $u_{i,j}(x^0) = v^0_{i,j}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_j$.*

Proof. The verification of this result is only a slight modification of the proof of Theorem 1. Corresponding to the regular separable system $\{x_k\}$ we construct the system of equations (1.7). If the initial conditions are chosen such that $H(x^0, v^0) = 0$ the, from (1.25), $D_1^{m_1} D_2^{m_2} \ldots D_n^{m_n} \cdot \mathscr{F}_{ij}(x^0, v^0) = 0$, $m_i \geqslant 0$, and these are the integrability conditions for the system (1.7), ([10], Chapter 1). (Q.E.D.)

Again we observe that if equations (1.25) are not satisfied identically, separable solutions still may exist but will depend on fewer than $\sum_i m_i$ independent parameters. This is *nonregular* separation.

Examples 1-4 above for $E = 0$ are instances of regular and nonregular separation. Less trivial is

EXAMPLE 5. $H = (x_2 - x_3)\, u_{11} + (x_3 - x_2)\, u_{22} + (x_1 - x_2)\, u_{33}$. Equations (1.25) are satisfied with $P_{i,j} \neq 0$, so x_k is a regular separable system for $H = 0$, though not for $H = E$. The general separable solution depends on 6 parameters and is given by

$$u = \frac{\alpha}{6}(x_1^3 + x_2^3 + x_3^3) + \frac{\beta}{2}(x_1^2 + x_2^2 + x_3^2) + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \delta. \tag{1.26}$$

We can define the concepts of *separation equation* and *normal* separation for $H = 0$ in exact analogy with the definitions for $H = E$. The separation equations depend on q independent parameters and we have:

THEOREM 4. *If the separation equations for $H = 0$ are normal in the coordinates $\{x_k\}$, then $q = n - 1$ and for each set of solutions $u^{(i)}(x_i)$, $i = 1, \ldots, n$ of the separation equations, the function $u = \Sigma\, u^{(i)}(x_i)$ is a separable solution of $H = 0$.*

The proof is virtually the same as that of Theorem 2. The only difference is that, since $E = 0$, there is one less parameter in the separation equations. It is easy to check that the separation equations for Example 5 are normal:

$$u_{ii} + \alpha x_i + \beta = 0, \quad i = 1, 2, 3.$$

2. – Separability for Hamilton-Jacobi, Helmholtz and Laplace Equations

We now apply the results of Section 1 to determine the possible regular separable coordinate systems for the Hamilton-Jacobi equation on a pseudo–Riemannian manifold (V^n, g):

$$H(x^i, u_i) \equiv \sum_{i,j=1}^{n} g^{ij} u_i u_j = E, \tag{2.1}$$

where $u_i = \partial_i u$. In the local coordinates $\{x^i\}$ the metric on V^n is $ds^2 = \Sigma\, g_{ij} dx^i dx^j$ and $\sum_j g_{ij} g^{jk} = \delta_i^k$, $g = \det(g_{ij}) \neq 0$. (We adopt the notation of Eisenhart's book [11] and assume that all functions on V^n are locally analytic). Initially we limit ourselves to *orthogonal* coordinates $\{x^i\}$, i.e., coordinates for which $ds^2 = \sum_i H_i^2 (dx^i)^2$, so that $g_{ij} = 0$ *if* $i \neq j$. Thus (2.1) becomes

$$\sum_{i=1}^{n} H_i^{-2} u_i^2 = E \tag{2.2}$$

and from the integrability conditions (1.6) we see that $\{x^i\}$ is a regular separable system if and only if

$$\partial_{jk} H_i^{-2} = \partial_j H_i^{-2}\, \partial_k \ln H_j^{-2} + \partial_k H_i^{-2}\, \partial_j \ln H_k^{-2}, \quad j \neq k. \tag{2.3}$$

These are the standard Levi-Civita separability conditions and are well known to be equivalent to the requirement that the metric coefficients be in *Stäckel form* with respect to the coordinates $\{x^i\}$, [2], [12]. That is, there exists an $n \times n$ matrix $S_{ji}(x^j)$, whose j th row depends only on x^j, such that $S = \det(S_{ji}) \neq 0$, (a *Stäckel matrix*) and

$$H_j^{-2} = \frac{S^{j1}}{S} \tag{2.4}$$

where S^{j1} is the (j, 1) minor of (S_{ij}). It is not difficult to show that the separation equations are normal and take the form

$$u_i^2 + \sum_{j=1}^{n} \lambda_j S_{ij}(x^i) = 0, \quad \lambda_1 = -E. \tag{2.5}$$

For the Hamilton-Jacobi equation with potential

$$\sum_{i=1}^{n} H_i^{-2} u_i^2 + V(x) = E, \tag{2.6}$$

the results are similar. The integrability conditions reduce to (2.3) and

$$\partial_{jk} V - \partial_k \ell n H_j^{-2} \partial_j V - \partial_j \ell n H_k^{-2} \partial_k V = 0, \quad j \neq k. \tag{2.7}$$

As shown in reference [4], this last condition means precisely that the potential function can be expressed in the form

$$V = \sum_{i=1}^{n} f^{(i)}(x^i) H_i^{-2} \tag{2.8}$$

Again the separation equations are normal.

A regular orthogonal separable system for the Hamilton-Jacobi equation (2.2) with $E = 0$ is characterized by the integrability conditions

$$\begin{aligned} \partial_{jk} H_i^{-2} - \partial_j H_i^{-2} \partial_k \ell n H_j^{-2} - \partial_k H_i^{-2} \partial_j \ell n H_k^{-2} \\ = \rho_{jk}(x) H_i^{-2} H_j^2 H_\ell^2, \quad j \neq k \end{aligned} \tag{2.9}$$

for some functions ρ_{jk}. These equations are equivalent to

$$\begin{aligned} \partial_{jk} \ell n K_i^{-2} + \partial_j \ell n K_i^{-2} \partial_k \ell n K_i^{-2} - \\ - \partial_j \ell n K_i^{-2} \partial_k \ell n K_j^{-2} - \partial_j \ell n K_k^{-2} \partial_k \ell n K_i^{-2} = 0 \end{aligned} \tag{2.10}$$

for $K_i^{-2} = H_i^{-2}/H_l^{-2}$. Furthermore, as shown in Ref. [13], the equations are equivalent to the requirement that $H_i^{-2} = Q(x)\, \mathscr{H}_i^{-2}$ where the metric $d\hat{s}^2 = \sum \mathscr{H}_j^2 (dx^j)^2$ is in Stäckel form. The separation equations have the appearance (2.5) with $E = 0$.

For the general case of regular nonorthogonal separation for the Hamilton-Jacobi equation (2.1) the integrability conditions are identical with those derived by Levi-Civita [1]. However, the basic types of nonorthogonal separation that can occur, as a result of solving these equations, were worked out only recently through Benenti's theory of separability structures, [14]. Benenti used the fact that separable coordinate systems occur in equivalence classes, each element of which determines the same separable solutions, and showed that every equivalence class contains a «canonical» separable system $\{x^a, x^r, x^\alpha\}$ with contravariant metric

$$
(2.11) \qquad (g^{ij}) = \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \left[\begin{array}{ccc} \delta^{ab} H_a^{-2} & 0 & 0 \\ 0 & 0 & H_r^{-2} B_r^\alpha(x^r) \\ 0 & H_r^{-2} B_r^\alpha(x^r) & g^{\alpha\beta} \end{array}\right] \end{array} \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} .
$$

Here $n_3 \geqslant n_2$ and the integer indices a, r, α vary in the ranges $1 \leqslant a, b \leqslant n_1$; $n_1 + 1 \leqslant r \leqslant n_1 + n_2$; $n_1 + n_2 + 1 \leqslant \alpha, \beta \leqslant n_1 + n_2 + n_3 = n$.. Furthermore, $\partial_\alpha g^{ij} = 0$ for each «ignorable variable» x^α. The integrability conditions further require that the metric

$$
(2.12) \qquad d\hat{s}^2 = \sum_{a=1}^{n_1} H_a^2 (dx^a)^2 + \sum_{r=n_1+1}^{n_1+n_2} H_r^2 (dx^r)^2
$$

is in Stäckel form and that each matrix element $g^{\alpha\beta}$ be expressible as

$$
(2.13) \qquad g^{\alpha\beta}(x^a, x^r) = \sum_{a=1}^{n_1} f_a^{\alpha\beta}(x^a)\, H_a^{-2} + \sum_{r=n_1+1}^{n_1+n_2} f_r^{\alpha\beta}(x^r)\, H_r^{-2}.
$$

This last requirement simply means that the equation (2.7) are satisfied for $1 \leqslant i, j, k \leqslant n_1 + n_2$ with $g^{\alpha\beta}$ in place of V. The separation equations take the form

$$
(2.14) \qquad \begin{aligned} & u_a^2 + \sum_{\alpha,\beta} f_a^{\alpha\beta}(x^a)\, \lambda_\alpha \lambda_\beta + \sum_{\varrho=1}^{n_1+n_2} \lambda_\varrho S_{a\varrho}(x^a) = 0, \\ & 2 \sum_\alpha B_r^\alpha(x^r)\, \lambda_\alpha u_r + \sum_{\alpha,\beta} f_r^{\alpha\beta}(x^r)\, \lambda_\alpha \lambda_\beta + \sum_{\varrho=1}^{n_1+n_2} \lambda_\varrho S_{r\varrho}(x^r) = 0, \end{aligned}
$$

$$u_\alpha = \lambda_\alpha, \lambda_1 = -E,$$

where (S_{ij}) is the $(n_1 + n_2) \times (n_1 + n_2)$ Stäckel matrix. (The similar equations for nonorthogonal separation of the Hamilton-Jacobi equations with $E = 0$ can be found in Ref. [13]).

Given a pseudo-Riemannian space V^n it is natural to ask for all regular separable coordinate systems of the Hamilton-Jacobi equation (2.1) on V^n. In his classic paper [12] Eisenhart developed the machinery to handle this problem for spaces of constant curvature and completely solved it for three dimensional Euclidean space (where there are 11 separable systems). In the past two decades complete lists of separable systems have been determined for a number of flat and constant curvature spaces of low dimension, [15]-[20]. Recently the authors have determined the separable systems for *all* Euclidean spaces E_n, spheres S_n and hyperbolic spaces H_n, [21], [22].

Next we study the problem of (multiplicative) separation of variables for the Helmholtz (or Schrödinger) equation

$$(\Delta + V(x))\,\Psi(x) = E\Psi(x) \tag{2.15}$$

on the pseudo-Riemannian manifold V^n.

Here

$$\Delta = \sum_{i,j} \frac{1}{\sqrt{g}}\,\partial_i(\sqrt{g}\,g^{ij}\,\partial_j) \tag{2.16}$$

is the Laplace-Beltrami operator on V^n, defined independent of local coordinates. To convert this product separation problem $\Psi = \Pi_{i=1}^n\,\Psi^{(i)}\,(x^i)$ to the standard additive separation form we introduce the new dependent variable $u = \ell n\ \Psi$. Further, we restrict ourselves to orthogonal separable systems

$$ds^2 = \sum_{i=1}^{n} H_i^2(dx^i)^2.$$

Then (2.15) becomes

$$H \equiv \sum_{\ell=1}^{n} [H_\ell^{-2}(u_{\ell\ell} + u_\ell^2) + S_\ell u_\ell] + V = E \tag{2.17}$$

where

$$S_\ell = \frac{1}{\sqrt{g}}\,\partial_\ell(\sqrt{g}\,H_\ell^{-2}),\ \sqrt{g} = H_1 H_2 \ldots H_r. \tag{2.18}$$

The integrability conditions (1.6) for regular separation lead to (2.3), (Stäckel form), upon comparison of the coefficients of u_ℓ^2. Comparison of the coefficients of u_{ii} in (1.6) yields the *Robertson condition* [23]:

$$(2.19) \qquad \partial_{ij}\, \ell n(\sqrt{g}\, H_i^{-2}) = 0, \quad i \neq j.$$

Comparison of the constant terms in (1.6) yields the conditions (2.7) on the potential $V(x)$, i.e., the potential must be expressible in the form (2.8) to permit separation. There are no additional consequences of the integrability conditions. Again the separation equations are normal and take the form

$$(2.20) \qquad u_{ii} + u_i^2 + g_i(x^i)\, u_i + f_i(x^i) + \sum_{j=1}^{n} \lambda_j S_{ij}(x^i) = 0,$$

where $\lambda_1 = -E$.

It follows that every orthogonal coordinate system permitting product separation of the Helmholtz equation (2.15) corresponds to a Stäckel form; hence permits additive separation of the Hamilton-Jacobi equation (2.6). Eisenhart has shown [12] that the additional Robertson condition for product separation is equivalent to the requirement $R_{ij} = 0$ for $i \neq j$ where R_{ij} is the Ricci tensor of V^n expressed in the Stäckel coordinates $\{x^i\}$. It follows that the Robertson condition is automatically satisfied in Euclidean space, a space of constant curvature or any Einstein space.

More generally we can introduce the notion of R-separation for the Helmholtz equation (2.15) in orthogonal coordinates $\{x^i\}$. Here, R-separable solutions take the form $\Psi = e^{R(x)} \Pi_{i=1}^n \Psi^{(i)}(x^i) = e^R\, \Theta$ where $R(x)$ is a fixed function, independent of parameters. If $R \equiv 0$ we have *separation* and if $R(x) = \Sigma_{i=1}^n R^{(i)}(x^i)$ we have *trivial* R-separation. Otherwise the R-separation is *nontrivial*. Writing $u = \ell n \Theta = R - \ell n \Psi$, we have the following generalization of (2.17):

$$(2.21) \qquad H \equiv \sum_{\ell=1}^{n} [H_\ell^{-2}(u_{\ell\ell} + u_\ell^2) + (2H_\ell^{-2}\, \partial_\ell R + S_\ell)\, u_\ell + $$
$$+ H_\ell^{-2}(\partial_{\ell\ell} R + (\partial_\ell R)^2) + S_\ell\, \partial_\ell R] + V = E.$$

Comparing the coefficients of u_ℓ^2 in the integrability conditions (1.6) we again find that the metric $ds^2 = \sum_i H_i^2 (dx^i)^2$ must be in Stäckel form. Comparison of the coefficients of u_{ii} yields

$$(2.22) \qquad \partial_{ij}[2R + \ell n(\sqrt{g}\, H_i^{-2})] = 0, \quad j \neq i.$$

Finally comparison of the constant terms in (1.6) and use of (2.22) leads to the requirement (2.7) for the «modified potential»

$$\tilde{V} = V - \frac{1}{2} \sum_{i=1}^{n} H_i^{-2} \left(\partial_i \ell_i + \frac{1}{2} \ell_i^2 \right) \tag{2.23}$$

where

$$\ell_i = \partial_i \ell n(\sqrt{g}\, H_i^{-2}) = \partial_i \ell n(\sqrt{g}/S). \tag{2.24}$$

We see that whenever $\tilde{V}$ satisfies (2.7) (hence (2.8)), equation (2.15) permits orthogonal R-separation with

$$R = -\frac{1}{2} \ell n(h/S) + \sum_{i=1}^{n} L^{(i)}(x^i) \tag{2.25}$$

where the functions $L^{(i)}$ are arbitrary. Thus through appropriate choice of V, every additively separable coordinate system $\{x^i\}$ for the zero-potential Hamilton-Jacobi equation can occur as a multiplicatively separable system for the Helmholtz equation. In all these cases the separation equations are normal. Details are given in reference [24].

The question arises whether nontrivial R-separation occurs for $V = 0$. From (2.19), (2.22) and Eisenhart's formulation of Robertson's condition as $R_{ij} = 0$, $i \neq j$, we see that only trivial orthogonal R-separation can occur in an Einstein space. However, as the authors have shown [24], [25], nontrivial R-separation can occur for $V = 0$; even in conformally flat spaces. An example is

$$\begin{aligned} ds^2 &= (x+y+z)\, [(x-y)(x-z)\, dx^2 + \\ &\quad + (y-z)(y-x)\, dy^2 + (z-x)(z-y)\, dz^2], \\ e^R &= (x+y+z)^{-1/4} . \end{aligned} \tag{2.26}$$

Nonorthogonal R-separation for the Helmholtz equation can be treated in an analogous manner. Necessary and sufficient conditions for R-separation are that the metric can be expressed in the form (2.11) and that the modified potential

$$\tilde{V} = V - \frac{1}{2} \sum_{a=1}^{n_1} H_a^{-2} \left(\partial_a \ell_a + \frac{1}{2} \ell_a^2 \right) \tag{2.27}$$

satisfy (2.7), (hence (2.8)) for $j \neq k$, $1 \leqslant k \leqslant n_1 + n_2$. Here

$$\ell_a = \partial_a \ell n(\sqrt{g}\, H_a^{-2}) = \partial_a \ell n(\sqrt{g}/S) \quad 1 \leqslant a \leqslant n_1 + n_2 .$$

If these conditions are satisfied the most general multiplier is given by

$$R(x) = -\frac{1}{2} \ell n(\sqrt{g}/S) + Q(x^s) + \sum_{a=1}^{n_1} A_a(x^a) \tag{2.28}$$

where

$$\hat{V}^{\alpha} = \sum_{r=n_1+1}^{n_1+n_2} g^{r\alpha}\, \partial_r Q \tag{2.28}$$

satisfies (2.7) for each α, $n_1 + n_2 + 1 \leqslant \alpha \leqslant n_1 + n_2 + n_3$. See reference [26] for more details and examples of nontrivial R-separation.

Next we take up orthogonal R-separation for the Laplace equation on V^n:

$$\Delta\, \Psi(x) = 0. \tag{2.29}$$

Here the Laplace-Beltrami operator is given by (2.16). We are interested in solutions of the form $\Psi(x) = e^{R(x)}\Theta(x)$ where $\Theta(x) = \Pi_{i=1}^{n} \Psi^{(i)}(x^i)$ and the metric becomes $ds^2 = \Sigma_{i=1}^{n} H_i^{-2}(dx^i)^2$ in the coordinates $\{x^j\}$. Writing $u = \ell n\, \Theta$ we can write (2.29) in the standard form

$$H \equiv \sum_{\ell=1}^{n} [H_\ell^{-2}(u_{\ell\ell} + u_\ell^2) + (2H_\ell^{-2}\,\partial_\ell R + S_\ell)\, u_\ell + \tag{2.30}$$

$$+ H_\ell^{-2}(\partial_{\ell\ell} R + (\partial_\ell R)^2) + S_\ell \partial_\ell R] = 0$$

where

$$S_\ell = \frac{1}{\sqrt{g}}\, \partial_\ell(\sqrt{g}\, H_\ell^{-2}), \quad \sqrt{g} = H_1 \ldots H_n. \tag{2.31}$$

We now substitute these expressions into the integrability conditions (1.25) to find the requirements that $\{x^j\}$ be a regular separable coordinate system. Equating the coefficients of u_i^2 we obtain the conditions (2.9), hence (2.10), on the metric components H_ℓ^{-2}. Thus there exists a function $Q(x)$ such that $H_i^{-2} =$ $= Q\mathscr{H}_i^{-2}$, $i = 1, \ldots, n$, where the metric $ds^2 = \Sigma_{i=1}^{n} \mathscr{H}_i^{-2}(dx^i)^2$ is in Stäckel form. Let $(S_{ij}(x^i))$ be a Stäckel matrix associated with this form. Comparison of the coefficients of u_{ii} yields

$$\partial_{ij}\left[2R + \ell n\left(\frac{Q^{(2-n)/2}h}{S}\right)\right] = 0, \quad i \neq j \tag{2.32}$$

where $h = \mathscr{H}_1\mathscr{H}_2, \ldots, \mathscr{H}_n$. Comparison of the constant terms in (1.25) and use of (2.32) leads to the remaining requirement that the «potential»

$$\tilde{V} = \Sigma_{i=1}^{n} \mathscr{H}_i^{-2}\left(\partial_i \ell_i + \frac{1}{2}\ell_i^2\right), \ \ell_i = \partial_i \ell n\left(\frac{Q^{(2-n)/2}h}{S}\right) \tag{2.33}$$

satisfies

(2.34) $$\partial_{jk}\tilde{V} - \partial_k \ln \mathcal{H}_j^{-2}\partial_j \tilde{V} - \partial_j \ln \mathcal{H}_k^{-2}\partial_k V = 0, \quad j \neq k;$$

hence $\tilde{V}$ is a *Stäckel multiplier*:

(2.35) $$\tilde{V} = \Sigma_{i=1}^n f^{(i)}(x^i)\,\mathcal{H}_i^{-2}.$$

If conditions (2.10) and (2.43) are satisfied then $\{x^j\}$ is R-separable with

(2.36) $$R(x) = -\frac{1}{2}\ln\left(\frac{Q^{(2-n)/2}h}{S}\right) + \sum_{i=1}^{n} L^{(i)}(x^i)$$

where the $L^{(i)}$ are arbitrary.

An important problem for each pseudo-Riemannian space V^n is to determine those coordinate systems on V^n that permit R-separation of the Helmholtz or Laplace equation. The differential geometric techniques for finding separable systems for the Helmholtz equation were introduced by Eisenhart [12] who showed how to find the solutions of equations (2.3) for flat spaces and spaces of nonzero constant curvature. In his book [27] Böcher developed geometric methods for finding R-separable orthogonal coordinate systems for the Laplace equations on these same spaces. The authors, in collaboration with C. P. Boyer, have extended both the theory of Eisenhart for Helmholtz equations (to obtain lists of separble coordinates for all 3 and 4 dimensional space and for E_n, S_n, H_n in all dimensions, [3], [21], [22], [28]) and Böcher's theory for Laplace equations, [29] - [31]. In particular, for E_n we have shown recently, [31], that Böcher's method yields all R-separation systems. (However, his method obviously fails for Minkowski space, i.e., the wave equation, where nonorthogonal R-separation becomes important).

The basic results on nonorthogonal R-separation for Helmholtz and Laplace equations and detailed lists of coordinates are due almost entirely to the authors and our collaborators: [26], [32] - [34]. We mention in particular Ref. [34] where it is shown that the Helmholtz equation on $\mathbb{C}P(2)$, complex projective space, admits separation in exactly 2 nonorthogonal coordinate systems, and fails to separate in any orthogonal system.

3. – Intrinsic Characterization of Variable Separation

For the Hamilton-Jacobi, Helmholtz and Laplace equations on V^n, introduced in the previous section, (R-) separable coordinate systems can always be characterized intrinsically, i.e., in a coordinate-free manner.

To describe this characterization we need to use the $2n$-dimensional *cotangent bundle* $\overset{*}{V}{}^n$ associated with V^n. If $\{x^i\}$ is a local coordinate system on V^n (with metric tensor g_{ij}) then there is a local coordinate system $\{x^i, p_j\}$ on $\overset{*}{V}{}^n$. If $\{x^{i\prime}\}$ is another coordinate system on V^n, $x^{i\prime} = f^i(x^\ell)$, it induces a new local coordinate system $\{x^{i\prime}, p_j{}'\}$ on $\overset{*}{V}{}^n$ where $p_j' = \sum_\ell \dfrac{\partial x^\ell}{\partial x^{j\prime}} p_\ell$. Thus the *Hamiltonian*

(3.1) $$\mathscr{H} = \sum_{i,j} g^{ij} p_i p_j$$

is independent of coordinates. A function $\mathscr{L}(\underset{\sim}{x}, \underset{\sim}{p})$ on $\overset{*}{V}{}^n$ is a *constant of the motion* (associated with Hamiltonian $\mathscr{H}$) if $\{\mathscr{L}, \mathscr{H}\} = 0$ where

(3.2) $$\{\mathscr{F}(x, p), \mathscr{I}(x, p)\} = \sum_{i=1}^{n} (\partial_{p_i} \mathscr{F} \partial_{x^i} \mathscr{I} - \partial_{p_i} \mathscr{I} \partial_{x^i} \mathscr{F})$$

is the *Poisson bracket* of functions $\mathscr{F}, \mathscr{I}$ on $\overset{*}{V}{}^n$. (As is well known the Poisson bracket is defined independent of canonical changes of coordinates). This notation arises from the fact that $\{\mathscr{L}, \mathscr{H}\} = 0$ if and only if $\dfrac{d}{dt} \mathscr{L}(x(t), p(t)) = 0$ for any solution $x(t)$, $p(t)$ of Hamilton's equations $\dot{p}_i = -\partial_{x^i} \mathscr{H}, \dot{x}^i = \partial_{p^i} \mathscr{H}$. If $\mathscr{L}$ is a polynomial in the p_i, then it is a *Killing tensor;* if linear in the p_i, a *Killing vector.*

Now consider the problem of additive orthogonal separation for the Hamilton-Jacobi equation (2.1). From the separation equations (2.5) we are led to the quadratic forms

(3.3) $$\mathscr{A}^\ell = \sum_{j=1}^{n} \frac{S^{j\ell}}{S} p_\ell^2, \qquad \ell = 1, \ldots, n.$$

From expressions (2.2) - (2.5) we can verify the following properties:

1) $\mathscr{A}^1 = \mathscr{H}$.

2) The n element set $(\mathscr{A}^\ell)$ is linearly independent (as a set of quadratic forms).

3) $(\mathscr{A}^\ell)$ is in involution, i.e., $\{\mathscr{A}^i, \mathscr{A}^j\} = 0$ and each $\mathscr{A}^i$ is a Killing tensor.

4) The differential of the separable coordinates $\omega^j = dx^j$ constitute a simultaneous eigenbasis for the $(\mathscr{A}^\ell)$. (Here, ρ is a *root* of a quadratic form $\mathscr{A} = (a^{ij})$ with respect to the metric g^{ij} if $(a^{ij} - \rho g^{ij}) = 0$, and $\omega = \Sigma_\ell \lambda_\ell dx^\ell$ is an *eigenform* corresponding to ρ if $\omega \neq 0$ and $\Sigma_j (a^{ij} - \rho g^{ij}) \lambda_j = 0$).

5) $\mathscr{A}^\ell(x, p) = -\lambda_\ell$ for each additively separable solution $u = \Sigma_i u^{(i)}(x^i, \lambda)$ of (2.2), where $p_j = u_j = \partial_j u$.

In [35] the authors proved the following converse of these statements:

THEOREM 5. *Let $(\mathscr{A}^\ell)$, $\mathscr{A}^1 = \mathscr{H}$, be a linearly independent set of n second order Killing tensors such that*

1) $\{\mathscr{A}^\ell, \mathscr{A}^m\} = 0, 1 \leqslant \ell, m \leqslant n.$

2) The $(\mathscr{A}^\ell)$ *have a common eigenbasis* $\{\omega^{(j)}\}$.

Then there is a separable coordinate system $\{x^j\}$ *for the Hamilton-Jacobi equation* $\mathscr{H}(y, \partial_y v) = E$ *on* V^n *such that* $\omega^{(j)} = f^{(j)}(x)\, dx^j$ *for some functions* $f^{(j)}$. *The separable solutions u are determined by* $\mathscr{A}^\ell(x, p) = -\lambda_\ell$, $p_i = \partial_{x^i} u$.

The main point of this theorem is that, under the required hypotheses the eigenforms $\omega^{(\ell)}$ of the quadratic forms a^{ij} are *normalizable*, i.e., that up to multiplication by a nonzero function, $\omega^{(\ell)}$ is the differential of a coordinate. This fact, which is proved through use of the commutation relations $\{\mathscr{A}^\ell, \mathscr{A}^m\} = 0$ to verify appropriate integrability conditions, permits us to compute the coordinates directly from a knowledge of the symmetry operators. (Earlier versions of this theorem, e.g. [6], page 31, assume also that $\omega^{(j)}$ is a closed eigenform. We show that this assumption is unnecessary).

These results extend to R-separation of the Helmholtz equation on V^n

(3.4) $$\Delta\Psi(x) = E\,\Psi(x)$$

where in local coordinates

(3.5) $$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \partial_i(\sqrt{g}\, g^{ij}\, \partial_j).$$

Here a linear differential operator A on V^n is a *symmetry operator* for Δ if

(3.6) $$[\Delta, A] \equiv \Delta A - A\Delta = 0.$$

Note that uniquely associated with every second order symmetry operator

$$A = \sum_{i,j=1}^{n} a^{ij}\, \partial_{ij} + \sum_i b^i\, \partial_i + c$$

in local coordinates $\{y^\ell\}$ is the second order Killing tensor

$$\mathscr{A} = \sum_{i,j} a^{ij}\, p_i p_j.$$

Indeed, $[\Delta, A] = 0$ implies $\{\mathscr{H}, \mathscr{A}\} = 0$, though the converse is false.

THEOREM 6. [24], [36]. *Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system* $\{x^j\}$ *for the Helmholtz equation* $\Delta\Psi = E\Psi$ *are that there exists n second order differential operators* $A^1 = \Delta$, $A^2, \ldots, A^n$ *such that*

1) $[A^i, A^j] = 0, 1 \leqslant i, j \leqslant n$.

2) The associated set of Killing tensors $(\mathscr{A}^i)$ *is linearly independent.*

3) There is a basis $\{\omega^{(j)} : 1 \leqslant j \leqslant n\}$ *of simultaneous eigenforms for the* $\mathscr{A}^i$.

If these conditions are satisfied then there exists functions $f^j(x)$ *such that* $\omega^{(j)} = f^j dx^j, 1 \leqslant j \leqslant n$.

Theorem 6 follows from Theorem 5 through exploitation of the commutation relations 1). Indeed, these relations can be used to show that the separation conditions (2.7) for $\tilde{V}$ in (2.23) are valid.

COROLLARY 2. [24]. *Suppose the second order differential operators* $A^1 = \Delta$, $A^2, \ldots, A^n$ *satisfy conditions 1)-3) of Theorem 6, and in addition that they are in self-adjoint form with respect to the measure* $\sqrt{g}\, dy^1 dy^2 \ldots dy^n$:

$$A^\ell = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_{y^i}(\sqrt{g}\, a^{ij}_{(\ell)}\, \partial_{y^j}) + c_\ell(y), \qquad \ell = 1, \ldots, n, \tag{3.7}$$

(a form which is independent of the choice of local coordinates $\{y^j\}$*). Then the R-separable solutions* $\Psi = e^R\, \Pi_{i=1}^n\, \Psi^{(i)}(x^i)$ *of* $\Delta\Psi = E\Psi$ *are characterized as the eigenfunctions of the* A^ℓ:

$$A^\ell \Psi = -\lambda_\ell \Psi, \ \ell = 1, \ldots, n, \tag{3.8}$$

where $\lambda_1 = -E$ *and* $\lambda_2, \ldots, \lambda_n$ *are separation constants.*

For the Hamilton-Jacobi equation

$$\mathscr{H}(x, \partial_i u) = 0, \tag{3.9}$$

i.e., $E = 0$, there is an analogous characterization of additive separation by conformal symmetries. A function $\mathscr{L}(x, p)$ on $\overset{*}{V}{}^n$, a polynomial in the p_i, is said to be a *confomal Killing tensor* provided there is a function $q(x, p)$ on $\overset{*}{V}{}^n$ such that

$$\{\mathscr{L}, \mathscr{H}\} = q\mathscr{H}. \tag{3.10}$$

THEOREM 7. [13]. *Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system* $\{x^j\}$ *for the Hamilton-Jacobi equation (3.9) are that there exist* $n - 1$ *quadratic functions* $\mathscr{B}^k = \sum_{i,j} b^{ij}_{(k)}(x)\, p_i p_j$ *on* $\overset{*}{V}{}^n$, $2 \leqslant k \leqslant n$, *such that*

1) Each $\mathscr{B}^k$ *is a conformal Killing tensor.*

2) $\{\mathscr{B}^i, \mathscr{B}^j\} = 0, 2 \leqslant i, j \leqslant n$.

3) The set $(\mathscr{H}, \mathscr{B}^k)$ *is linearly independent (as n quadratic forms).*

4) There is a basis $\{\omega^{(j)} : 1 \leqslant j \leqslant n\}$ *of simultaneous eigenforms for the* $\mathscr{B}^k$.

If conditions 1)-4) are satisfied then there exists function $f^j(x)$ *such that* $\omega^{(j)} = f^j dx^j$, $1 \leqslant j \leqslant n$.

Again, these results extend to R-separation of the Laplace equation on V^n:

$$\Delta\Psi(x) = 0 \tag{3.11}$$

A linear differential operator B on V^n is a *conformal symmetry operator* for Δ if there exists a linear differential operator C such that

$$[\Delta, B] = \Delta B - B\Delta = C\Delta. \tag{3.12}$$

Uniquely associated with every second order conformal symmetry operator

$$B = \sum_{i,j=1}^{n} b^{ij}\, \partial_{ij} + \sum_i b^i\, \partial_i + c$$

(in local coordinates $\{y^\ell\}$) is the second order conformal Killing tensor

$$\mathscr{B} = \sum_{i,j} b^{ji}\, p_i\, p_j\,.$$

THEOREM 8. [36]. *Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system* $\{x^j\}$ *for the Laplace equation (3.11) are that there exist* $n-1$ *second order differential operators* $B^2, \ldots, B^n$ *on* V^n *such that*

1) Each B^k *is a conformal symmetry operator.*

2) $[B^i, B^j] = 0$, $2 \leqslant i, j \leqslant n$.

3) The set $(\mathscr{H}, \mathscr{B}^2, \ldots, \mathscr{B}^n)$ *is linearly independent.*

4) There is a basis $\{\omega^{(j)} : 1 \leqslant j \leqslant n\}$ *of simultaneous eigenforms for the* $\mathscr{B}^k$.

If conditions 1)-4) are satisfied then there exists functions $f^j(x)$ *such that* $\omega^{(j)} = f^j(x)\, dx^j$, $j = 1, \ldots, n$.

The preceding theorems characterize orthogonal separation and R-separation for Hamilton-Jacobi, Helmholtz and Laplace equations in terms of symmetries. There are similar results for nonorthogonal separation of these equations, though the results are more complicated to state and prove: [4], [26].

We can reach several important conclusions concerning variable separation and R-separation for Hamilton-Jacobi, Helmholtz and Laplace (or wave) equations. First, one must recognise the intrinsic geometric nature of R-separation. The apparently technical conditions for R-separation are equivalent to the existence of an n-dimensional family of commuting symmetry operators which can be simultaneously diagonalized. In spaces, such as those of constant curvature, for which all symmetry operators can be constructed from the Lie symmetry algebra, all R-separation questions become problems in algebra [5], [35].

Second, comparing Theorems 5 and 6, it is obvious that R-separation, not ordinary separation, for the Helmholtz equation is the natural analogy of additive separation for the Hamilton-Jacobi equation. Finally, we note the close relationship between variable separation and quantization theory. Corresponding to a separable system $\{x^j\}$ for the Hamilton-Jacobi equation we have an involutive family $\{\mathscr{A}^\ell\}$ of quadratic constants of the motion. The Helmholtz equation R-separates in these same coordinates if and only if second order operators $\{A^\ell\}$ can be found (with the pure second order terms in A^ℓ agreeing with those of $\mathscr{A}^\ell$) such that the A^ℓ pairwise commute.

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