Lecture Notes and Background Materials on Lebesgue Theory from a Hilbert and Banach Space Perspective, Including an Application to Fractal Image Compression

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Contents

Comment These are lecture notes and background materials for the course. I have included supplementary material, for those students who wish to delve deeper into some of the topics mentioned in class.

Chapter 1

Vector Spaces with Inner Product.

1.1 Definitions

Let F be either the field of real numbers R or the field of complex number C.

Definition 1 *A* **vector space** V *over* F *is a collection of elements (vectors) with the following properties:*

- *For every pair* $u, v \in V$ *there is defined a unique vector* $w = u + v \in V$ *(the* **sum** *of* u *and* v*)*
- *For every* $\alpha \in F$*,* $u \in V$ *there is defined a unique vector* $z = \alpha u \in V$ *(product of* α *and* u*)*
- *Commutative, Associative and Distributive laws*
  1. $u + v = v + u$
  2. $(u + v) + w = u + (v + w)$
  3. *There exists a vector* $\Theta \in V$ *such that* $u + \Theta = u$ *for all* $u \in V$
  4. *For every* $u \in V$ *there is a* $-u \in V$ *such that* $u + (-u) = \Theta$
  5. $1u = u$ *for all* $u \in V$
  6. $\alpha(\beta u) = (\alpha\beta)u$ *for all* $\alpha, \beta \in F$

7. $(\alpha + \beta)u = \alpha u + \beta u$
8. $\alpha(u + v) = \alpha u + \alpha v$

Definition 2 *A non-empty set W in V is a* **subspace** *of V if $\alpha u + \beta v \in W$ for all $\alpha, \beta \in F$ and $u, v \in W$.*

Note that W is itself a vector space over F.

Lemma 1 *Let $u_1, u_2, \cdots, u_m$ be a set of vectors in the vector space V. Denote by $[u_1, u_2, \cdots, u_m]$ the set of all vectors of the form $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$ for $\alpha_i \in F$. The set $[u_1, u_2, \cdots, u_m]$ is a subspace of V.*

PROOF: Let $u, v \in [u_1, u_2, \cdots, u_m]$. Thus,

$$u = \sum_{i=1}^{m} \alpha_i u_i, \qquad v = \sum_{i=1}^{m} \beta_i u_i$$

so

$$\alpha u + \beta v = \sum_{i=1}^{m} (\alpha\alpha_i + \beta\beta_i) u_i \in [u_1, u_2, \cdots, u_m].$$

Q.E.D.

Definition 3 *The elements $u_1, u_2, \cdots, u_p$ of V are* **linearly independent** *if the relation $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_p u_p = \Theta$ for $\alpha_i \in F$ holds* **only** *for $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$. Otherwise $u_1, \cdots, u_p$ are* **linearly dependent**

Definition 4 *V is* **n-dimensional** *if there exist n linearly independent vectors in V and any $n + 1$ vector in V are linearly dependent.*

Definition 5 *V is* **finite-dimensional** *if V is n-dimensional for some integer n. Otherwise V is* **infinite dimensional**.

Remark: If there exist vectors $u_1, \cdots, u_n$, linearly independent in V and such that every vector $u \in V$ can be written in the form

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n, \qquad \alpha_i \in F,$$

($\{u_1, \cdots, u_n\}$ **spans** V), then V is n-dimensional. Such a set $\{u_1, \cdots, u_n\}$ is called a **basis** for V.

Theorem 1 *Let V be an n-dimensional vector space and $u_1, \cdots, u_n$ a linearly independent set in V. Then $u_1, \cdots, u_n$ is a basis for V and every $u \in V$ can be written* **uniquely** *in the form*

$$u = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n.$$

PROOF: let $u \in V$. then the set $u_1, \cdots, u_n, u$ is linearly dependent. Thus there exist $\alpha_1, \cdots, \alpha_{n+1} \in F$, not all zero, such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \alpha_{n+1} u = \Theta.$$

If $\alpha_{n+1} = 0$ then $\alpha_1 = \cdots = \alpha_n = 0$. Impossible! Therefore $\alpha_{n+1} \neq 0$ and

$$u = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n, \qquad \beta_i = -\frac{\alpha_i}{\alpha_{n+1}}.$$

Now suppose

$$u = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n = \gamma_1 u_1 + \gamma_2 u_2 + \cdots + \gamma_n u_n.$$

Then

$$(\beta_1 - \gamma_1) u_1 + \cdots + (\beta_n - \gamma_n) u_n = \Theta.$$

But the u_i form a linearly independent set, so $\beta_1 - \gamma_1 = 0, \cdots, \beta_n - \gamma_n = 0$. Q.E.D.

Examples 1

- *V_n, the space of all (real or complex) n-tuples $(\alpha_1, \cdots, \alpha_n)$, $\alpha_i \in F$. Here, $\Theta = (0, \cdots, 0)$. A standard basis is:*

  $$u_1 = (1, 0 \cdots, 0), \quad u_2 = (0, 1, 0, \cdots, 0), \cdots, u_n = (0, 0, \cdots, 1).$$

  PROOF:

  $$(\alpha_1, \cdots, \alpha_n) = \alpha_1 u_1 + \cdots + \alpha_n u_n,$$

  so the vectors span. They are linearly independent because

  $$(\beta_1, \cdots, \beta_n) = \beta_1 u_1 + \cdots + \beta_n u_n = \Theta = (0, \cdots, 0)$$

  if and only if $\beta_1 = \cdots = \beta_n = 0$. Q.E.D.

- *V_∞, the space of all (real or complex) infinity-tuples*

  $$(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots).$$

  This is an infinite-dimensional space.

- $C^{(n)}[a,b]$: *Set of all complex-valued functions with continuous derivatives of orders* $0, 1, 2, \cdots n$ *on the closed interval* $[a,b]$ *of the real line. Let* $t \in [a,b]$, *i.e.,* $a \leq t \leq b$ *with* $a < b$. *Vector addition and scalar multiplication of functions* $u, v \in C^{(n)}[a,b]$ *are defined by*

$$[u+v](t) = u(t) + v(t) \qquad [\alpha u](t) = \alpha u(t).$$

  The zero vector is the function $\Theta(t) \equiv 0$. *The space is infinite-dimensional.*

- $S(J)$: *Space of all complex-valued step functions on the (bounded or unbounded) interval* J *on the real line.* s *is a* **step function** *on* J *if there are a finite number of non-intersecting bounded intervals* $J_1, \cdots, J_m$ *and complex numbers* $c_1, \cdots, c_m$ *such that* $s(t) = c_k$ *for* $t \in J_k$, $k = 1, \cdots, m$ *and* $s(t) = 0$ *for* $t \in J - \cup_{k=1}^m J_k$. *Vector addition and scalar multiplication of step functions* $s_1, s_2 \in S(J)$ *are defined by*

$$[s_1 + s_2](t) = s_1(t) + s_2(t) \qquad [\alpha s_1](t) = \alpha s_1(t).$$

  (One needs to check that $s_1 + s_2$ *and* αs_1 *are step functions.) The zero vector is the function* $\Theta(t) \equiv 0$. *The space is infinite-dimensional.*

1.2 Schwarz inequality

Definition 6 *A vector space* $\mathcal{N}$ *over* F *is a* **normed linear space** *(pre-Banach space) if to every* $u, \in \mathcal{N}$ *there corresponds a real scalar* $||u||$ *such that*

1. $||u|| \geq 0$ *and* $||u|| = 0$ *if and only if* $u = 0$.
2. $||\alpha u|| = |\alpha| \; ||u||$ *for all* $\alpha \in F$.
3. *Triangle inequality.* $||u + v|| \leq ||u|| + ||v||$ *for all* $u, v \in \mathcal{N}$.

Examples 2

- $C^{(n)}[a,b]$: *Set of all complex-valued functions with continuous derivatives of orders* $0, 1, 2, \cdots n$ *on the closed interval* $[a,b]$ *of the real line. Let* $t \in [a,b]$, *i.e.,* $a \leq t \leq b$ *with* $a < b$. *Vector addition and scalar multiplication of functions* $u, v \in C^{(n)}[a,b]$ *are defined by*

$$[u+v](t) = u(t) + v(t) \qquad [\alpha u](t) = \alpha u(t).$$

  The zero vector is the function $\Theta(t) \equiv 0$. *The norm is defined by* $||u|| = \int_a^b |u(t)| \, dt$.

- $S^1(J)$: *Set of all complex-valued step functions on the (bounded or unbounded) interval J on the real line. s is a* **step function** *on J if there are a finite number of non-intersecting bounded intervals $J_1, \cdots, J_m$ and real numbers $c_1, \cdots, c_m$ such that $s(t) = c_k$ for $t \in J_k$, $k = 1, \cdots, m$ and $s(t) = 0$ for $t \in J - \cup_{k=1}^m J_k$. Vector addition and scalar multiplication of step functions $s_1, s_2 \in S(J)$ are defined by*

$$[s_1 + s_2](t) = s_1(t) + s_2(t) \qquad [\alpha s_1](t) = \alpha s_1(t).$$

(One needs to check that $s_1 + s_2$ and αs_1 are step functions.) The zero vector is the function $\Theta(t) \equiv 0$. The space is infinite-dimensional. We define the integral of a step function as the "area under the curve",i.e., $\int_J s(t)dt \equiv \sum_{k=1}^m c_k \ell(J_k)$ where $\ell(J_k) =$ length of $J_k = b-a$ if $J_k = [a,b]$ or $[a,b)$, or $(a,b]$ or (a,b). Note that

  1. $s \in S(J) \Longrightarrow |s| \in S(J)$.
  2. $|\int_J s(t)dt| \leq \int_J |s(t)|dt$.
  3. $s_1, s_2 \in S(J) \Longrightarrow \alpha_1 s_1 + \alpha_2 s_2 \in S(J)$ *and* $\int_J (\alpha_1 s_1 + \alpha_2 s_2)(t)dt = \alpha_1 \int_J s_1(t)dt + \alpha_2 \int_J s2(t)dt$.

Now we define the norm by $||s|| = \int_J |s(t)|dt$. Finally, we adopt the rule that we identify $s_1, s_2 \in S(J)$, $s_1 \sim s_2$ if $s_1(t) = s_2(t)$ except at a finite number of points. (This is needed to satisfy property 1. of the norm.) Now we let $S^1(J)$ be the space of equivalence classes of step functions in $S(J)$. Then $S^1(J)$ is a normed linear space with norm $||\cdot||$.

Definition 7 *A vector space $\mathcal{H}$ over F is an* **inner product space** *(pre-Hilbert space) if to every ordered pair $u, v \in \mathcal{H}$ there corresponds a scalar $(u,v) \in F$ such that*

Case 1: *$F = C$, Complex field*

- $(u,v) = \overline{(v,u)}$
- $(u+v,w) = (u,w) + (v,w)$
- $(\alpha u, v) = \alpha(u,v)$, *for all* $\alpha \in C$
- $(u,u) \geq 0$, *and* $(u,u) = 0$ *if and only if* $u = 0$

Note: $(u, \alpha v) = \bar{\alpha}(u, v)$

Case 2: *$F = R$, Real field*

1. $(u, v) = (v, u)$
2. $(u + v, w) = (u, w) + (v, w)$
3. $(\alpha u, v) = \alpha(u, v)$, *for all* $\alpha \in R$
4. $(u, u) \geq 0$, *and* $(u, u) = 0$ *if and only if* $u = 0$

Note: $(u, \alpha v) = \alpha(u, v)$

Unless stated otherwise, we will consider complex inner product spaces from now on. The real case is usually an obvious restriction.

Definition 8 *let $\mathcal{H}$ be an inner product space with inner product (u, v). The norm $||u||$ of $u \in \mathcal{H}$ is the non-negative number $||u|| = \sqrt{(u, u)}$.*

Theorem 2 *Schwarz inequality. Let $\mathcal{H}$ be an inner product space and $u, v \in \mathcal{H}$. Then*

$$|(u, v)| \leq ||u||\ ||v||.$$

Equality holds if and only if u, v are linearly dependent.

PROOF: We can suppose $u, v \neq \Theta$. Set $w = u + \alpha v$, for $\alpha \in C$. Then $(w, w) \geq 0$ and $= 0$ if and only if $u + \alpha v = 0$. Hence

$$(w, w) = (u + \alpha v, u + \alpha v) = ||u||^2 + |\alpha|^2\ ||v||^2 + \alpha(v, u) + \bar{\alpha}(u, v) \geq 0.$$

Set $\alpha = -(u, v)/||v||^2$. Then

$$||u||^2 + \frac{|(u, v)|^2}{||v||^2} - 2\frac{|(u, v)|^2}{||v||^2} \geq 0.$$

Thus $|(u, v)|^2 \leq ||u||^2\ ||v||^2$. Q.E.D.

Theorem 3 *Properties of the norm. Let $\mathcal{H}$ be an inner product space with inner product (u, v). Then*

- $||u|| \geq 0$ *and* $||u|| = 0$ *if and only if* $u = 0$.

- $||\alpha u|| = |\alpha|\ ||u||$.
- *Triangle inequality.* $||u+v|| \le ||u|| + ||v||$. *PROOF:*

$$||u+v||^2 = (u+v, u+v) = ||u||^2 + (u,v) + (v,u) + ||v||^2$$
$$\le ||u||^2 + 2||u||\ ||v|| + ||v||^2 = (||u|| + ||v||)^2.$$

Examples:

- $\mathcal{H}_n$ This is the space of complex n-tuples V_n with inner product

$$(u,v) = \sum_{i=1}^{n} \alpha_i \overline{\beta}_i$$

  for vectors

$$u = (\alpha_1, \cdots, \alpha_n), \qquad v = (\beta_1, \cdots, \beta_n), \qquad \alpha_i, \beta_i \in C.$$

- R_n This is the space of real n-tuples V_n with inner product

$$(u,v) = \sum_{i=1}^{n} \alpha_i \beta_i$$

  for vectors

$$u = (\alpha_1, \cdots, \alpha_n), \qquad v = (\beta_1, \cdots, \beta_n), \qquad \alpha_i, \beta_i \in R.$$

  Note that (u,v) is just the dot product. In particular for R_3 (Euclidean 3-space) $(u,v) = ||u||\ ||v|| \cos\phi$ where $||u|| = \sqrt{\alpha_i^2 + \alpha_2^2 + \alpha_3^2}$ (the length of u), and $\cos\phi$ is the cosine of the angle between vectors u and v. The triangle inequality $||u+v|| \le ||u|| + ||v||$ says in this case that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides.

- $\hat{\mathcal{H}}_\infty$, the space of all complex infinity-tuples

$$u = (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots).$$

  such that only a finite number of the α_i are nonzero. $(u,v) = \sum_{i=1}^{\infty} \alpha_i \overline{\beta}_i$.

- $\mathcal{H}_\infty$, the space of all complex infinity-tuples

$$u = (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots).$$

such that $\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty$. Here, $(u, v) = \sum_{i=1}^{\infty} \alpha_i \overline{\beta}_i$. (need to verify that this is a vector space.)

- ℓ^2, the space of all complex infinity-tuples

$$u = (\cdots, \alpha_{-1}, \alpha_0, \alpha_1, \cdots, \alpha_n, \cdots).$$

such that $\sum_{i=-\infty}^{\infty} |\alpha_i|^2 < \infty$. Here, $(u, v) = \sum_{i=-\infty}^{\infty} \alpha_i \overline{\beta}_i$. (need to verify that this is a vector space.)

- $C_2^{(n)}[a, b]$: Set of all complex-valued functions $u(t)$ with continuous derivatives of orders $0, 1, 2, \cdots n$ on the closed interval $[a, b]$ of the real line. We define an inner product by

$$(u, v) = \int_a^b u(t)\overline{v}(t)\ dt, \qquad u, v \in C_2^{(n)}[a, b].$$

- $C_2^{(n)}(a, b)$: Set of all complex-valued functions $u(t)$ with continuous derivatives of orders $0, 1, 2, \cdots n$ on the open interval (a, b) of the real line, such that $\int_a^b |u(t)|^2\ dt < \infty$, (Riemann integral). We define an inner product by

$$(u, v) = \int_a^b u(t)\overline{v}(t)\ dt, \qquad u, v \in C_2^{(n)}(a, b).$$

Note: $u(t) = t^{-1/3} \in C_2^{(2)}(0, 1)$, but $v(t) = t^{-1}$ doesn't belong to this space.

- $L_0^2[a, b]$: Set of all complex-valued functions $u(t)$ on the closed interval $[a, b]$ of the real line, such that $\int_a^b |u(t)|^2\ dt < \infty$, (Riemann integral). We define an inner product by

$$(u, v) = \int_a^b u(t)\overline{v}(t)\ dt, \qquad u, v \in L^2[a, b].$$

Note: There are problems here. Strictly speaking, this isn't an inner product. Indeed the nonzero function $u(0) = 1, u(t) = 0$ for $t > 0$ belongs to $L_0^2[0, 1]$, but $||u|| = 0$. However the other properties of the inner product hold.

- $S^2(J)$: Space of all complex-valued step functions on the (bounded or unbounded) interval J on the real line. s is a **step function** on J if there are a finite number of non-intersecting bounded intervals $J_1, \cdots, J_m$ and numbers $c_1, \cdots, c_m$ such that $s(t) = c_k$ for $t \in J_k$, $k = 1, \cdots, m$ and $s(t) = 0$ for $t \in J - \cup_{k=1}^m$. Vector addition and scalar multiplication of step functions $s_1, s_2 \in S(J)$ are defined by

  $$[s_1 + s_2](t) = s_1(t) + s_2(t) \quad [\alpha s_1](t) = \alpha s_1(t).$$

  (One needs to check that $s_1 + s_2$ and αs_1 are step functions.) The zero vector is the function $\Theta(t) \equiv 0$. Note also that the product of step functions, defined by $s_1 s_2(t) \equiv s_1(t)s_2(t)$ is a step function, as are $|s_1|$ and $\bar{s_1}$. We define the integral of a step function as $\int_J s(t)dt \equiv \sum_{k=1}^m c_k \ell(J_k)$ where $\ell(J_k)$ = length of $J_k = b - a$ if $J_k = [a, b]$ or $[a, b)$, or $(a, b]$ or (a, b). Now we define the inner product by $(s_1, s_2) = \int_J s_1(t)\overline{s_2(t)}dt$. Finally, we adopt the rule that we identify $s_1, s_2 \in S(J)$, $s_1 \sim s_2$ if $s_1(t) = s_2(t)$ except at a finite number of points. (This is needed to satisfy property 4. of the inner product.) Now we let $S^2(J)$ be the space of equivalence classes of step functions in $S(J)$. Then $S^2(J)$ is an inner product space.

1.3 An aside on metric spaces and the completion of inner product spaces

This is supplementary material for the course. For motivation, consider the space R of the real numbers. You may remember from earlier courses that R can be constucted from the more basic space R' of rational numbers. The norm of a rational number r is just the absolute value $|r|$. Every rational number can be expressed as a ratio of integers $r = n/m$. The rationals are closed under addition, subtraction, multiplication and division by nonzero numbers. Why don't we stick with the rationals and not bother with real numbers? The basic problem is that we can't do analysis (calculus, etc.) with the rationals because they are not closed under limiting processes. For example $\sqrt{2}$ wouldn't exist. The Cauchy sequence $1, 1.4, 1.41, 1.414, \cdots$ wouldn't diverge, but would fail to converge to a rational number. There is a "hole" in the field of rational numbers and we label this hole by $\sqrt{2}$. We say that the Cauchy sequence above and all other sequences approaching the same hole are converging to $\sqrt{2}$. Each hole can be identified with the equivalence class of Cauchy sequences approaching the hole. The reals are just the space of equivalence classes of these sequences with appropriate definitions for addition and multiplication. Each rational number r corresponds to a constant Cauchy sequence $r, r, r, \cdots$ so the rational numbers can be embedded as a subset of the reals. Then one can show that the reals are **closed**: every Cauchy sequence of real numbers converges to a real number. We have filled in all of the holes between the rationals. The reals are the **closure** of the rationals.

The same idea works for inner product spaces and it also underlies the relation between the Riemann integral of your calculus classes and the Lebesgue integral. To see how this goes, it is convenient to introduce the simple but general concept of a metric space. We will carry out the basic closure construction for metric spaces and then specialize to inner product and normed spaces.

Definition 9 *A set $\mathcal{M}$ is called a* **metric space** *if for each $u, v \in \mathcal{M}$ there is a real number $\rho(u, v)$ (the metric) such that*

1. $\rho(u, v) \geq 0, \qquad \rho(u, v) = 0$ *if and only if* $u = v$
2. $\rho(u, v) = \rho(v, u)$
3. $\rho(u, w) \leq \rho(u, v) + \rho(v, w)$ *(triangle inequality).*

REMARK: Normed spaces are metric spaces: $\rho(u, v) = ||u - v||$.

Definition 10 *A sequence $u_1, u_2, \cdots$ in $\mathcal{M}$ is called a Cauchy sequence if for every $\epsilon > 0$ there exists an integer $N(\epsilon)$ such that $\rho(u_n, u_m) < \epsilon$ whenever $n, m > N(\epsilon)$.*

Definition 11 *A sequence $u_1, u_2, \cdots$ in $\mathcal{M}$ is convergent if for every $\epsilon > 0$ there exists an integer $M(\epsilon)$ such that $\rho(u_n, u) < \epsilon$ whenever $n, m > M(\epsilon)$. here u is the* **limit** *of the sequence, and we write $u = \lim_{n\to\infty} u_n$.*

Lemma 2 *1) The limit of a convergent sequence is unique.*

2) Every convergent sequence is Cauchy.

PROOF: 1) Suppose $u = \lim_{n\to\infty} u_n$, $v = \lim_{n\to\infty} u_n$. Then $\rho(u,v) \leq \rho(u,u_n) + \rho(u_n,v) \to 0$ as $n \to \infty$. Therefore $\rho(u,v) = 0$, so $u = v$. 2) $\{u_n\}$ converges to u implies $\rho(u_n,u_m) \leq \rho(u_n,u) + \rho(u_m,u) \to 0$ as $n,m \to \infty$. Q.E.D

Definition 12 *A metric space $\mathcal{M}$ is* **complete** *if every Cauchy sequence in $\mathcal{M}$ converges.*

Examples 3 *Some particular metric spaces:*

- *Any normed space. $\rho(u,v) = ||u-v||$. Finite-dimensional inner product spaces are complete.*
- *$\mathcal{M}$ as the set of all rationals on the real line. $\rho(u,v) = |u-v|$ for rational numbers u,v. (absolute value) Here $\mathcal{M}$ is not complete.*
- *The Hausdorff metric. (This is an important tool in the study of fractals.) Let $V_n \equiv R_n$ be the space of real n-tuples with the usual distance norm $||\cdot||$, i.e., n-dimensional Euclidean space. Let X be a closed subset of R_n. Recall that $A \subset X$ is* **compact** *if A is a closed, bounded subset of X. Let $K(X)$ be the collection of all nonempty compact subsets of X. If $x \in X$ and $A \in K(X)$ we define the distance from x to A by*
$$\text{dist}(x,A) = \inf_{a\in A} ||x-a||.$$
(Note that this is the minimum distance from x to a point in A. Since A is closed there exists $\tilde{a} \in A$ such that $\text{dist}(x,A) = ||x-\tilde{a}||$.) Now we define the **Hausdorff metric** *on $K(X)$ by*
$$d_H(A,B) = \max\{\sup_{a\in A} \text{dist}(a,B), \sup_{b\in B} \text{dist}(b,A)\}$$
for $A,B \in K(X)$. (Note that there is a point $\tilde{a} \in A$ that is maximally distant from B, and a point $\tilde{b} \in B$ that is maximally distant from A. $d_H(A,B)$ is the largest of those two distances.) We can verify that d_H is a metric on $K(X)$ (we will show later that this metric space is complete):

  1. *(positive definiteness) A is closed $\Longrightarrow \text{dist}(x,A) = 0$ if and only if $x \in A$. Thus $d_H(A,B) = 0$ if and only if $A = B$.*
  2. *(symmetry) Obvious.*

3. *(triangle inequality). Let $A, B, C \in K(X)$. Given $a \in A$ we can find $b \in B$ such that* $\text{dist}(a, B) = ||a - b||$ *and $c \in C$ such that* $\text{dist}(b, C) = ||b - c||$. *Thus,*

$$\text{dist}(a, C) \leq ||a - c|| \leq ||a - b|| + ||b - c|| = \text{dist}(a, B) + \text{dist}(b, C)$$

$$\leq d_H(A, B) + d_H(B, C).$$

$\implies \sup_{a \in A} \text{dist}(a, C) \leq d_H(A, B) + d_H(B, C)$. *Similarly (switching the roles of A and C) we have* $\sup_{c \in C} \text{dist}(c, A) \leq d_H(A, B) + d_H(B, C)$. *Hence,*

$$d_H(A, C) \leq d_H(A, B) + d_H(B, C).$$

Q.E.D.

Definition 13 *A subset $\mathcal{M}'$ of the metric space $\mathcal{M}$ is* **dense** *in $\mathcal{M}$ if for every $u \in \mathcal{M}$ there exists a sequence $\{u_n\} \subset \mathcal{M}$ such that $u = \lim_{n \to \infty} u_n$.*

1.3.1 An aside on compact metric spaces

In this section we collect several results on compactness that will be useful in the remainder of the course. This is all very standard material in analysis courses. (Indeed, we will follow the treatment in the book by Davidson and Donsig.) We collect it here for reference and completeness. Most of these results can be stated and proved for general metric spaces. A few are specific to Euclidean spaces where the metric is the Euclidean distance. We shall demonstrate that several versions of compactness are, in fact, equivalent. We give proofs for some of the results and leave the more standard proofs to the reader.

Let $\mathcal{M}$ be a metric space with metric $\rho(u, v)$.

Definition 14 *A subset $\mathcal{O}$ of $\mathcal{M}$ is* **open** *if for every $u \in \mathcal{O}$ there is an $\epsilon > 0$ such that all points $v \in \mathcal{M}$ contained in the ball $\rho(u, v) < \epsilon$ also lie in $\mathcal{O}$. A subset $\mathcal{C}$ is* **closed** *if all convergent Cauchy sequences in $\mathcal{C}$ converge to points in $\mathcal{C}$.*

Lemma 3 *$\mathcal{C} \subset \mathcal{M}$ is closed if and only if $\mathcal{O} = \mathcal{M} - \mathcal{C}$ (the complement of $\mathcal{C}$ is open.*

PROOF: Suppose $\mathcal{C}$ is closed and let $u \in \mathcal{O}$. Let $r_u = \inf_{v \in \mathcal{C}} \rho(u, v)$. If $r_u = 0$ we can find a sequence $\{v_n\}$ of points in $\mathcal{C}$ such that $\lim_{n\to\infty} \rho(u, v_n) = 0$. Thus $\{v_n\}$ is a Cauchy sequence converging to $u \in \mathcal{O}$. But $\mathcal{C}$ is closed, so $u \in \mathcal{C}$. This is impossible, so that we must have $r_u > 0$. Thus every point v in the ball $\rho(u, v) < r_u$ must lie in $\mathcal{O}$. This means that $\mathcal{O}$ is open. Conversely, suppose $\mathcal{O}$ is open and let $\{v_n\}$ be a convergent sequence of points in $\mathcal{C}$, converging to v. If $v \in \mathcal{O}$ then for every $\epsilon > 0$ we can find points $v_n \in \mathcal{C}$ such that $\rho(v, v_n) < \epsilon$. This is impossible since $\mathcal{O}$ is open. Thus $v \in \mathcal{C}$ and $\mathcal{C}$ is closed. Q.E.D.

Definition 15 *Let f be a function $f : \mathcal{M} \to \mathcal{M}'$ from the metric space $\mathcal{M}$ to the metric space $\mathcal{M}'$, with metrics ρ, ρ', respectively. We say f is* **continuous** *on $\mathcal{M}$ if for every point $u \in \mathcal{M}$ and every $\epsilon > 0$ there is a $\delta(\epsilon, u) > 0$ such that $\rho'(f(u), f(v)) < \epsilon$ whenever $v \in \mathcal{M}$ satisfies $\rho(u, v) < \delta(\epsilon, u)$.*

Theorem 4 *The following are equivalent:*

1. *$f : \mathcal{M} \to \mathcal{M}'$ is continuous.*
2. *If $v_n \to v$ in $\mathcal{M}$ then $f(v_n) \to f(v)$ in $\mathcal{M}'$.*
3. *The set $f^{-1}(\mathcal{O}') = \{v \in \mathcal{M} : f(v) \in \mathcal{O}'\}$ is an open subset of $\mathcal{M}$ for every open subset $\mathcal{O}'$ of $\mathcal{M}'$.*

PROOF: The equivalence of 1 and 2 is let to the reader. We show that 1 is equivalent to 3. Suppose f is continuous and $\mathcal{O}'$ is an open subset of $\mathcal{M}'$. Let $v_0 \in f^{-1}(\mathcal{O}')$. Then $f(v_0) = v_0' \in \mathcal{O}'$. Since $\mathcal{O}'$ is open, there is an $\epsilon > 0$ such that if $\rho'(v_0', v') < \epsilon$ then $v' \in \mathcal{O}'$. Since f is continuous there is a $\delta(\epsilon, v_0) > 0$ such that $\rho'(f(v_0), f(v)) < \epsilon$ wherever $\rho(v_0, v) < \delta$. Thus every point v in the ball $\rho(v_0, v) < \delta$ belongs to $f^{-1}(\mathcal{O}')$.

Conversely, suppose 3 holds and $f(v_0) = v_0'$. Given $\epsilon > 0$ consider the open ball $B'_\epsilon = \{v' \in \mathcal{M}' : \rho'(v_0', v') < \epsilon\}$. Then $f^{-1}(B'_\epsilon)$ is open in $\mathcal{M}$ and $v_0 \in f^{-1}(B'_\epsilon)$. Thus we can find a $\delta(\epsilon, v_0) > 0$ such that the ball $B_\delta = \{v \in \mathcal{M} : \rho(v_0, v) < \delta\}$ is contained in $f^{-1}(B'_\epsilon)$. This means that f is continuous. Q.E.D.

We will give several definitions of compactness and show that for metric spaces (hence for normed linear spaces) they are equivalent.

Definition 16

- *A metric space $\mathcal{M}$ is* **sequentially compact** *if every sequence $\{u_n\}$ in $\mathcal{M}$ has a convergent subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u \in \mathcal{M}$ as $k \to \infty$.*

- *An* **open covering** *of a subset $\mathcal{N}$ of $\mathcal{M}$ is a collection of open sets $\{\mathcal{O}_\alpha : \alpha \in A\}$ such that $\mathcal{N} \subseteq \cup_{\alpha\in A}\mathcal{O}_\alpha$. The set A is the index set for the covering. (Note that A may not be countable so that the covering can't be indexed by the integers.) A* **subcover** *of $\{\mathcal{O}_\alpha : \alpha \in A\}$ is a subcollection $\{\mathcal{O}_\alpha : \alpha \in B\}$, where $B \subset A$, that is still a covering of $\mathcal{N}$. If B is a finite set then it defines a* **finite subcover**. *We say that the metric space $\mathcal{M}$ is* **compact** *if every open cover of $\mathcal{M}$ has a finite subcover.*

- *A collection of closed sets $\{\mathcal{C}_\alpha : \alpha \in A\}$ in $\mathcal{M}$ has the* **finite intersection property** *if every* **finite** *subcollection of these sets has nonempty intersection.*

- *A metric space $\mathcal{M}$ is* **totally bounded** *if for every $\epsilon > 0$ there are* **finitely** *many points $v_1, v_2, \cdots, v_n$ in $\mathcal{M}$ such that the collection of balls $\{B_\epsilon(v_i) : 1 \le i \le n\}$ is an open cover. Here $B_\epsilon(v_i) = \{v \in \mathcal{M} : \rho(v_i, v) < \epsilon\}$.*

Theorem 5 *(Borel-Lebesgue lemma) The following are equivalent.*

1. *$\mathcal{M}$ is a compact metric space.*
2. *Every collection of closed subsets of $\mathcal{M}$ with the finite intersection property has a nonempty intersection, i.e., if* **every finite** *subcollection of the closed sets $\{\mathcal{C}_\alpha : \alpha \in A\}$ has nonempty intersection then $\cap_{\alpha\in A}\mathcal{C}_\alpha \neq \emptyset$, i.e., is nonempty.*
3. *$\mathcal{M}$ is sequentially compact.*
4. *$\mathcal{M}$ is complete and totally bounded.*

PROOF:

- $1 \Longrightarrow 2$. Let $\{\mathcal{C}_\alpha : \alpha \in A\}$ be a collection of closed sets in $\mathcal{M}$ such that $\cap_{\alpha\in A}\mathcal{C}_\alpha = \emptyset$. Let $\mathcal{O}_\alpha = \mathcal{M} - \mathcal{C}_\alpha$ (the complement of $\mathcal{C}_\alpha$.) Then $\mathcal{O}_\alpha$ is an open set and $\cup_{\alpha\in A}\mathcal{O}_\alpha = \mathcal{M}$. Since $\mathcal{M}$ is compact there is a finite subcover $\mathcal{M} = \mathcal{O}_{\alpha_1} \cup \mathcal{O}_{alpha_2} \cup \cdots \cup \mathcal{O}_{\alpha_n}$. But then $\mathcal{C}_{\alpha_1} \cap \mathcal{C}_{alpha_2} \cap \cdots \cap \mathcal{C}_{\alpha_n} = \emptyset$, so $\{\mathcal{C}_\alpha : \alpha \in A\}$ doesn't have the finite intersection property.

- $2 \Longrightarrow 3$. Let $\{v_i\}$ be a sequence in $\mathcal{M}$ and set $\mathcal{C}_n = \overline{\{v_i : i \geq n\}}$, i.e., $\mathcal{C}_n$ is the closure of the set $\{v_i : i \geq n\}$. Then $\mathcal{C}_{n+1} \subset \mathcal{C}_n$, so we have a decreasing sequence of closed sets. Each finite subcollection of these sets $\mathcal{C}_{n_1}, \cdots, \mathcal{C}_{n_k}$ with $n_1 < n_2 < \cdots < n_k$ contains the point v_{n_k}, so the full collection has the finite intersection property. By assumption 2 there is a point $v \in \cap_{n=1}^{\infty} \mathcal{C}_n$. Now we choose a subsequence v_{i_k} of the sequence v_i recursively as follows. Set $i_1 = 1$. If i_k has been chosen, choose i_{k+1} such that $v_{i_{k+1}} \in \overline{\{v_i : i > n_k\}}$ with $\rho(v, v_{i_{k+1}}) < \frac{1}{k+1}$. Then $v_{i_k} \to v$ as $k \to \infty$.

- $3 \Longrightarrow 4$. Let $\{v_i\}$ be a Cauchy sequence in $\mathcal{M}$. Then we can find a convergent subsequence $\{v_{i_k}\}$: $v_{i_k} \to v \in \mathcal{M}$ as $k \to \infty$. It is straightforward to show that $v_i \to v$ as $i \to \infty$. Now suppose $\mathcal{M}$ is not totally bounded. Then there is an $\epsilon > 0$ such that no finite collection of open balls $B_\epsilon(u_j)$ can cover $\mathcal{M}$. We can select the points u_j recursively so that $u_{j+1} \notin B_\epsilon(u_1) \cup \cdots \cup B_\epsilon(u_j)$ for all $j \geq 1$. However, the sequence u_j can contain can contain no covergent subsequence u_{j_k}, for if it could there would be a finite positive integer N_ϵ such that $\rho(u_{j_k}, u_{j_\ell}) < \epsilon$ for all $j, \ell > N_\epsilon$. Then we would have $u_{j+k} \in B_\epsilon(u_{j_\ell})$, which contradicts the construction of the sequence u_j.

- $4 \Longrightarrow 1$. (The hard part!) For each $k \geq 1$ we can find a finite set $v_1^k, \cdots, v_{n_k}^k$ such that $B_{1/k}(v_1^k) \cup \cdots B_{1/k}(v_{n_k}^k) = \mathcal{M}$. Suppose $\mathcal{M}$ has an open cover $\mathcal{O} = \{\mathcal{O}_\alpha : \alpha \in A\}$ with no finite subcover. Now we construct, recursively, a sequence of points $u_k = v_{j_k}^k$ so that $\cap_{j=1}^k \overline{B_{1/j}(u_j)}$ has no finite subcover from $\mathcal{O}$. To start with, $\mathcal{M}$ has no finite subcover. Suppose that we have carried out the construction to stage $k-1$. If every $\cap_{j=1}^{k-1} \overline{B_{1/j}(u_j)} \cap \overline{B_{1/k}(v_i^k)}$ had a finite subcover from $\mathcal{O}$, then by combining them for $i = 1, \cdots, n_k$ we would have a finite subcover for $\cap_{j=1}^{k-1} \overline{B_{1/j}(u_j)}$, a contradiction. Thus we can choose some $u_k = v_{j_k}^k$ such that $\cap_{j=1}^k \overline{B_{1/j}(u_j)}$ has no finite subcover from $\mathcal{O}$.

  We will show that the sequence u_j is Cauchy, and this will lead to a contradiction. Note that $\cap_{j=1}^k \overline{B_{1/j}(u_j)}$ must be nonempty. (Indeed, if it were empty then it would have a finite subcover.) Hence for any $j < k$ there is a point $v \in \overline{B_{1/j}(u_j)} \cap \overline{B_{1/k}(u_k)}$, so

$$\rho(u_j, u_k) \leq \rho(u_j, v) + \rho(v, u_k) \leq \frac{1}{j} + \frac{1}{k}.$$

It follows easily that the sequence u_j is Cauchy. Since $\mathcal{M}$ is complete, by 4, we have $u_j \to u \in \mathcal{M}$ as $j \to \infty$. Now $u \in \mathcal{O}_\alpha$ for some α, since $\mathcal{O}$ is an open covering. Thus there is an $\epsilon > 0$ such that $B_\epsilon(u) \subset \mathcal{O}_\alpha$. Now we can choose k so large that $1/k < \epsilon/2$ and $\rho(u, u_k) < \epsilon/2$. Then by the triangle inequality $\overline{B_{1/k}(u_k)} \subset B_\epsilon(u) \subset \mathcal{O}_\alpha$. This contradicts the fact that $\overline{B_{1/k}(u_k)}$ has no finite subcover from $\mathcal{O}$.

Q.E.D.

Definition 17 *Let $\mathcal{M}$ be a metric space with metric $\rho(\cdot,\cdot)$, and let $\mathcal{N}$ be a nonempty subspace of $\mathcal{M}$. We say that $\mathcal{N}$ is a* **compact subspace** *of $\mathcal{M}$ if, considered as a metric space with inherited metric ρ, N is compact.*

Thus the preceding theorem gives equivalent conditions for a subspace of a metric space to be compact. Note: In the special case where $\mathcal{N}$ is a subset of Euclidean n-space R^n with the usual distance metric, then it is elementary to show that $\mathcal{N}$ is closed and bounded if and only if it is complete and totally bounded. Hence a closed bounded subset of R^n is compact. (However, a closed bounded subet of V_∞ need not be compact.)

Theorem 6 *(Cantor intersection theorem) Let $\mathcal{M}$ be a metric space with metric $\rho(\cdot,\cdot)$, and let $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots$ be a decreasing sequence of nonempty compact sets in $\mathcal{M}$. Then $\mathcal{C} = \cap_{n=1}^{\infty} \mathcal{C}_n$ is a nonempty compact set.*

PROOF: The sets $\mathcal{C}_n$ are closed and have the finite intersection property. Hence, by the Borel-Lebesgue lemma, $\mathcal{C}$ is nonempty. Now let $\{v_\ell\}$ be a sequence of points in $\mathcal{C}$. Then, since $\mathcal{C}_1 \supset \mathcal{C}$ is compact there is a convergent subsequence $\{v_{\ell_k}\}$ such that $v_{\ell_k} \to v \in \mathcal{C}_1$ as $k \to \infty$. Since $\{v_{\ell_k}\} \subset \mathcal{C}_n$ for all $n = 1, 2, \cdots$ and each $\mathcal{C}_n$ is compact, it follows that $v \in \mathcal{C}_n$, hence $v \in \mathcal{C}$. Thus $\mathcal{C}$ is compact. Q.E.D.

Lemma 4 *A finite union of compact sets is compact.*

PROOF: Let $\mathcal{N}_1, \cdots, \mathcal{N}_r$ be compact subsets of the metric space $\mathcal{M}$, and let $\mathcal{O} = \{\mathcal{O}_\alpha : \alpha \in A\}$ be an open covering of $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_r$ with index set A. Since each $\mathcal{N}_j$ is compact it has a finite subcovering $\mathcal{O}_\alpha,\ \alpha \in A^{(j)}$, where $A^{(j)}$ is a *finite* subset of the index set A. Then $\{\mathcal{O}_\alpha,\ \alpha \in A^{(1)} \cup A^{(2)} \cup \cdots \cup A^{(r)}\}$ is a finite subcovering for $\mathcal{N}$. Thus $\mathcal{N}$ is compact. Q.E.D.

Theorem 7 *Let f be a continuous function $f : \mathcal{M} \to \mathcal{M}'$ from the metric space $\mathcal{M}$ to the metric space $\mathcal{M}'$, with metrics ρ, ρ', respectively, and let $\mathcal{C}$ be a compact subset of $\mathcal{M}$. Then the image set $f(\mathcal{C})$ is compact in $\mathcal{M}'$.*

PROOF: Let $\{v'n = f(v_n) : v_n \in \mathcal{C}\}$ be a sequence of points in $f(\mathcal{C})$. Since $\mathcal{C}$ is compact there is a convergent subsequence v_{n_k} in $\mathcal{C}$ such that $v_{n_k} \to v \in \mathcal{C}$ as $k \to \infty$. Since f is continuous, $f(v_{n_k}) \to f(v)$. Thus the subsequence $f(v_{n_k})$ converges to $f(v)$ and $f(\mathcal{C})$ is compact. Q.E.D.

Corollary 1 *(Extreme value theorem) Let f be a real-valued continuous function $f : \mathcal{M} \to R$ from the metric space $\mathcal{M}$ with metric ρ and let $\mathcal{C}$ be a compact subset of $\mathcal{M}$. Let $M = \sup_{v\in\mathcal{C}} f(v)$, $m = inf_{v\in calC} f(v)$. Then there are points $v_0, v_1 \in \mathcal{C}$ such that $f(v_0) = M$, $f(v_1) = m$.*

PROOF: Since f is continuous and $\mathcal{C}$ is compact, then $f(\mathcal{C})$ is a compact subset of the reals. This subset must be bounded, so $M = \sup_{v\in\mathcal{C}} f(v)$ is a finite number. By definition, we can find a sequence of points $v_n \in \mathcal{C}$ such that $f(v_n) \to M$ as $n \to \infty$. Then there exiasts a convergent subsequence v_{n_k} in $\mathcal{C}$ such that $v_{n_k} \to v_0 \in \mathcal{C}$ as $k \to \infty$ while $f(v_{n_k}) \to M$. Since f is continuous we must have $f(v_0) = M$. A similar proof works for the minimum value. Q.E.D.

Definition 18 *Let f be a continuous function $f : \mathcal{M} \to \mathcal{M}'$ from the metric space $\mathcal{M}$ to the metric space $\mathcal{M}'$, with metrics ρ, ρ', respectively, and let $\mathcal{N}$ be a subset of $\mathcal{M}$. We say f is* **uniformly continuous** *on $\mathcal{N}$ if for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $\rho'(f(u), f(v)) < \epsilon$ whenever $u, v \in \mathcal{N}$ satisfy $\rho(u, v) < \delta(\epsilon)$.*

Theorem 8 *Let f be a continuous function $f : \mathcal{M} \to \mathcal{M}'$ from the metric space $\mathcal{M}$ to the metric space $\mathcal{M}'$, with metrics ρ, ρ', respectively, and let $\mathcal{C}$ be a compact subset of $\mathcal{M}$. Then f is uniformly continuous on $\mathcal{C}$.*

PROOF: Suppose f is continuous on $\mathcal{C}$, but not uniformly continuous. Then for some $\epsilon > 0$ there would be no $\delta > 0$ satisfying the required property. That means that for each $\delta_n = \frac{1}{n}$ and $n = 1, 2, \cdots$ we can find points $u_n, v_n \in \mathcal{C}$ such that $\rho(u_n, v_n) < \frac{1}{n}$ and $\rho'(f(u_n), f(v_n)) \geq \epsilon$. Now $\mathcal{C}$ is compact so there is a convergent subsequence of u_{n_k} of u_n such that $u_{n_k} \to u \in \mathcal{C}$ as $k \to \infty$. Since $\rho(u_{n_k}, v_{n_k}) < \frac{1}{n_k}$ it follows that also $v_{n_k} \to u$ as $k \to \infty$. Since f is continuous we have $f(u_{n_k}) \to f(u)$ and $f(v_{n_k}) \to f(u)$ as $k \to \infty$. But this contradicts the assumption that $\rho'(f(u_{n_k}), f(v_{n_k})) \geq \epsilon$ for all k. Q.E.D.

1.3.2 Completion of metric spaces

Definition 19 *Two metric spaces $\mathcal{M}_\infty, \mathcal{M}_\in$ are* **isometric** *if there is a 1-1 onto map $f : \mathcal{M}_\infty \to \mathcal{M}_\in$ such that $\rho_2(f(u), f(v)) = \rho_1(u, v)$ for all $u, v \in \mathcal{M}_\infty$*

Remark: We identify isometric spaces.

Theorem 9 *Given an incomplete metric space $\mathcal{M}$ we can extend it to a complete metric space $\overline{\mathcal{M}}$ (the completion of $\mathcal{M}$) such that 1) $\mathcal{M}$ is dense in $\overline{\mathcal{M}}$. 2) Any two such completions $\overline{\mathcal{M}}'$, $\overline{\mathcal{M}}''$ are isometric.*

PROOF: (divided into parts)

1. **Definition 20** *Two Cauchy sequences $\{u_n\}, \{\tilde{u}_n\}$ in $\mathcal{M}$ are* **equivalent** *($\{u_n\} \sim \{\tilde{u}_n\}$) if $\rho(u_n, \tilde{u}_n) \to 0$ as $n \to \infty$.*

   Clearly $\sim$ is an equivalence relation, i.e.,

   (a) $\{u_n\} \sim \{u_n\}$, reflexive
   (b) If $\{u_n\} \sim \{v_n\}$ then $\{v_n\} \sim \{u_n\}$, symmetric
   (c) If $\{u_n\} \sim \{v_n\}$ and $\{v_n\} \sim \{w_n\}$ then $\{u_n\} \sim \{v_n\}$. transitive

   Let $\overline{\mathcal{M}}$ be the set of all equivalence classes of Cauchy sequences. An equivalence class $\overline{u}$ consists of all Cauchy sequences equivalent to a given $\{u_n\}$.

2. $\overline{\mathcal{M}}$ is a metric space. Define $\overline{\rho}(\overline{u}, \overline{v}) = \lim_{n\to\infty} \rho(u_n, v_n)$, where $\{u_n\} \in \overline{u}, \{v_n\} \in \overline{v}$.

   (a) $\overline{\rho}(\overline{u}, \overline{v})$ exists. PROOF:

   $$\rho(u_n, v_n) \le \rho(u_n, u_m) + \rho(u_m, v_m) + \rho(v_m, v_n),$$

   so

   $$\rho(u_n, v_n) - \rho(u_m, v_m) \le \rho(u_n, u_m) + \rho(v_m, v_n),$$

   and

   $$|\rho(u_n, v_n) - \rho(u_m, v_m)| \le \rho(u_n, u_m) + \rho(v_m, v_n) \to 0$$

   as $n, m \to \infty$.

(b) $\overline{\rho}(\overline{u},\overline{v})$ is well defined. PROOF: Let $\{u_n\},\{u'_n\} \in \overline{u}$, $\{v_n\},\{v'_n\} \in \overline{v}$. Does $\lim_{n\to\infty} \rho(u_n, v_n) = \lim_{n\to\infty} \rho(u'_n, v'_n)$? Yes, because

$$\rho(u_n, v_n) \le \rho(u_n, u'_n) + \rho(u'_n, v'_n) + \rho(v'_n, v_n),$$

so

$$|\rho(u_n, v_n) - \rho(u'_n, v'_n)| \le \rho(u_n, u'_n) + \rho(v'_n, v_n) \to 0$$

as $n \to \infty$.

(c) $\overline{\rho}$ is a metric on $\overline{\mathcal{M}}$, i.e.

  i. $\overline{\rho}(\overline{u},\overline{v}) \ge 0$, and $= 0$ if and only if $\overline{u} = \overline{v}$ PROOF: $\overline{\rho}(\overline{u},\overline{v}) = \lim_{n\to\infty} \rho(u_n, v_n) \ge 0$ and $= 0$ if and only if $\{u_n\} \sim \{v_n\}$, i.e., if and only if $\overline{u} = \overline{v}$.

  ii. $\overline{\rho}(\overline{u},\overline{v}) = \overline{\rho}(\overline{v},\overline{u})$ obvious

  iii. $\overline{\rho}(\overline{u},\overline{v}) \le \overline{\rho}(\overline{u},\overline{w}) + \overline{\rho}(\overline{w},\overline{v})$ easy

(d) $\mathcal{M}$ is isometric to a metric subset $\overline{\mathcal{S}}$ of $\overline{\mathcal{M}}$. PROOF: Consider the set $\mathcal{S}$ of equivalence classes $\overline{u}$ all of whose Cauchy sequences converge to elements of $\mathcal{M}$. If $\overline{u}$ is such a class then there exists $u \in \mathcal{M}$ such that $\lim_{n\to\infty} u_n = u$ if $\{u_n\} \in \overline{u}$. Note that $u, u, \cdots, u, \cdots \in \overline{u}$ (stationary sequence). The map $u \leftrightarrow \overline{u}$ is a 1-1 map of $\mathcal{M}$ onto $\overline{\mathcal{S}}$. It is an isometry since

$$\overline{\rho}(\overline{u},\overline{v}) = \lim_{n\to\infty} \rho(u_n, v_n) = \rho(u, v)$$

for $\overline{u},\overline{v} \in \overline{\mathcal{S}}$, with $\{u_n\} = \{u\} \in \overline{u}$, $\{v_n\} = \{v\} \in \overline{v}$.

(e) $\mathcal{M}$ is dense in $\overline{\mathcal{M}}$. PROOF: Let $\overline{u} \in \overline{\mathcal{M}}$, $\{u_n\} \in \overline{u}$. Consider $\overline{s}_k = \{u_k, u_k, \cdots, u_k, \cdots\} \in \overline{\mathcal{S}} = \mathcal{M}$, $k = 1, 2, \cdots$. Then $\overline{\rho}(\overline{u}, \overline{s}_k) = \lim_{n\to\infty} \rho(u_n, u_k)$. But $\{u_n\}$ is Cauchy in $\mathcal{M}$. Therefore, given $\epsilon > 0$, if we choose $k > N(\epsilon)$ we have $\overline{\rho}(\overline{u}, \overline{s}_k) < \epsilon$. Q.E.D.

(f) $\overline{\mathcal{M}}$ is complete. PROOF: Let $\{\overline{v}_k\}$ be a Cauchy sequence in $\overline{\mathcal{M}}$. For each k choose $\overline{s}_k = \{u_k, u_k, \cdots, u_k, \cdots\} \in \overline{\mathcal{S}} = \mathcal{M}$, such that $\overline{\rho}(\overline{v}_k, \overline{s}_k) < 1/k$, $k = 1, 2, \cdots$. Then

$$\rho(u_j, u_k) = \overline{\rho}(\overline{s}_j, \overline{s}_k) \le \overline{\rho}(\overline{s}_j, \overline{v}_j) + \overline{\rho}(\overline{v}_j, \overline{v}_k) + \overline{\rho}(\overline{v}_k, \overline{s}_k) \to 0$$

as $j, k \to \infty$. Therefore $\overline{u} = \{u_k\}$ is Cauchy in $\mathcal{M}$. Now

$$\overline{\rho}(\overline{u}, \overline{v}_k) \le \overline{\rho}(\overline{u}, \overline{s}_k) + \overline{\rho}(\overline{s}_k, \overline{v}_k) \to 0$$

as $k \to \infty$. Therefore $\lim_{k\to\infty} \overline{v}_k = \overline{u}$. Q.E.D.

1.3.3 Completion of a normed linear space

Here $\mathcal{B}$ is a normed linear space with norm $\rho(u,v) = ||u - v||$. We will show how to extend it to a complete normed linear space, called a **Banach Space**.

Definition 21 *Let $\mathcal{S}$ be a subspace of the normed linear space $\mathcal{B}$. $\mathcal{S}$ is a* **dense** *subspace of $\mathcal{B}$ if it is a dense subset of $\mathcal{B}$. $\mathcal{S}$ is a* **closed subspace** *of $\mathcal{B}$ if every convergent sequence $\{u_n\}$ in $\mathcal{S}$ converges to an element of $\mathcal{S}$. (Note: If $\mathcal{B}$ is a Banach space then so is $\mathcal{S}$.)*

Theorem 10 *An incomplete normed linear space $\mathcal{B}$ can be extended to a Banach space $\overline{\mathcal{B}}$ such that $\mathcal{B}$ is a dense subspace of $\overline{\mathcal{B}}$.*

PROOF: By the previous theorem we can extend the metric space $\mathcal{B}$ to a complete metric space $\overline{\mathcal{B}}$ such that $\mathcal{B}$ is dense in $\overline{\mathcal{B}}$.

1. $\overline{\mathcal{B}}$ is a vector space.

   (a) $\overline{u}, \overline{v} \in \overline{\mathcal{B}} \Longrightarrow \overline{u} + \overline{v} \in \overline{\mathcal{B}}$.
   If $\{u_n\} \in \overline{u}, \quad \{v_n\} \in \overline{v}$, define $\overline{u} + \overline{v} = \overline{u + v}$ as the equivalence class containing $\{u_n + v_n\}$. Now $\{u_n + v_n\}$ is cauchy because $||(u_n + v_n) - (u_m - v_m)|| \le ||u_n - u_m|| + ||v_n - v_m|| \to 0$ as $n, m \to \infty$. Easy to check that addition is well defined.

   (b) $\alpha \in C, \overline{u} \in \overline{\mathcal{B}} \Longrightarrow \alpha\overline{u} \in \overline{\mathcal{B}}$.
   If $\{u_n\} \in \overline{u}$, define $\alpha\overline{u} \in \overline{\mathcal{B}}$ as the equivalence class containing $\{\alpha u_n\}$, Cauchy because $||\alpha u_n - \alpha u_m|| \le |\alpha|||u_n - u_m||$.

2. $\overline{\mathcal{B}}$ is a Banach space.

   Define the norm $||\overline{u}||'$ on $\overline{\mathcal{B}}$ by $||\overline{u}||' = \overline{\rho}(\overline{u}, \overline{\Theta}) = \lim_{n\to\infty} ||u_n||$ where $\overline{\Theta}$ is the equivalence class containing $\{\Theta, \Theta, \cdots\}$. positivity is easy. Let $\alpha \in C$, $\{u_n\} \in \overline{u}$. Then $||\alpha\overline{u}||' = \overline{\rho}(\alpha\overline{u}, \overline{\Theta}) = \lim_{n\to\infty} ||\alpha u_n|| = |\alpha| \lim_{n\to\infty} ||u_n|| = |\alpha|\overline{\rho}(\overline{u}, \overline{\Theta}) = |\alpha|||\overline{u}||'$.

   $||\overline{u} + \overline{v}||' = \overline{\rho}(\overline{u} + \overline{v}, \overline{\Theta}) \le \overline{\rho}(\overline{u} + \overline{v}, \overline{v}) + \overline{\rho}(\overline{v}, \overline{\Theta}) = ||\overline{u}||' + ||\overline{v}||'$, because $\overline{\rho}(\overline{u} + \overline{v}, \overline{v}) = \lim_{n\to\infty} ||(u_n + v_n) - v_n|| = \lim_{n\to\infty} ||u_n|| = ||\overline{u}||'$. Q.E.D.

1.3.4 Completion of an inner product space

Here $\mathcal{H}$ is an inner product space with inner product (u,v) and norm $\rho(u,v) = ||u-v||$. We will show how to extend it to a complete inner product space, called a **Hilbert Space**.

Theorem 11 *Let $\mathcal{H}$ be an inner product space and $\{u_n\}, \{v_n\}$ convergent sequences in $\mathcal{H}$ with $\lim_{n\to\infty} u_n = u, \lim_{n\to\infty} v_n = v$. Then $\lim_{n\to\infty}(u_n, v_n) = (u,v)$.*

PROOF: Must first show that $||u_n||$ is bounded for all n. $\{u_n\}$ converges $\Longrightarrow$ $||u_n|| \le ||u_n - u|| + ||u|| < \epsilon + ||u||$ for $n > N(\epsilon)$. Set $K = \max\{||u_1||, \cdots, ||u_{N(\epsilon)||}, \epsilon + ||u||\}$. Then $||u_n|| \le K$ for all n. Then $|(u,v)-(u_n,v_n)| = |(u-u_n, v) + (u_n, v-v_n)| \le ||u-u_n|| \cdot ||v|| + ||u_n|| \cdot ||v-v_n|| \to 0$ as $n \to \infty$. Q.E.D.

Theorem 12 *Let $\mathcal{H}$ be an incomplete inner product space. We can extend $\mathcal{H}$ to a Hilbert space $\overline{\mathcal{H}}$ such that $\mathcal{H}$ is a dense subspace of $\overline{\mathcal{H}}$.*

PROOF: $\mathcal{H}$ is a normed linear space with norm $||u|| = \sqrt{(u,u)}$. Therefore we can extend $\mathcal{H}$ to a Banach space $\overline{\mathcal{H}}$ such that $\mathcal{H}$ is dense in $\overline{\mathcal{H}}$. Claim that $\overline{\mathcal{H}}$ is a Hilbert space. Let $\overline{u}, \overline{v} \in \overline{\mathcal{H}}$ and let $\{u_n\}, \{\tilde{u}_n\} \in \overline{u}$, $\{v_n\}, \{\tilde{v}_n\} \in \overline{v}$. We define an inner product on $\overline{\mathcal{H}}$ by $(\overline{u}, \overline{v})' = \lim_{n\to\infty}(u_n, v_n)$. The limit exists since $|(u_n,v_n) - (u_m,v_m)| = |(u_m, v_n - v_m) + (u_n - u_m, v_m) + (u_n - u_m, v_n - v_m)| \le ||u_m|| \cdot ||v_n - v_m|| + ||u_n - u_m|| \cdot ||v_m|| + ||u_n - u_m|| \cdot ||v_n - v_m|| \to 0$ as $n, m \to \infty$. The limit is unique because $|(u_n, v_n) - (\tilde{u}_n, \tilde{v}_n)| \to 0$ as $n, m \to \infty$. can easily verify that $(\cdot,\cdot)'$ is an inner product on $\overline{\mathcal{H}}$ and $||\cdot||' = \sqrt{(\cdot,\cdot)'}$. Q.E.D.

1.4 Hilbert spaces, L^2 and ℓ^2

A Hilbert space is an inner product space for which every Cauchy sequence in the norm converges to an element of the space.

EXAMPLE: ℓ^2

The elements take the form

$$u = (\cdots, \alpha_{-1}, \alpha_0, \alpha_1, \cdots), \quad \alpha_i \in C$$

such that $\sum_{i=-\infty}^{\infty} |\alpha_i|^2 < \infty$. For

$$v = (\cdots, \beta_{-1}, \beta_0, \beta_1, \cdots) \in \ell^2,$$

we define vector addition and scalar multiplication by

$$u + v = (\cdots, \alpha_{-1} + \beta_{-1}, \alpha_0 + \beta_0, \alpha_1 + \beta_1, \cdots)$$

and

$$\alpha u = (\cdots, \alpha\alpha_{-1}, \alpha\alpha_0, \alpha\alpha_1, \cdots).$$

The zero vector is $\Theta = (\cdots, 0, 0, 0, \cdots)$ and the inner product is defined by $(u.v) = \sum_{i=-\infty}^{\infty} \alpha_i \bar{\beta}_i$. We have to verify that these definitions make sense. Note that $2|ab| \leq |a|^2 + |b|^2$ for any $a, b \in C$. The inner product is well defined because $|(u, v)| \leq \sum_{i=-\infty}^{\infty} |\alpha_i \bar{\beta}_i| \leq \frac{1}{2}(\sum_{i=-\infty}^{\infty} |\alpha_i|^2 + \sum_{i=-\infty}^{\infty} |\beta_i|^2) < \infty$. Note that $|\alpha_i + \beta_i|^2 \leq |\alpha_i|^2 + 2|\alpha_i| \cdot |\beta_i| + |\beta_i|^2 \leq 2(|\alpha_i|^2 + |\beta_i|^2)$. Thus if $u, v \in \ell^2$ we have $||u + v||^2 \leq 2||u||^2 + 2||v||^2 < \infty$, so $u + v \in \ell^2$.

Theorem 13 *ℓ^2 is a Hilbert space.*

PROOF: We have to show that ℓ^2 is complete. Let $\{u_n\}$ be Cauchy in ℓ^2,

$$u_n = (\cdots, \alpha_{-1}^{(n)}, \alpha_0^{(n)}, \alpha_1^{(n)}, \cdots).$$

Thus, given any $\epsilon > 0$ there exists an integer $N(\epsilon)$ such that $||u_n - u_m|| < \epsilon$ whenever $n, m > N(\epsilon)$. Thus

$$\sum_{i=-\infty}^{\infty} |\alpha_i^{(n)} - \alpha_i^{(m)}|^2 < \epsilon^2. \tag{1.1}$$

Hence, for fixed i we have $|\alpha_i^{(n)} - \alpha_i^{(m)}| < \epsilon$. This means that for each i, $\{\alpha_i^{(n)}\}$ is a Cauchy sequence in C. Since C is complete, there exists $\alpha_i \in C$ such that $\lim_{n\to\infty} \alpha_i^{(n)} = \alpha_i$ for all integers i. Now set $u = (\cdots, \alpha_{-1}, \alpha_0, \alpha_1, \cdots)$. Claim that $u \in \ell^2$ and $\lim_{n\to\infty} u_n = u$. It follows from (1.1) that for any fixed k, $\sum_{i=-k}^{k} |\alpha_i^{(n)} - \alpha_i^{(m)}|^2 < \epsilon^2$ for $n, m > N(\epsilon)$. Now let $m \to \infty$ and get $\sum_{i=-k}^{k} |\alpha_i^{(n)} - \alpha_i|^2 \leq \epsilon^2$ for all k and for $n > N(\epsilon)$. Next let $k \to \infty$ and get $\sum_{i=-\infty}^{\infty} |\alpha_i^{(n)} - \alpha_i|^2 \leq \epsilon^2$ for $n > N(\epsilon)$. This implies

$$||u_n - u|| \leq \epsilon \tag{1.2}$$

for $n > N(\epsilon)$. Thus, $u_n - u \in \ell^2$ for $n > N(\epsilon)$, so $u = (u - u_n) + u_n \in \ell^2$. Finally, (1.2) implies that $\lim_{n\to\infty} u_n = u$. Q.E.D.

EXAMPLE: $L^2[a,b]$

Recall that $C_2(a,b)$ is the set of all complex-valued functions $u(t)$ continuous on the open interval (a,b) of the real line, such that $\int_a^b |u(t)|^2\ dt < \infty$, (Riemann integral). We define an inner product by

$$(u,v) = \int_a^b u(t)\overline{v}(t)\ dt, \qquad u,v \in C_2^{(n)}(a,b).$$

We verify that this is an inner product space. First, from the inequality $|u(x)+v(x)|^2 \le 2|u(x)|^2 + 2|v(x)|^2$ we have $||u+v||^2 \le 2||u||^2 + 2||v||^2$, so if $u,v \in C_2(a,b)$ then $u+v \in C_2(a,b)$. Second, $|u(x)\overline{v}(x)| \le \frac{1}{2}(|u(x)|^2 + |v(x)|^2)$, so $|(u,v)| \le \int_a^b |u(t)\overline{v}(t)|\ dt \le \frac{1}{2}(||u||^2 + ||v||^2) < \infty$ and the inner product is well defined.

Now $C_2(a,b)$ is not complete, but it is dense in a Hilbert space $\overline{C}_2(a,b) = \overline{L}_0^2[a,b] = L^2[a,b]$ In most of this course we will normalize to the case $a = 0, b = 2\pi$. We will show that the functions $e_n(t) = e^{int}/\sqrt{2\pi}$, $n = 0, \pm 1, \pm 2, \cdots$ form a basis for $L^2[0,2\pi]$. This is a countable (rather than a continuum) basis. Hilbert spaces with countable bases are called **separable**, and we will be concerned only with separable Hilbert spaces in this course.

1.4.1 The Riemann integral and the Lebesgue integral

Recall that $S^1(J)$ is the normed linear space space of all real or complex-valued step functions on the (bounded or unbounded) interval J on the real line. s is a **step function** on J if there are a finite number of non-intersecting bounded intervals $J_1, \cdots, J_m$ and numbers $c_1, \cdots, c_m$ such that $s(t) = c_k$ for $t \in J_k$, $k = 1, \cdots, m$ and $s(t) = 0$ for $t \in J - \cup_{k=1}^m J_k$. The integral of a step function is the $\int_J s(t)dt \equiv \sum_{k=1}^m c_k \ell(J_k)$ where $\ell(J_k) =$ length of $J_k = b - a$ if $J_k = [a,b]$ or $[a,b)$, or $(a,b]$ or (a,b). The norm is defined by $||s|| = \int_J |s(t)|dt$. We identify $s_1, s_2 \in S(J)$, $s_1 \sim s_2$ if $s_1(t) = s_2(t)$ except at a finite number of points. (This is needed to satisfy property 1. of the norm.) We let $S^1(J)$ be the space of equivalence classes of step functions in $S(J)$. Then $S^1(J)$ is a normed linear space with norm $||\cdot||$.

The space of **Lebesgue integrable functions** on J, ($L^1(J)$) is the completion of $S^1(J)$ in this norm. $L^1(J)$ is a Banach space. Every element u of $L^1(J)$ is an equivalence class of Cauchy sequences of step functions $\{s_n\}$,

$\int_J |s_j - s_k|dt \to 0$ as $j, k \to \infty$. (Recall $\{s'_n\} \sim \{s_n\}$ if $\int_J |s'_k - s_n|dt \to 0$ as $n \to \infty$.

In the next section we shall show that, in fact, we can associate equivalence classes of functions $f(t)$ on J with each equivalence class of step functions $\{s_n\}$. The Lebesgue integral of f is defined by

$$\int_{J \text{ Lebesgue}} f(t)dt = \lim_{n\to\infty} \int_J s_n(t)dt,$$

and its norm by

$$||f|| = \int_{J \text{ Lebesgue}} |f(t)|dt = \lim_{n\to\infty} \int_J |s_n(t)|dt.$$

How does this definition relate to Riemann integrable functions? To see this we take $J = [a, b]$, a closed bounded interval, and let $f(t)$ be a real bounded function on $[a, b]$. Recall that we have already defined the integral of a step function.

Definition 22 *f is* **Riemann integrable** *on $[a, b]$ if for every $\epsilon > 0$ there exist step functions $r, s \in S[a, b]$ such that $r(t) \le f(t) \le s(t)$ for all $t \in [a, b]$, and $0 \le \int_a^b (s - r)dt < \epsilon$.*

EXAMPLE. Divide $[a, b]$ by a grid of n points $a = t_0 < t_1 < \cdots < t_n = b$ such that $t_j - t_{j-1} = (b - a)/n$, $j = 1, \cdots, n$. Let $M_j = \sup_{t\in[t_{j-1},t_j]} f(t)$, $m_j = \inf_{t\in[t_{j-1},t_j]} f(t)$ and set

$$s_n(t) = \{ \begin{array}{ll} M_j & t \in [t_{j-1}, t_j) \\ 0 & t \notin [a, b) \end{array}$$

$$r_n(t) = \{ \begin{array}{ll} m_j & t \in [t_{j-1}, t_j) \\ 0 & t \notin [a, b) \end{array}$$

$\int_a^b s_n(t)dt$ is an **upper Darboux sum**. $\int_a^b r_n(t)dt$ is a **lower Darboux sum**. If f is Riemann integrable then the sequences of step functions $\{r_n\}, \{s_n\}$ satisfy $r_n \le f \le s_n$ on $[a, b]$, for $n = 1, 2, \cdots$ and $\int_a^b (s_n - r_n)dt \to 0$ as $n \to \infty$. The Riemann integral is defined by

$$\int_{a\text{Riemann}}^b f \; dt = \lim_{n\to\infty} \int s_n \; dt = \lim_{n\to\infty} \int r_n \; dt =$$

$$\inf_{\text{upper Darboux sums}} \int s\ dt = \sup_{\text{lower Darboux sums}} \int t\ dt.$$

Note that

$$\sum_{j=1}^{n} M_j(t_j - t_{j-1}) \geq \int_a^b {}_{\text{Riemann}} f\ dt \geq \sum_{j=1}^{n} m_j(t_j - t_{j-1}).$$

Note also that

$$r_j - r_k \leq s_k - r_k, \qquad r_k - r_j \leq s_j - r_j$$

because every "upper" function is $\geq$ every "lower" function. Thus

$$\int |r_j - r_k| dt \leq \int (s_k - r_k) dt + \int (s_j - r_j) dt \to 0$$

as $j, k \to \infty$. Thus $\{r_n\}$ and similarly $\{s_n\}$ are Cauchy sequences in the norm, equivalent because $\lim_{n\to\infty} \int (s_n - r_n) dt = 0$.

Theorem 14 *If f is Riemann integrable on $J = [a, b]$ then it is also Lebesgue integrable and*

$$\int_{J\ \text{Riemann}} f(t) dt = \int_{J\ \text{Lebesgue}} f(t) dt = \lim_{n\to\infty} \int_J s_n(t) dt$$

.

The following is a simple example to show that the space of Riemann integrable functions isn't complete. Consider the closed interval $J = [0, 1]$ and let $r_1, r_2, \cdots$ be an enumeration of the rational numbers in $[0, 1]$. Define the sequence of step functions $\{s_n\}$ by

$$s_n(t) = \{ \begin{array}{ll} 1 & t = r_1, r_2, \cdots, r_n \\ 0 & \text{otherwise.} \end{array}$$

Note that

- $s_1(t) \leq s_2(t) \leq \cdots$ for all $t \in [0, 1]$.
- s_n is a step function.
- The pointwise limit

$$f(t) = \lim_{n\to\infty} s_n(t) = \{ \begin{array}{ll} 1 & \text{if } t \text{ is rational} \\ 0 & \text{otherwise.} \end{array}$$

- $\{s_n\}$ is Cauchy in the norm. Indeed $\int_0^1 |s_j - s_k| dt = 0$ for all $j, k = 1, 2, \cdots$.
- f is Lebesgue integrable with $\int_0^1{}_{\text{Lebesgue}} f(t)dt = \lim_{n\to\infty} \int_0^1 s_n(t)dt = 0$.
- f is not Riemann integrable because *every* upper Darboux sum for f is 1 and every lower Darboux sum is 0. Can't make $1 - 0 < \epsilon$ for $\epsilon < 1$.

Recall that $S^2(J)$ is the space of all real or complex-valued step functions on the (bounded or unbounded) interval J on the real line with real inner product $(s_1, s_2) = \int_J s_1(t)\bar{s}_2(t)dt$. We identify $s_1, s_2 \in S(J)$, $s_1 \sim s_2$ if $s_1(t) = s_2(t)$ except at a finite number of points. (This is needed to satisfy property 4. of the inner product.) Now we let $S^2(J)$ be the space of equivalence classes of step functions in $S(J)$. Then $S^2(J)$ is an inner product space with norm $||s||^2 = \int_J |s(t)|^2 dt$.

The space of **Lebesgue square-integrable functions** on J, ($L^2(J)$) is the completion of $S^2(J)$ in this norm. $L^2(J)$ is a Hilbert space. Every element u of $L^2(J)$ is an equivalence class of Cauchy sequences of step functions $\{s_n\}$, $\int_J |s_j - s_k|^2 dt \to 0$ as $j, k \to \infty$. (Recall $\{s'_n\} \sim \{s_n\}$ if $\int_J |s'_k - s_n|^2 dt \to 0$ as $n \to \infty$.

In the next section we shall show that we can associate an equivalence class of functions $f(t)$ on J with each equivalence class of step functions $\{s_n\}$. The Lebesgue integral of $f_1, f_2 \in L^2(J)$ is defined by $(f_1, f_2) = \int_{J\text{Lebesgue}} f_1(t) f_2\ dt = \lim_{n\to\infty} \int_J s_n^{(1)}(t) s_n^{(2)}(t)dt$.

How does this definition relate to Riemann square integrable functions? In a manner similar to our treatment of $L^1(J)$ one can show that if the function f is Riemann square integrable on J, then it is Lebesgue square integrable and $\int_{J\text{Lebesgue}} |f(t)|^2 dt = \int_{J\text{Riemann}} |f(t)|^2 dt$.

1.4.2 Some technical results

In the preceedinfg section we obtained the space of Lebesgue integrable functions $L^1(J)$ as the completion of the normed linear space space $S(J)$ of all real or complex-valued step functions on the (bounded or unbounded) interval J on the real line. We let $S^1(J)$ be the space of equivalence classes of step functions in $S(J)$. Then $S^1(J)$ is a normed linear space with norm $||s|| = \int_J |s(t)| dt$. The space of **Lebesgue integrable functions** on J, ($L^1(J)$) is the completion of $S^1(J)$ in this norm. $L^1(J)$ is a Banach space.

Every element u of $L^1(J)$ is an equivalence class of Cauchy sequences of step functions $\{s_n\}$, $\int_J |s_j - s_k| dt \to 0$ as $j, k \to \infty$. (Recall $\{s'_n\} \sim \{s_n\}$ if $\int_J |s'_k - s_n| dt \to 0$ as $n \to \infty$.

We turn now from the abstract notion of the elements of the Lebesgue space as equivalence class of Cauchy sequences of step functions to an identification with actual functions. How should we identify a function with an equivalence class of Cauchy sequences of step functions in $S(J)$? Let $\{s_j\}$ be a Cauchy sequence in $S(J)$.

- Does $f(t) = \lim_{j\to\infty} s_j(t)$ always exist point wise for all $t \in J$? ANSWER: no.
- Is there a subsequence $\{s_{j'}\}$ of $\{s_j\}$ such that $f(t) = \lim_{j'\to\infty} s_{j'}(t)$ exists? ANSWER: almost.

Example 1 *Let $J = (0, 1]$ and $f(t) = 0$ for $t \in J$. We will define a particular Cauchy sequence $\{s_j\}$ of step functions that conferge to f in the norm. Any positive integer j can be written uniquely as $j = 2^p + q$ for $p = 0, 1, \cdots$ and $q = 0, 1, \cdots, 2^p - 1$. Now set*

$$s_{2^p+q}(t) = \begin{cases} 1 & \text{for } \frac{q}{2^p} < t \le fracq + 12^p \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_J |f - s_j| dt = \int_J |s_{2^p+q}| dt = \frac{1}{2^p} \to 0$$

as $p \to \infty$ (or $j \to \infty$) so $s_j \to f$ in the norm as $j \to \infty$. However $\lim_{j\to\infty} s_j(t)$ doesn't exist for any $t \in J$. Indeed for any fixed $t_0 \in J$ and any j there are always integers $j_1, j_2 > j$ such that $s_{j_1}(t_0) = 0$ and $s_{j_2}(t_0) = 1$. Note, however, that we can find pointwise convergent subsequences of $\{s_j\}$. In particular, for the subsequence of values $j = 2^p$ we have $\lim_{p\to\infty} s_{2^p}(t) = f(t) = 0$ for all $t \in J$.

Definition 23 *A subset N of the real line is a null set (set of measure zero) if for every $\epsilon > 0$ there exists a countable collection of open intervals $\{I_n = (a_n, b_n)\}$ such that $N \subset \cup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} \ell(I_n) \le \epsilon$. (Here $\ell(I_n) = b_n - a_n$.)*

EXAMPLES:

1. Any finite set is null.

2. Any countable set is null. PROOF: Let $N = \{r_1, r_2, \cdots, r_n, \cdots\}$. Given $\epsilon > 0$ let $I_n = (r_n - \frac{\epsilon}{2^{n+1}}, r_n + \frac{\epsilon}{2^{n+1}})$, $n = 1, 2, \cdots$ Then $N \subset \cup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$.

3. The rationals form a null set.

4. The set $I = (a, b)$ with $a < b$ is not null.

5. A countable union of null sets is null.

6. There exist null sets which are not countable. The most famous example is probably the Cantor set.

EXAMPLE: The Cantor set

$$C = \{x \in [0,1] :\ x = \sum_{n=1}^{\infty} \frac{c_n}{3^n},\ c_n = 0, 2\}. \tag{1.3}$$

Note that any real number $x \in [0,1]$ can be given a ternary representation $x = \sum_{n=1}^{\infty} \frac{c_n}{3^n} = .c_1c_2c_3c_4\cdots$, for $c_n = 0, 1, 2$. For example $1 = \sum_{n=1}^{\infty} \frac{2}{3^n} = .2222\cdots$. (Just as with decimal expansions of real numbers, this representation is not quite unique, e.g., $\frac{1}{3} = .10000\cdots = .02222\cdots$, but we could adopt a convention to make it unique.) The Cantor set consists of those real numbers whose ternary representation doesn't contain any $c_n = 1$. We are more familiar with the binary expansion of any real number $y \in [0,1]$, $y = \sum_{n=1}^{\infty} \frac{b_n}{2^n} = .b_1b_2b_3b_4\cdots$, where $b_n = 0, 1$.

- C is uncountable. PROOF: The map $C \Longrightarrow [0,1]$ defined by

$$\sum_{n=1}^{\infty} \frac{c_n}{3^n} \Longrightarrow \sum_{n=1}^{\infty} \frac{\frac{1}{2}c_n}{2^n}$$

  is one-to-one and onto. Since the real line contains an uncountable number of points, C is also uncountable.

- C is a null set. PROOF: We can see this geometrically, from the triadic representation. The points in the open middle third $(\frac{1}{3}, \frac{2}{3})$ of the interval $[0,1]$ don't lie in the Cantor set, so we can remove this third. Then we can remove the open middle thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ of the remaining intervals, etc. After k steps there remains a set C_k that is the union of 2^k intervals of total length $(\frac{2}{3})^k$. $C = \cap_{k=1}^{\infty} C_k$, so for each k, C can be covered by 2^k open intervals of length $\leq 2(\frac{2}{3})^k$. Since this goes to zero as $k \to \infty$ we see that C is a null set. Q.E.D.

Definition 24 *If a property holds for all real numbers t except for $t \in N$, a null set, we say that the property holds almost everywhere (a.e.).*

The following technical lemmas show us how to associate a function with a Cauchy sequence of step functions.

Lemma 5 *Let $\{s_k\}$ be a Cauchy sequence in $S(J)$. Then there exist strictly increasing sequences $\{n_k\}$ of positive integers such that*

$$\sum_{k=1}^{\infty} \int_J |s_{n_{k+1}} - s_{n_k}| dt < \infty. \tag{1.4}$$

For every such sequence $\{n_k\}$ the subsequence $\{s_{n_k}(t)\}$ converges pointwise a.e. in J.

PROOF: Choose $n_k > n_{k-1}$, and so large that $\int_J |s_m - s_n| dt \leq \frac{1}{2^k}$ for all $m, n \geq n_k$, $k = 1, 2, \cdots$. Then $\int_J |s_{n_{k+1}} - s_{n_k}| dt \leq \frac{1}{2^k}$, $k = 1, 2, \cdots$ and $\sum_{k=1}^{\infty} \int_J |s_{n_{k+1}} - s_{n_k}| dt \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$. Now assume $\{s_{n_k}\}$ is an arbitrary subsequence of $\{s_n\}$ such that (1.4) converges. Set

$$u_k = |s_{n_1}| + |s_{n_2} - s_{n_1}| + \cdots + |s_{n_k} - s_{n_{k-1}}|, \quad k \geq 2, \qquad u_1 = |s_{n_1}|.$$

Then $0 \leq u_1(t) \leq u_2(t) \leq \cdots$ for all $t \in J$ and u_k is a step function. By (1.4) there exists $M > 0$ such that $\int_J u_k dt \leq M$ for all k. We will show that $\lim_{k\to\infty} u_k(t)$ exists for almost all $t \in J$. Given $\epsilon > 0$, let $\mathcal{R}_k(\epsilon) = \{t \in J : u_k(t) \geq \frac{M}{\epsilon}\}$. Clearly:

1. $\mathcal{R}_k(\epsilon) = \mathcal{R}_k$ is the union of finitely many nonintersecting intervals.

2. let ϵ_k be the sum of the lengths of the intervals in $\mathcal{R}_k$. Then $\epsilon_k \leq \epsilon$ because $u_k(t) \geq \frac{M}{\epsilon}\chi_{\mathcal{R}_k}(t) \Longrightarrow \int u_k dt \geq \int \frac{M}{\epsilon}\chi_{\mathcal{R}_k} dt \Longrightarrow M \geq \frac{M}{\epsilon}\epsilon_k$ where $\chi_S(t)$ is the *characteristic function* of the set S, i.e.,

$$\chi_S(t) = \begin{cases} 1 & \text{if } t \in S \\ 0 & \text{if } t \notin S. \end{cases}$$

3. $u_k \geq u_{k-1} \Longrightarrow \mathcal{R}_k \supseteq \mathcal{R}_{k-1}$.

4. Let

$$\mathcal{R} = \mathcal{R}(\epsilon) = \cup_{k\geq 1}\mathcal{R}_k(\epsilon) = \mathcal{R}_1 \cup (\mathcal{R}_2 - \mathcal{R}_1) \cup \cdots \cup (\mathcal{R}_k - \mathcal{R}_{k-1}) \cup \cdots$$

   Then $\mathcal{R}_k - \mathcal{R}_{k-1}$ can be represented as the union of finitely many nonintersecting intervals of total length $\epsilon_k - \epsilon_{k-1}$.

5. It follows that $\mathcal{R}(\epsilon)$ is the union of countably many non-intersecting intervals. The sum of the lengths of the intervals in $\mathcal{R}(\epsilon)$ is

$$\epsilon_1 + (\epsilon_2 - \epsilon_1) + \cdots + (\epsilon_k - \epsilon_{k-1}) + \cdots = \lim_{k\to\infty} \epsilon_k \le \epsilon.$$

CONCLUSION: $u_k(t) < \frac{M}{\epsilon}$, $k = 1, 2, \cdots$ for all $t \in J - \mathcal{R}(\epsilon) \Longrightarrow \lim_{k\to\infty} u_k(t)$ exists for all $t \in J - \mathcal{R}(\epsilon)$. The points of divergence are covered by the intervals of $\mathcal{R}(\epsilon)$ (of total length $\le \epsilon$). But ϵ is arbitrary so the points of divergence form a null set N. $\Longrightarrow \lim_{k\to\infty} u_k(t)$ exists a.e.

Consider

$$s_{n_1}(t) + (s_{n_2}(t) - s_{n_1}(t)) + \cdots + (s_{n_k}(t) - s_{n_{k-1}}(t)) + \cdots \tag{1.5}$$

Now

$$|s_{n_1}(t)| + |s_{n_2}(t) - s_{n_1}(t)| + \cdots + |s_{n_k}(t) - s_{n_{k-1}}(t)| + \cdots = \lim_{k\to\infty} u_k(t)$$

exists for all $t \notin N$. Therefore (1.5) converges for all $t \notin N \Longrightarrow \lim_{k\to\infty} s_{n_k}(t)$ exists a.e. Q.E.D.

Lemma 6 *Let $\{s_k\}$, $\{s'_k\}$ be equivalent Cauchy sequences in $S(J)$, possibly the same sequence. Let $\{s_{p_k}\}$, $\{s'_{q_k}\}$ be subsequences of $\{s_k\}$ and $\{s'_k\}$ that converge pointwise a.e. on J. Then $\lim_{k\to\infty}\Big(s_{p_k}(t) - s'_{q_k}(t)\Big) = 0$ pointwise a.e. on J.*

PROOF: Let $v_k = s_{p_k} - s'_{q_k}$, $k = 1, 2, \cdots$. Then $\{v_k\}$ is a Cauchy sequence and $\lim_{k\to\infty} v_k(t)$ exists pointwise for all $t \notin N_1$ where N_1 is some null set. Also,

$$\int_J |v_k|dt = \int_J |s_{p_k} - s'_{q_k}|dt \le \int_J |s_{p_k} - s_k|dt + \int_J |s_k - s'_k|dt + \int_J |s'_k - s'_{q_k}|dt \to 0$$

as $k \to \infty$. let $\{k_\ell\}$ be an increasing sequence of positive integers such that $\sum_{\ell=1}^{\infty} \int_J |v_{k_\ell}|dt < \infty$. Then by lemma 5, $\sum_{\ell=1}^{\infty} |v_{k_\ell}(t)|$ converges for all $t \notin N_2$ where N_2 is a null set. $\Longrightarrow \lim_{\ell=\infty} v_{k_\ell}(t) = 0$ pointwise for all $t \notin N_2 \Longrightarrow$ For all $t \notin N_1 \cup N_2$ (a null set) we have $\lim_{k\to\infty} v_k(t) = \lim_{\ell\to\infty} v_{k_\ell}(t) = 0$. Q.E.D.

We want to associate an equivalence class of functions (equal a.e.) with each equivalence class of Cauchy sequences of step functions. How can we do this uniquely? In particular, let $\{s_k\}$ be a sequence in $S(J)$ such that $s_k \to 0$ a.e. as $k \to \infty$. How can we guarantee that $\int_J s_k dt \to 0$ as $k \to \infty$?

EXAMPLE: let $\alpha > 0$, $J = (0,1)$. Set

$$s_k(t) = \begin{cases} 0 & \text{if } t \geq \frac{1}{k} \\ \alpha k & \text{if } 0 < t < \frac{1}{k},\ k \text{ odd} \\ k & \text{if } 0 < t < \frac{1}{k},\ k \text{ even.} \end{cases}$$

REMARKS:

- $s_k(t) \to 0$ as $k \to \infty$, for every t.
- $$\int_J s_k dt = \begin{cases} \alpha, & k \text{ odd} \\ 1, & k \text{ even.} \end{cases}$$
- $\lim_{k\to\infty} \int_J s_k dt = 1$, if $\alpha = 1$. Otherwise the limit doesn't exist.

The next lemma and the basic theorem to follow gives conditions that guarantee uniqueness.

Lemma 7 *Let $\{s_k\}$, $\{t_k\}$ be Cauchy sequences in $S(J)$ that converge pointwise a.e. in J to limit functions that are equal a.e. in J. Then $\{s_k\} \sim \{t_k\}$, i.e., $\int_J |s_k - t_k| dt \to 0$ as $k \to \infty$.*

PROOF: This is an immediate consequence of the following basic theorem.

Theorem 15 *Let $\{s_k\}$ be a sequence in $S(J)$ such that $\lim_{k\to\infty} s_k(t) = 0$ a.e. in J. Suppose either*

1. *The s_j are real and*

$$s_1(t) \geq s_2(t) \geq \cdots \geq s_n(t) \geq \cdots \geq 0$$

for all $t \in J$, or

2. *For every $\epsilon > 0$ there exists an integer $N(\epsilon)$ such that $\int_J |s_j - s_k| dt < \epsilon$ whenever, $j, k \geq N(\epsilon)$, i.e., $\{s_k\}$ is Cauchy in the norm.*

Then $\int_J |s_k| dt \to 0$ as $k \to \infty$. (Note: $|\int_J s_k dt| \leq \int_J |s_k| dt$, so $\int_J s_k dt \to 0$ as $k \to \infty$.)

PROOF:

1. $$s_1 \geq s_2 \geq \cdots \geq 0, \qquad s_k(t) \to 0 \text{ a.e.as } k \to \infty.$$

   Given $\epsilon > 0$, let $M = \max s_1(t)$, and let $[a, b]$ be the smallest closed interval outside of which $s_1(t) = 0$. We can assume $a \neq b$.

   Let N be the null set consisting of the points t where either $\lim_{k\to\infty} s_k(t)$ is not zero or where the limit doesn't exist, plus the points where one or more of the functions $s_1.s_2, \cdots$ is discontinuous. Let $\mathcal{I} = \{I_1, I_2, \cdots\}$ be a countable family of open intervals that cover N and such that $\sum_{k=1}^{\infty} \ell(I_k) \leq \epsilon$. Choose $t_0 \in (a, b)$ such that $t_0 \notin \cup_{k\geq 1} I_k$. Since $s_k(t_0) \to 0$ as $K \to \infty$ there exists a smallest index $h = h(\epsilon, t_0)$ such that $s_k(t_0) \leq \epsilon$ for all $k \geq h$. Since t_0 is a point of continuity of s_h there exists an open interval in $[a, b]$ that contains t_0 and on which $s_h(t) \leq \epsilon$. let $J(t_0) = J(t_0, \epsilon)$ be the largest such interval. Then $s_k(t) \leq \epsilon$ on $J(t_0)$ for all $k \geq h(t_0)$. Let $\mathcal{G} = \{J(t) : t \in [a, b] - \cup_n I_n\}$. Let $\mathcal{H} = \mathcal{H}(\epsilon)$ be the family consisting of the intervalsd of $\mathcal{I}$ and those of $\mathcal{G}$. Now $\mathcal{H}$ forms a covering of $[a, b]$ by open intervals. Therefore, by the Heine-Borel theorem (see the digression below, if you are not familiar with this theorem) we can find a finite subfamilty $\mathcal{H}'$ of $\mathcal{H}$ that covers $[a, b]$.

   $$\mathcal{H}' = \{J(t_1), J(t_2), \cdots, J(t_n), I_1, I_2, \cdots, I_m\}.$$

   On $J(t_1)$, $1 \leq i \leq n$, we have $s_k(t) \leq \epsilon$ for all $k \geq h(t_i)$. Let $p = \max\{h(t_1), \cdots, h(t_n)\}$. Then $s_k(t) \leq \epsilon$ for $k \geq p$ on every interval $J(t_i)$, $1 \leq i \leq n$. On I_k, $1 \leq k \leq m$, we know only $s_k(t) \leq s_1(t) \leq M$. Therefore, for $k \geq p$,

   $$\int_J s_k dt \leq \int_J s_p dt \leq \int_J \left[\epsilon\chi_{J(t_1)\cup\cdots\cup J(t_n)} + M\chi_{I_1\cup\cdots\cup I_m}\right] dt$$

   $$\leq \epsilon(b-a) + M\epsilon \qquad \text{since} \quad s_p \leq \epsilon\chi_{J(t_1)\cup\cdots\cup J(t_n)} + M\chi_{I_1\cup\cdots\cup I_m}.$$

   (Note that the latter are step functions.) But ϵ is arbitrary, so $\lim_{k\to\infty} s_k dt = 0$.

2. $$\lim_{k\to\infty} s_k(t) = 0 \quad \text{a.e.}, \quad \int_j |s_k - s_\ell| dt \to 0 \text{ as } k, \ell \to \infty.$$

Now

$$\left|\int_J |s_k|dt - \int |s_\ell|dt\right| = \left|\int_J (|s_k| - |s_\ell|)dt\right| \le \int_J ||s_k| - |s_\ell||$$

$$\le \int_J |s_k - s_\ell|dt \to 0$$

as $k, \ell \to \infty$. $\Longrightarrow \{\int_J |s_k|dt\}$ is a Cauchy sequence of real numbers $\Longrightarrow$ $\int_J |s_k|dt \to A$ as $k \to \infty$.

We will show that $A = 0$ by making use of part 1 of the proof. Let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasinfg sequence of positive integers. Set $v_1(t) = |s_{n_1}(t)|$, and $v_k = |s_{n_1}| \wedge |s_{n_2}| \wedge \cdots \wedge |s_{n_k}|$, $k = 2, 3, \cdots$, i.e., $v_k(t) = \min\{|s_{n_1}(t)|, |s_{n_2}(t)|, \cdots, |s_{n_k}(t)|$. REMARKS:

- $v_k \in S(J)$.
- $v_k = v_{k-1} \wedge |s_{n_k}|$.
- $v_1 \ge v_2 \ge v_3 \ge \cdots \ge 0$.
- $v_k(t) \le |s_{n_k}(t)| \to 0$ a.e. as $k \to \infty$. Therefore, by part 1, $\lim_{k\to\infty} \int_J v_k dt = 0$.
- $-v_k + v_{k-1} = -v_{k-1} \wedge |s_{n_k}| + v_{k-1} \Longrightarrow$

$$v_{k-1}(t) - v_k(t) =$$

$$\begin{cases} 0 & \text{if } \ |s_{n_k}(t)| \ge v_{k-1}(t) \\ t_{k-1}(t) - |s_{n_k}(t)| \le |s_{n_{k-1}}(t)| - |s_{n_k}(t)| & \\ \qquad \le |s_{n_{k-1}}(t) - s_{n_k}(t)| & \text{otherwise.} \end{cases}$$

- $\int_J (v_{k-1} - v_k)dt \le \int_J |s_{n_{k-1}} - s_{n_k}|dt = \delta_k$ (definition).

Therefore,

$$\int_J |s_{n_1}|dt = \int_J v_1 dt = \int_J [v_2 + (v_1 - v_2)]dt \le \int_J v_2 dt + \delta_2 \tag{1.6}$$

$$= \int_J [v_3 + (v_2 - v_3)]dt + \delta_2 \le \int_J v_3 dt + \delta_2 + \delta_3 \le \cdots \le \int_J v_p dt + \delta_2 + \delta_3 + \cdots + \delta_p,$$

for $p = 2, 3, \cdots$. Given $\epsilon > 0$ choose n_1 so large that $\int_J |s_{n_1}|dt > A - \epsilon$ and $\int_J |s_{n_1} - s_n|dt < \frac{\epsilon}{2}$ for all $n > n_1$. For $k \ge 2$ choose $n_k > n_{k-1}$

so large that $\int_J |s_{n_k} - s_n| dt < \frac{\epsilon}{2^k}$ for all $n > n_k$. Then $\delta_k < \frac{\epsilon}{2^{k-1}}$, for $k = 2, 3, \cdots$. Choose p so large that $\int_J v_p dt < \epsilon$. Then (1.6) $\Longrightarrow$

$$A - \epsilon < \int_J |s_{n_1}| dt < \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{2^2} + \cdots + \frac{\epsilon}{2^{p-1}} < 2\epsilon,$$

$\Longrightarrow 0 \le A < 3\epsilon$. But ϵ is arbitrary, so $A = 0$. Q.E.D.

DIGRESSION: The Heine-Borel theorem.

Theorem 16 *let $[a, b]$ be a bounded closed interval on the real line, and $\mathcal{I}$ a collection of open intervals that cover $[a, b]$, i.e., $[a, b] \subset \cup\{I : I \in \mathcal{I}\}$. Then one can find a finite number $I_1, \cdots, I_n$ of intervals in $\mathcal{I}$ such that $[a, b] \subset I_1 \cup I_2 \cup \cdots \cup I_n$.*

PROOF: Let

$$S = \{c \in [a, b] : [a, c] \text{ can be covered by a finite number of intervals in } \mathcal{I}\}.$$

S is not empty because $a \in S$. Let $c_0 = \sup S$. We will show that $c_0 \in S$. Choose $I \in \mathcal{I}$ such that $c_0 \in I$. Then we can find a $c_1 \in I$ such that $c_1 \le c_0$ and $c_1 \in S$. $\Longrightarrow [a, c_1] \subset I_1 \cup I_2 \cup \cdots \cup I_n$ with $I_i \in \mathcal{I}$, $\Longrightarrow c_0 \in S$. I claim that $c_0 = b$. If not, we can find $c_2 \in I$, with $c_2 > c_0$

$$\Longrightarrow [a, c_2] \subset I_1 \cup I_2 \cup \cdots \cup I_n \cup I \Longrightarrow c_2 \in S, \; c_2 > c_0.$$

Impossible! Therefore $c_0 = b$. Q.E.D.

Theorem 17 *N is a null set in $J \Longleftrightarrow$ there exists a sequence $\{s_k\}$ is $S(J)$ such that $\sum_{k=1}^{\infty} s_k(t)$ diverges for each $t \in N$ and $\sum_{k=1}^{\infty} \int_J |s_k| dt < \infty$.*

PROOF: $\Longleftarrow$ Done in lemma 5.

$\Longrightarrow$ Given $\epsilon > 0$ let $C_k = \{I_1^k, I_2^k, \cdots, I_n^k, \cdots\}$ be, for each k, a countable covering of N by open intervals of total length $\sum_{n=1}^{\infty} \ell(I_n^k) < \frac{\epsilon}{2^k}$, $k = 1, 2, \cdots$. Note that $\chi_{I_n^k}(t)$ is a step function. Any $t_0 \in N$ is contained in an infinite number of the I_n^k, so $\sum_{n,k=1}^{\infty} \chi_{I_n^k}(t_0)$ diverges. But $\sum_{n,k=1}^{\infty} \int_J \chi_{I_n^k}(t) dt < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon < \infty$. Q.E.D.

Lemma 8 *$f \in L^1(J) \Longleftrightarrow$ there exists a sequence $\{u_k\}$ in $S(J)$ such that $f(t) = \sum_{k=1}^{\infty} u_k(t)$ a.e. and $\sum_{k=1}^{\infty} \int_J |u_k| dt < \infty$.*

PROOF: $\Longleftarrow$ Set $s_n(t) = \sum_{k=1}^{n} u_k(t)$ for $n = 1, 2, \cdots$. Then $s_n \in S(J)$, $\lim_{n\to\infty} s_n(t) = f(t)$ a.e. and, for $m > n$,

$$\int_J |s_m - s_n| dt = \int_J |\sum_{k=n+1}^{m} u_k| dt \leq \sum_{k=n+1}^{m} \int_J |u_k| dt \to 0$$

as $m, n \to \infty$. Therefore $\{s_n\}$ is Cauchy and $f \in L^1(J)$.

$\Longrightarrow$ Suppose $\{s_n\}$ is a Cauchy sequence in $S(J)$ such that $\lim_{n\to\infty} s_n(t) = f(t)$ a.e. By passing to a subsequence if necessary, we can assume $\sum_{n=1}^{\infty} \int_J |s_{n+1} - s_n| dt < \infty$. Let $u_1(t) = s_1(t)$, and $U - k(t) = s_k(t) - s_{k-1}(t)$ for $k \geq 2$. Then $u_k \in S(J)$, $f(t) = \sum_{k=1}^{\infty} u_k(t)$, a.e. and $\sum_{k=1}^{\infty} \int_J |u_k| dt < \infty$. Q.E.D.

Lemma 9 *$f \in L^1(J) \Longleftrightarrow$ For every $\epsilon > 0$ there exists a sequence $\{w_k\}$ in $S(J)$ such that*

1. *$f = \sum_{k=1}^{\infty} w_k$ a.e.*
2. *$\sum_{k=2}^{\infty} \int_J |w_k| dt \leq \epsilon$.*
3. *$\int_J |f - w_1| dt \leq \epsilon$.*

In this case $\int_J f\ dt = \sum_{k=1}^{\infty} w_k dt$ and $\sum_{k=1}^{\infty} \int_J |w_k| dt \leq \int_J |f| dt + 2\epsilon$.

PROOF: $\Longleftarrow$ Immediate.

$\Longrightarrow$ By lemma 8, $f \in L^1(J) \Longrightarrow$ there exists $\{u_k\}$ in $S(J)$ such that $f(t) = \sum_{k=1}^{\infty} u_k(t)$ a.e. and $\sum_{k=1}^{\infty} \int_J |u_k| dt < \infty$. Choose k_0 so large that $\sum_{k=k_0}^{\infty} \int_J |u_k| dt \leq \epsilon$ and set $w_1 = \sum_{\ell=1}^{k_0-1} u_\ell$, $w_k = u_{k_0+k-2}$ for $k \geq 2$. Then $\sum_{k=2}^{\infty} \int_J |w_k| dt \leq \epsilon$ and

$$\int_J |f - w_1| dt = \int_J |\sum_{k=2}^{\infty} w_k| dt \leq \sum_{k=2}^{\infty} \int_J |w_k| dt \leq \epsilon.$$

Also

$$\sum_{k=1}^{\infty} \int_J |w_k| dt = \int_J |w_1| dt + \sum_{k=2}^{\infty} \int_J |w_k| dt \leq \int_J |f| dt + \int_J |f - w_1| dt + \epsilon \leq \int_J |f| dt + 2\epsilon.$$

Q.E.D.

Theorem 18 *Let $f \in L^1(J)$. Then for every $\epsilon > 0$ we can find a $s \in S(J)$ and a countable family $\mathcal{I}_\epsilon$ of intervals with total length $\leq \epsilon$ such that*

1. $\int_J |f - s|dt < \epsilon$.

2. $|f(t) - s(t)| \leq \epsilon$ *on* $J - \cup(\mathcal{I}_\epsilon)$, *where* $\cup(\mathcal{I}_\epsilon)$ *is the union of the intervals in* $\mathcal{I}_\epsilon$.

PROOF: Part 1 follows from lemma 9. For part 2, we again refer to lemma 9 and define step functions $s_j = \sum_{k=1}^{j} w_k$ for $j = 1, 2, \cdots$. Then for every $t \notin N$ (a null set) we have $s_j \to f$ a.e. as $n \to \infty$ and $\lim_{j\to\infty} \int_J |f - s_j|dt = 0$. Now let $j_1 < j_2 < \cdots < j_n < \cdots$ be a increasing sequence of integers. Then

$$f = s_{j_1} + [s_{j_2} - s_{j_1}] + \cdots + [s_{j_{n+1}} - s_{j_n}] \cdots.$$

Set $v_n(t) = s_{j_{n+1}}(t) - s_{j_n}(t)$. Then

$$f(t) - s_{j_h}(t) = \sum_{n=h}^{\infty} v_n(t), \qquad t \in J - N, \quad h = 1, 2, \cdots.$$

Now choose j_1 so large that $\int_J |f - s_{j_1}|dt < \epsilon$, and in addition require that the j_n be chosen such that $\int_J |s_j - s_{j_n}|dt \leq \frac{1}{4^n}$ for all $j > j_n$ and $n = 1, 2, \cdots$. Let $\mathcal{G}_n$ be the finite family of intervals consisting of the maximal subintervals of J on which $|v_n(t)| \geq \frac{1}{2^n}$. Since $\int_J |v_n|dt \leq \frac{1}{4^n}$ it follows that $\ell(\mathcal{G}_n) \leq \frac{1}{2^n}$. Note that $|v_n(t)| < \frac{1}{2^n}$ outside $\cup(\mathcal{G}_n)$.

For each positive integer p, let $\mathcal{H}_p$ be the countable family of all intervals in $\mathcal{G}_p, \mathcal{G}_{p+1}, \mathcal{G}_{p+2}, \cdots$. Now

$$\ell(\mathcal{H}_p \leq \frac{1}{2^p} + \frac{1}{2^{p+1}} + \cdots + \frac{1}{2^{p+n}} + \cdots = \frac{1}{2^{p-1}},$$

and, for $n \geq p$ we have $|v_n(t)| < \frac{1}{2^n}$ outside $\cup(\mathcal{H}_p)$. Thus

$$|f(t) - s_{j_p}(t)| \leq \sum_{n=p}^{\infty} |v_n(t)| < \sum_{n=p}^{\infty} \frac{1}{2^n} = \frac{1}{2^{p-1}}$$

for $t \notin \cup(\mathcal{H}_p) \cup N$, $\ell(\mathcal{H}_p) \leq \frac{1}{2^{p-1}}$. Now cover N by a countable family of intervals of total length $\leq (\frac{1}{2})^{p-1}$ and add these intervals to $\mathcal{H}_p$ to form $\mathcal{K}_p$, $\ell(\mathcal{K}_p) \leq \frac{1}{2^{p-2}}$, so that

$$|f(t) - s_{j_p}(t)| \leq \frac{1}{2^{p-1}}, \qquad t \notin J - \cup(\mathcal{K}_p).$$

We obtain the statement of the theorem by choosing p so large that $\epsilon \leq \frac{1}{2^{p-2}}$, and setting $\mathcal{I}_\epsilon = \mathcal{K}_p$. Q,E.D.

Theorem 19 *1. Let $\{g_n\}$ be a Cauchy sequence in $L^1(J)$ such that $g_n(t) \to f(t)$ a.e. as $n \to \infty$. Then $f \in L^1(J)$ and $\int_J f\ dt = \lim_{n\to\infty} \int_J g_n dt$. Also, $\lim_{n\to\infty} \int_J |f - g_n| dt = 0$.*

2. Let $\{f_n\}$ be a sequence in $L^1(J)$ such that $f(t) = \sum_{n=1}^\infty f_n(t)$ a.e. and $\sum_{n=1}^\infty \int_J |f_n| dt < \infty$. Then $f \in L^1(J)$ and $\int_J f\ dt = \sum_{n=1}^\infty \int_J f_n dt$.

PROOF: Assertion 2 $\Longrightarrow$ assertion 1, by the proof of lemma 8. It is enough to prove assertion 2. By lemma 9 there exist step functions s_{nk} such that $f_n(t) = \sum_{k=1}^\infty s_{nk}(t)$ for $t \notin N_n$ (where N_n is a null set), and

$$\sum_{k=1}^\infty \int_J |s_{nk}| dt \le \int_J |f_n| dt + \frac{1}{2^n}, \qquad n = 1, 2, \cdots.$$

(Set $\epsilon_n = \frac{1}{2^{n+1}}$.) Note that $\int_J f_n dt = \sum_{k=1}^\infty \int_J s_{nk} dt$. Then

$$\sum_{n,k=1}^\infty \int_J |s_{nk}| dt \le \sum_{n=1}^\infty \int_J |f_n| dt + \sum_{n=1}^\infty \frac{1}{2^n} < \infty.$$

This implies that $\sum_{n,k=1}^\infty s_{nk}(t)$ converges absolutely for all $t \notin N$, where N is a null set. Therefore, if $t \notin (\cup_n N_n) \cup N$ then $\sum_{n=1}^\infty f_n(t) = \sum_{n,k=1}^\infty s_{nk}(t) = f(t) \Longrightarrow$

$$f \in L^1(J) \qquad \text{and} \qquad \int_J f\ dt = \sum_{n=1}^\infty \sum_{k=1}^\infty \int_J s_{nk} dt = \sum_{n=1}^\infty \int_J f_n dt.$$

Q.E.D.

1.5 Lebesgue measure and integration

We have already shown that we can obtain the spaces $L^1(J)$, $L^2(J)$ of Lebesgue integrable and square integrable functions as the closures of the space $S(J)$ of step functions on J. The closures are taken in the respective norms $\int_J |s| dt$ and $\sqrt{\int_J |s|^2 dt}$. The "elements" of these Banach spaces are equivalence classes of Cauchy sequences of step functions. In the last (technical) section we showed that to each equivalence class of Cauchy sequences in the $L^1(J)$ norm we could associate a class of a.e. pointwise convergent functions. Two functions in the same class converge to the same values a.e. Here we will use these results to recast Lebesgue theory as a theory of integrals of true functions.

Definition 25 *A function f is measurable on J if there exists a sequence $\{s_n\}$ in $S(J)$ such that* $\lim_{n\to\infty} s_n = f$*, a.e.*

NOTE: If f is associated with $L^1(J)$ then there is a *Cauchy* sequence $\{s_n\}$ in $S(J)$ such that $\lim_{n\to\infty} s_n = f$, a.e.

Definition 26 *Let f be a function on J. If there exists a sequence $\{s_n\}$ such that*

1. $\lim_{n\to\infty} s_n = f$ *a.e.*

2. $\int_J |s_n - s_m| dt \to 0$ *as $n, m \to \infty$, (i.e., if $\{s_n\}$ is Cauchy)*

then we say that f is Lebesgue integrable and we define the Lebesgue integral of f on J by $\int_J f(t)dt = \lim_{n\to\infty} \int_j s_n(t)dt$.

JUSTIFICATION:

1. Limit exists: $\left|\int_J s_n dt - \int_j s_m dt\right| \le \int_J |s_n - s_m| dt \to 0$ as $n, m \to \infty \Longrightarrow$ $\{\int_J s_n dt\}$ is Cauchy in C (or R) $\Longrightarrow \int_J f\ dt$ exists.

2. Limit is unique: By lemma 7, if $\{s_n\}$, $\{r_n\}$ are Cauchy in $S(J)$ and $s_n \to f$, $r_n \to f$ a.e. as $n \to \infty$ then $\{s_n\} \sim \{r_n\}$

$$\Longrightarrow \int_J |s_n - r_n| dt \to 0 \quad \text{as } n \to \infty$$

$$\Longrightarrow \left|\int_J s_n - \int_J r_n dt\right| \le \int_J |s_n - r_n| dt \to 0 \quad \text{as } n \to \infty$$

$$\Longrightarrow \lim_{n\to\infty} \int_J s_n dt = \lim_{n\to\infty} \int_J r_n dt.$$

Note: Let $F(J)$ be the space of equivalence class of a.e. equal Lebesgue integrable functions on J and recall that $L^1(J)$ is the space of equivalence classes (in the norm) of Cauchy sequences of step functions on J. There is a 1-1 correspondence between these spaces. Indeed

1. Let $\{s_n\}$ be a Cauchy sequence in $S^1(J)$. Then there is a subsequence $\{s_{n_k}\}$ such that $\lim_{k\to\infty} s_{n_k} \to f$ a.e. and $||f|| = \lim_{n\to\infty} \int_J |s_n| dt = \lim_{k\to\infty} \int_J |s_{n_k}| dt = \int_{J,\ \text{Lebesgue}} |f| dt$.

2. If f, g are Lebesgue integrable and $f \sim g$, then $f = g$ a.e. on J, so $\int_J f\ dt = \int_J g\ dt$.

Let $\mathcal{L}(J)$ be the set of all real Lebesgue integrable functions on J. The following theorem is very simple to prove.

Theorem 20 *Let $f, g \in \mathcal{L}(J)$, α, β scalars. Then*

1. *$\alpha f + \beta g \in \mathcal{L}(J)$ and*

$$\int_J (\alpha f + \beta g) dt = \alpha \int_J f\ dt + \beta \int_J g\ dt.$$

2. *Let $f \in \mathcal{L}(J)$ and suppose $\{s_n\}$ is a Cauchy sequence in the norm of $S(J)$ converging to f, Then $\{|s_n|\}$ is also a Cauchy sequence in the norm of $S(J)$, $|f| \in \mathcal{L}(J)$ and $\int_J |f| dt = \lim_{n\to\infty} \int_J |s_n| dt$.*

3. *$f \geq 0 \Longrightarrow \int_j f\ dt \geq 0$.*

We define the norm of the Lebesgue integrable function f by $||f|| = \int_J |f| dt$. Also we introduce an equivalence relation on $\mathcal{L}(J)$ by defining $f, g \in \mathcal{L}(J)$ as **equivalent** ($f \sim g$) provided $f = g$ a.e. Note that this equivalence relation preserves the vector space structure: if $f_j \sim g_j$ for $j = 1, 2$ and α, β are scalars, then $\alpha f_1 + \beta f_2 \sim \alpha g_1 + \beta g_2$. Also, all elements of the equivalence class F have the same norm, so we can define $||F|| = \int_J |f| dt$ for any $f \in F$.

Now let $L(J)$ be the normed vector space of equivalence classes of functions in $\mathcal{L}(J)$, equal a.e. It follows that $L(J)$ is isomorphic to $L^1(J) = \overline{S(J)}$. Hence $L(J)$ is complete, i.e., if $\{f_n\}$ is a Cauchy sequence of integrable functions on J, then there exists an integrable function f such that $f_n \to f$ in the norm: $\lim_{n\to\infty} \int_J |f_n - f| dt = 0$.

Theorem 21 1. *$f \in L(J)$, $\int_J |f| dt = 0 \Longleftrightarrow f = 0$ a.e.*

2. *$f \in L(J) \Longrightarrow$ there exists $\{s_n\} \subset S(J)$ such that $\lim_{n\to\infty} \int_J |f - s_n| dt = 0$.*

PROOF:

1. $f = 0$ a.e $\Longrightarrow |f| = 0$ a.e. $\Longrightarrow \int_J |f| dt = 0$. Conversely, $\int_J |f| dt = 0 \Longrightarrow f \in$ equivalence class of zero function $\Longrightarrow f = 0$ a.e.

2. $f \in \overline{S(J)} \Longrightarrow$ there exists $\{s_n\} \subset S(J)$ such that $\lim_{n\to\infty} ||f - s_n|| = 0$.

Q.E.D.

REMARK: Every $f \in \mathcal{L}(J)$ is a limit in the mean, not only of step functions, but also of continuous finctions, of differentiable functions, and of C^∞ or $\overset{\circ}{C}{}^{\infty}$ functions. Further, we have

Theorem 22 *Let $\{f_k\}$ be a Cauchy sequence in $\mathcal{L}^1(J)$. Then there exists a subsequence $\{f_{k_n}\}$ such that $\lim_{n\to\infty} f_{k_n}(t) = f(t)$ is finite a.e. For any such subsequence, $f \in \mathcal{L}^1(J)$, $\int_J f\ dt = \lim_{k\to\infty} \int_J f_k dt$, and $\int_J |f - f_k| dt \to 0$ as $k \to \infty$. The f's obtained from any two subsequences are equal a.e.*

PROOF: From theorem 18, for each integer k there exists a step function $s_k \in S(J)$ such that $\int_J |f_k - s_k| dt < \frac{1}{2^k}$ and a countable family of intervals $\mathcal{F}_k$ of total length $\ell(\mathcal{F}_k) < \frac{1}{2^k}$ such that $|f_k(t) - s_k(t)| < \frac{1}{2^k}$ for $t \notin \cup(\mathcal{F}_k)$. From lemma 5 we can find a subsequence $\{s_{k_n}\}$ of $\{s_k\}$ such that $\lim_{n\to\infty} s_{k_n}(t)$ exists and is finite a.e., i.e., outside of the null set N. Set $f(t) = \lim_{n\to\infty} s_{k_n}(t)$, outside N. Now

$$\int_J |s_j - s_k| dt = \int_J |s_j - f_j + f_j - f_k + f_k - s_k| dt$$

$$\leq \int_J |s_j - f_j| dt + \int_J |f_j - f_k| dt + \int_J |f_k - s_k| dt \to 0$$

as $j, k \to \infty$. Thus $\{s_k\}$ is Cauchy in $S(J)$, $f \in \mathcal{L}^1(J)$ and $\int_J f\ dt = \lim_{k\to\infty} \int_J s_k dt$. But $|\int_J s_k dt - \int_J f_k dt| \leq \int_J |s_k - f_k| dt \to 0$ as $k \to \infty$, so $\int_J f\ dt = \lim_{k\to\infty} \int_j f_k dt$.

Clearly $s_{k_n} \to f$ a.e. Let $H_p = \cup_{k=0}^{\infty} \mathcal{F}_{p+k}$. Then

$$\ell(H_p) < \frac{1}{2^p} + \frac{1}{2^{p+1}} + \cdots = \frac{1}{2^{p-1}}.$$

Hence $|f_k(t) - s_{k_n}(t)| < \frac{1}{2^k}$ for $t \notin H_p$ where $\ell(H_p) < \frac{1}{2^{p-1}}$ and $k \geq p$. Thus $f_k - s_k \to 0$ a.e. $\Longrightarrow f_{k_n} \to f$ a.e. as $n \to \infty$. Q.E.D.

1.5.1 Fundamental convergence theorems

This section includes the proofs of the most important Lebesgue convergence theorems that will be used as tools in the rest of the course. From now on we will identify the spaces $L(J), L^1(J)$ and $\mathcal{L}^1(J)$

Theorem 23 *(Monotone Convergence Theorem.) Let* $\{f_n\}$ *be an increasing sequence of real valued functions in* $L^1(J)$, $(f_1 \leq f_2 \leq \cdots)$ *such that* $\int_J f_n dt \leq M < \infty$ *for all* n. *Then* $\lim_{n\to\infty} f_n = f$ *exists and is finite a.e., and* $f \in L^1(J)$. *Further,* $\int_J f\ dt = \lim_{n\to\infty} \int_J f_n dt = K \leq M$

PROOF: Let $m \geq n$. Then

$$\int_J |f_M - f_n| dt = \int_J (f_m - f_n) dt = \int_J f_m dt - \int_J f_n dt \to 0$$

as $n, m \to \infty$. Therefore, $\{f_n\}$ is a Cauchy sequence in $L^1(J)$. The rest follows from theorem 22. Q.E.D.

Note: There is a similar result for monotone decreasing sequences.

Before presenting the next major theorem, we need to review some basic concepts from real analysis. Let $\{a_k : \quad k = 1, 2, \cdots\}$ be a sequence of real numbers, and let

$$b_n = \inf_{k \geq n} a_k, \qquad c_n = \sup_{k \geq n}\ a_k.$$

Note that b_n, c_n always exist, though they may be infinite.

Definition 27

$$\liminf\ a_k = \lim_{n\to\infty} \inf_{k\geq n}\ a_k = \lim_{n\to\infty}\ b_n = \underline{\lim}\ a_k,$$

$$\limsup\ a_k = \lim_{n\to\infty} \sup_{k\geq n}\ a_k = \lim_{n\to\infty}\ c_n = \overline{\lim}\ a_k.$$

For the proofs of our two major results, Fatou's lemma and the Lebesgue Dominated Convergence Theorem, we need the following preliminary result:

Lemma 10 *Let* f, g *be real valued functions in*$L^1(J)$ *and* $f \wedge g(t) = \min\{f(t), g(t)\}$, $f \vee g(t) = \max\{f(t), g(t)\}$. *Then* $f \wedge g, f \vee g \in L^1(J)$.

PROOF: We will lay out the steps in the straightforward proof and leave the details to the reader.

- Since $f \vee g = -(-f) \wedge (-g)$ it is enough to prove the stated result for $f \wedge g$.

- If $f, g \in L^1(J)$ there exist Cauchy sequences of step functions $\{f_n\}$, $\{g_n\}$ in $S(J)$ such that $f_n \to f$, a.e., and $g_n \to g$, a.e., and $\int_J f_n dt \to \int_J f\ dt$, $\int_J g_n dt \to \int_J g\ dt$, as $n \to \infty$.

- $f_n \wedge g_n \in S(J)$ and $f_n \wedge g_n \to f \wedge g$, a.e. as $n \to \infty$.
- If w_1, w_2, v_1, v_2 are real-valued functions on J then

$$|w_1 \wedge v_1(t) - w_2 \wedge v_2(t)| \leq |w_1(t) - w_2(t)| + |v_1(t) - v_2(t)|.$$

- $\{f_n \wedge g_n\}$ is a Cauchy sequence of step functions in the norm.
- Thus $\int_J f_n \wedge g_n dt \to \int_J f \wedge g \; dt$, $\int_J |f \wedge g - f_n \wedge g_n| dt \to 0$ as $n \to \infty$ and $f \wedge g \in L^1(J)$.

Q.E.D.

Properties of lim inf and lim sup:

1. $\underline{\lim} \; a_k, \overline{\lim} \; a_k$ always exist. They are either finite or $+\infty$ or $-\infty$. This follows from the properties

$$b_1 \leq b_2 \leq b_3 \leq \cdots,$$

$$c_1 \geq c_2 \geq c_3 \geq \cdots.$$

2. $c_n \geq b_n \Longrightarrow \lim_{n\to\infty} \; c_n \geq \lim_{n\to\infty} \; b_n$. Therefore, $\overline{\lim} \; a_k \geq \underline{\lim} \; a_k$.
3. $\lim_{k\to\infty} \; a_k$ exists $\Longleftrightarrow \underline{\lim} \; a_k = \overline{\lim} \; a_k$ and is finite, in which case

$$\lim_{k\to\infty} \; a_k = \underline{\lim} \; a_k = \overline{\lim} \; a_k.$$

   PROOF: $c_n \geq a_n \geq b_n$.

   (a) $\underline{\lim} \; a_k = \overline{\lim} \; a_k$ finite $\Longrightarrow$

$$\overline{\lim} \; a_k = \lim_n \; c_n \geq \lim_k \; a_k \geq \lim_n \; b_n \geq \underline{\lim} \; a_k.$$

   (b) $\lim_n \; a_n = a$ exists $\Longrightarrow$ for every $\epsilon > 0$ there exists an integer N_ϵ such that $|a - a_k| < \epsilon$ for $k \geq N_\epsilon$. This means that

$$a - \epsilon < a_k < a + \epsilon \qquad \text{if } k \geq N_\epsilon.$$

   Hence if $n \geq N_\epsilon$ then $b_n \geq a - \epsilon$ and $c_n \leq a + \epsilon$. This implies that $\underline{\lim} \; a_k \geq a - \epsilon$ and $\overline{\lim} \; a_k \leq a + \epsilon$ for all $\epsilon > 0$. Hence $\lim_{k\to\infty} \; a_k = \underline{\lim} \; a_k = \overline{\lim} \; a_k = a$. Q.E.D.

Theorem 24 *(Fatou's lemma) Suppose* $\{f_n\}$ *is a real sequence in* $L^1(J)$ *such that* $f_n \geq 0$ *a.e. for all* n*. Then*

$$\int_J \underline{\lim}\ f_n\ dt \leq \underline{\lim} \int_J f_n\ dt.$$

EXPLANATION: This means that if the right-hand side of this expression is finite then $\underline{\lim}\ f_n \in L^1(J)$ and the inequality holds.

PROOF: There is no loss in generality by assuming that $f_n \geq 0$ everywhere. Set $g_n(t) = \inf\{f_j(t) : j \geq n\}$. Now $g_n \in L^1(J)$ since the decreasing sequence $h_k^{(n)} = f_n \wedge \cdots \wedge f_{n+k-1} \to g_n$, everywhere as $k \to \infty$. (Note that $h_k^{(n)} \geq h_{k+1}^{(n)} \geq \cdots$.) Now $h_k^{(n)} \in L^1(J)$ and $\int_J h_k^{(n)} dt \geq 0 \Longrightarrow g_n \in L^1(J)$, by the monotone convergence theorem. But $g_1 \leq g_2 \leq g_3 \leq \cdots$ and $\int_J g_n dt \leq \int_J f_j dt$ for all $j \geq n$, $\Longrightarrow$

$$\int_J g_n \leq \inf_{j \geq n} \int_J f_j dt \leq \underline{\lim} \int_J f_j dt.$$

Therefore, if $\underline{\lim} \int_J f_j dt$ is finite, the monotone convergence theorem says

$$\lim_{n\to\infty} g_n(t) = \lim_{n\to\infty} \inf_{j\geq n}\{f_j(t)\} = \underline{\lim}\ f_j(t) \in L^1(J)$$

and $\int_J \underline{\lim}\ f_j dt \leq \underline{\lim} \int_J f_j dt$. Q.E.D.

Theorem 25 *(Lebesgue's Dominated Convergence Theorem) Let* $\{f_n\}$ *be a sequence in* $L^1(J)$ *such that* $\lim_{n\to\infty} f_n$ *converges a.e., and suppose there is a function* $g \in L^1(J)$ *such that* $|f_n| \leq g$ *a.e. for all* n*. If* $f = \lim_{n\to\infty} f_n$ *a.e., then* $f \in L^1(J)$ *and* $\int_J f\ dt = \int_J \lim_{n\to\infty}\ f_n dt = \lim_{n\to\infty} \int_J f_n dt$*. Furthermore* $\lim_{n\to\infty} \int_J |f - f_n| dt = 0$*.*

PROOF: (We will assume that the f_n are real functions. The extension to complex functions then follows easily by splitting f_n into real and imaginary parts.) We have $2g \geq g - f_n \geq 0$ a.e. for all n and $\int_J (g - f_n) dt \leq 2 \int_J g\ dt$. Therefore by Fatou's lemma

$$g - f = g - \lim_{n\to\infty}\ f_n = \underline{\lim}(g - f_n) \in L^1(J),$$

and

$$\int_J (g - f) dt \leq \underline{\lim} \int_J (g - f_n) dt = \underline{\lim}[\int_J g\ dt - \int_J f_n dt] \leq 2 \int_J g\ dt,$$

$\Longrightarrow$

$$\int_J g\ dt - \int_J f\ dt \leq \int_J g\ dt - \overline{\lim} \int_J f_n dt,$$

$\Longrightarrow \int_J f\ dt \geq \overline{\lim} \int_J f_n dt$.

Similarly, $2g \geq g + f_n \geq 0$ a.e., all n, and $\int_J (g + f_n) dt \leq 2 \int_J g\ dt$. Therefore by Fatou's lemma: $g + f = \underline{\lim}(g + f_n) \in L^1(J)$ and

$$\int_J (g+f) dt \leq \underline{\lim} \int_J (g + f_n) dt = \int_J g\ dt + \underline{\lim} \int_J f_n\ dt \leq 2 \int_J g\ dt,$$

$\Longrightarrow \int_J f\ dt \leq \underline{\lim} \int_J f_n dt$. Therefore $\int_J f\ dt = \lim_{n\to\infty} \int_J f_n dt$. Q.E.D.

1.5.2 The Hilbert space $L^2(J)$

Recall that $S^2(J)$ is the space of all real or complex-valued step functions on the (bounded or unbounded) interval J on the real line with real inner product $(s_1, s_2) = \int_J s_1(t)\bar{s}_2(t) dt$. (We identify $s_1, s_2 \in S(J)$, $s_1 \sim s_2$ if $s_1(t) = s_2(t)$ except at a finite number of points.) $S^2(J)$ is the space of equivalence classes of step functions in $S(J)$. Then $S^2(J)$ is an inner product space with norm $||s||^2 = \int_J |s(t)|^2 dt$.

The space of $L^2(J)$ is the completion of $S^2(J)$ in this norm. $L^2(J)$ is a Hilbert space. Every element u of $L^2(J)$ is an equivalence class of Cauchy sequences of step functions $\{s_n\}$, $\int_J |s_j - s_k|^2 dt \to 0$ as $j, k \to \infty$. (Recall $\{s'_n\} \sim \{s_n\}$ if $\int_J |s'_k - s_n|^2 dt \to 0$ as $n \to \infty$. Now by a slight modification of lemmas 5, 6 and 7 we can identify an equivalence class X of Cauchy sequences of step functions with an equivalence class F of functions $f(t)$ that are equal a.e. Indeed $f \in F \Longleftrightarrow$ there exists a Cauchy sequence $\{s_n\} \in X$ and an increasing sequence of integers $n_1 < n_2 \cdots$ such that $s_{n_k} \to f$ a.e. as $k \to \infty$.

In analogy with our treatment of the function space $\mathcal{L}^1(J)$ and the Banach space $L^1(J)$ of Cauchy sequences of step functions, we introduce the function space $\mathcal{L}^2(J)$.

Definition 28 $f \in \mathcal{L}^2(J)$ *if there exists a sequence of step functions* $\{s_n\}$ *in* $S^2(J)$ *such that*

1. $s_n \to f$ *pointwise a.e.*

2. $\int_J |s_N - s_m|^2 dt \to 0$ *as* $n, m \to \infty$, *i.e.,* $\{s_n\}$ *is Cauchy in* $S^2(J)$.

Lemma 11 $\mathcal{L}^2(J)$ *is a vector space.*

PROOF: Suppose $f, g \in \mathcal{L}^2(J)$, α, β scalars. Then there exist Cauchy sequences $\{s_n\}, \{r_n\} \in S^2(J)$ such that $s_n \to f$ a.e. and $r_n \to g$ a.e. as $n \to \infty$. It follows that $\alpha s_n + \beta r_n \to \alpha f + \beta g$ a.e. as $n \to \infty$, and

$$||(\alpha s_n + \beta r_n) - (\alpha s_m - \beta r_m)|| = ||\alpha(s_n - s_m) + \beta(r_n - r_m)|| \leq |\alpha| \cdot ||s_n - s_m|| + |\beta| \cdot ||r_n - r_m|| \to 0$$

as $n, m \to \infty$. Q.E.D.

Lemma 12 *If $\{s_n\}$ is a Cauchy sequence in $S^2(J)$ then $\{|s_n|^2\}$ is a Cauchy sequence in $S(J)$.*

PROOF: Given $\epsilon > 0$ we have

$$|s_n| = |s_m - (s_m - s_n)| \leq |s_m| + |s_m - s_n| \leq \begin{cases} (1+\epsilon)|s_m| & \text{if } |s_m - s_n| \leq \epsilon |s_m| \\ (1 + \frac{1}{\epsilon})|s_m - s_n| & \text{if } |s_m - s_n| \geq \epsilon |s_m|, \end{cases}$$

$\Longrightarrow$

$$|s_n|^2 \leq (1+\epsilon)^2 |s_m|^2 + (1 + \frac{1}{\epsilon})^2 |s_m - s_n|^2,$$

so

$$\Big| |s_n|^2 - |s_m|^2 \Big| \leq [(1+\epsilon)^2 - 1] |s_m|^2 + (1 + \frac{1}{\epsilon})^2 |s_m - s_n|^2.$$

Integrating this inequality over J and using that fact that since $\{s_m\}$ is Cauchy in $S^2(J)$ there exists a finite number M such that $\int_J |s_m|^2 dt \leq M$, we obtain

$$\int_J \Big| |s_n|^2 - |s_m|^2 \Big| \, dt \leq [(1+\epsilon)^2 - 1] M + (1 + \frac{1}{\epsilon})^2 ||s_m - s_n||^2.$$

Now, given $\epsilon' > 0$ we can choose ϵ so small that the first term on the right-hand side of this inequality is $< \epsilon'/2$. Then we can choose n, m so large that the second term on the right-hand side of this inequality is also $< \epsilon'/2$. Thus $\int_J ||s_n|^2 - |s_m|^2| \, dt \to 0$ as $n, m \to \infty$. Q.E.D.

Note that if $f \in \mathcal{L}^2(J)$ than f is measurable and $|f|^2 \in \mathcal{L}^1(J)$. Indeed $s_n \to f$ a.e. and $\{s_n\}$ is Cauchy in $S^2(J)$, so $\{|s_n|^2\}$ is Cauchy in $S(J)$, $|s_n|2 \to |f|^2$ a.e., and $\int_J |f|^2 dt = \lim_{n\to\infty} \int_J |s_n|^2 dt$.

Lemma 13 *If f is measurable on J and $|f|^2 \in \mathcal{L}^1(J)$, then $f \in \mathcal{L}^2(J)$, i.e., there exists a sequence $\{s_n\} \in \mathcal{S}(J)$ such that $s_n \to f$ a.e. as $n \to \infty$ and $\{s_n\}$ is Cauchy in $S^2(J)$.*

CONCLUSION: $\mathcal{L}^2(J)$ is the space of all measurable functions f on J such that $|f|^2 \in \mathcal{L}^1(J)$. There is a 1-1 relationship between equivalence classes X of Cauchy sequences $\{s_n\}$, i.e., elements $X \in L^2(J)$ and equivalence classes F of functions $\{f\}$ equal a.e. Here $||X||^2 = \lim_{n\to\infty} \int_J |s_n|^2 dt = \int_J |f|^2 dt$.

Theorem 26 *If $f, g \in \mathcal{L}^2(J)$ then $f \cdot g \in \mathcal{L}^1(J)$ and*

$$\left|\int_J f(t)g(t)dt\right|^2 \leq \int_J |f(t)|^2 dt \int_J |g(t)|^2 dt.$$

PROOF: There exist Cauchy sequences $\{s_n\}, \{r_n\} \in S^2(J)$ such that $s_n \to f$ a.e. and $r_n \to g$ a.e. as $n \to \infty$. Then $s_n \cdot r_n \to f \cdot g$ a.e. as $n \to \infty$. Now

$$\int_J |s_n r_n - s_m r_m| dt \leq \int_J |s_n r_n - s_n r_m| dt + \int_J |s_n r_m - s_m r_m| dt$$

$$\leq \int_J |s_n| \cdot |r_n - r_m| dt + \int_J |r_m| \cdot |s_n - s_m| dt$$
$$\leq ||s_n||_2 dot ||r_n - r_m||_2 + ||r_m||_2 \cdot ||s_n - s_m||_2 \to 0$$

as $n.m \to \infty$. (Here the norms are in $S^2(J)$ and the norms $||s_n||_2, ||r_m||_2$ are uniformly bounded.) Thus $\{s_n \cdot r_n\}$ is Cauchy in $S^1(J)$. $\Longrightarrow f \cdot g \in \mathcal{L}(J)$ and

$$|\int_J f \cdot g \; dt| = \lim_{n\to\infty} |\int_J s_n r_n \; dt| \leq \lim_{n\to\infty} ||s_n||_2 ||r_n||_2 = ||f||_2 ||g||_2.$$

Q.E.D.

1.5.3 An aside on measurable functions

Recall that a real function f is measurable on the interval J if there exists a sequence $\{s_n\} \in S(J)$ such that $s_n \to f$ a.e.

Theorem 27 *let f, g be measurable on J and let a, b be real constants. Then*

1. *$af + bg$ is measurable.*

2. *$|f|$ is measurable.*

3. *$f \vee g$, $f \wedge g$, f^+ and f^- are measurable.*

4. *fg is measurable.*

PROOF:

1. Obvious.
2. If $\{s_k\} \in S(J)$ and $s_k \to f$ a.e., then $\{|s_k|\} \in S(J)$ and $|s_k| \to |f|$ a.e.
3. $f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$, $f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$, $f+ = f \vee 0$, $f^- = (-f) \vee 0$.
4. Suppose $\{s_n\}, \{r_n\} \in S(J)$ such that $s_n \to f$ a.e. and $r_n \to g$ a.e. as $n \to \infty$. Then $s_n \cdot r_n \to f \cdot g$ a.e. as $n \to \infty$ and $s_n \cdot r_n \in S(J)$.

Q.E.D.

Theorem 28 *Let $\{f_n\}$ be a sequence in $L^1(R)$ such that $\lim_{n\to\infty} f_n$ converges pointwise a.e. to f. Then f is measurable.*

PROOF: From theorem 18, for each integer n there exists a step function $s_n \in S(J)$ such that $\int_J |f_k - s_n| dt < \frac{1}{2^n}$ and a countable family of intervals $\mathcal{F}_n$ of total length $\ell(\mathcal{F}_n) < \frac{1}{2^n}$ such that $|f_n(t) - s_n(t)| < \frac{1}{2^n}$ for $t \notin \cup(\mathcal{F}_n)$. Let $H_p = \cup_{k=0}^{\infty} \mathcal{F}_{p+n}$. Then

$$\ell(H_p) < \frac{1}{2^p} + \frac{1}{2^{p+1}} + \cdots = \frac{1}{2^{p-1}}.$$

Hence $|f_n(t) - s_n(t)| < \frac{1}{2^n}$ for $t \notin H_p$ where $\ell(H_p) < \frac{1}{2^{p-1}}$ and $n \geq p$. Thus $f_n - s_n \to 0$ a.e. $\Longrightarrow s_n \to f$ a.e. as $n \to \infty$, so f is measurable. Q.E.D.

Lemma 14 *f is measurable on the real line $\Longleftrightarrow \chi_{[-n,n]} f_n \in L^1(R)$ for $n = 1, 2, \cdots$, where*

$$f_n(t) = \begin{cases} f(t) & -n \leq f(t) \leq n \\ n & f(t) > n \\ -n & f(t) < -n. \end{cases}$$

PROOF: $\chi_{[-n,n]} f_n = \chi_{[-n,n]} \cdot (f^+ \wedge n - f^- \wedge n)$. Therefore, f measurable $\Longrightarrow$ $\chi_{[-n,n]} f_n$ bounded and measurable, with compact support $\Longrightarrow \chi_{[-n,n]} f_n \in L^1(R)$. Conversely, if $\chi_{[-n,n]} f_n \in L^1(R)$ for each n we see that $\chi_{[-n,n]} f_n \to f$ a.e. as $n \to \infty$, so by the preceeding theorem, f is measurable. Q.E.D.

Theorem 29 *Let $\{f_n\}$ be a sequence of measurable functions on the real line such that $\lim_{n\to\infty} f_n$ converges pointwise a.e. to f. Then f is measurable.*

PROOF: Let $g_n = \chi_{[-n,n]} \cdot (f_n^+ \wedge n - f_n^- \wedge n)$. Then $g_n \in L^1(R)$ for all n and $g_n \to f$ a.e. $\Longrightarrow f$ is measurable. Q.E.D.

1.5.4 Extensions of the theory

There are two important ways of extending the previous theory that we will use later. First we can replace the usual length measure on the real line by a more general measure. The easiest way to do this is to choose a weight function $\rho(t)$ on the interval J. Here ρ is a measurable function that is strictly positive on J. We say that the measurable function

$$f \in L^2(J, k) \Longleftrightarrow f\sqrt{k} \in L^2(J) \Longleftrightarrow |f|^2 k \in L^1(J).$$

Then $L^2(J, k)$ is a Hilbert space with the inner product $(f, g) = \int_J f(t)\overline{g(t)}k(t)dt$.

A second extension is obtained by replacing the interval J by a measurable set, or by a domain D in an n-dimensional Euclidean space. (For the time being, however, we will stay in one dimension.) Let S be a subset of the real line and let χ_S be the **characteristic function** of S. i.e.,

$$\chi_S(t) = \begin{cases} 1 & \text{if } t \in S \\ 0 & \text{otherwise} \end{cases}$$

We say the S is a **measurable** set if the function χ_S is a measureable function. We say that S is of **finite measure** if $\chi_S \in \mathcal{L}^1(R)$ where R is the real line. The *measure* of S is $m(S) = \int_R \chi_S(t)dt$. Now suppose that S is a measurable set and $S \subset J$. We define the integral on S of a measurable function f by

$$\int_S f \ dt \equiv \int_R f\chi_S dt.$$

Similarly we can define the Hilbert space $L^2(S)$ and the Banach space $L^1(S)$.

Lemma 15 *S is of measure zero $\Longleftrightarrow$ S is a null set.*

PROOF:

$$0 = m(S) \Longleftrightarrow \int_R |\chi_S| dt = 0 \Longleftrightarrow \chi_S = 0 \text{ a.e. } \longleftrightarrow S \text{ is a null set.}$$

Q.E.D.

1.5.5 An aside on differentiation and integration

Suppose $f(t)$ is integrable on the closed interval $[a, b]$ of the real line. We shall denote the integral of f on this interval variously as

$$\int_a^b f(t)dt = \int_{[a,b]} f(t)dt = \int f(t)\chi_{[a,b]}(t)dt$$

where $\chi_{[a,b]}(t)$ is the characteristic function of $[a, b]$ and the last integral is taken over the full real line. This section will be devoted to examing the fundamental theorem of calculus from a Lebesgue theory viewpoint. Given a function $f(t)$ with derivative $f'(t)$, which may not exist at each point t, we ask the following:

1. When does $\int_a^c f'(t)dt = f(c) - f(a)$?
2. When does $\frac{d}{dt}\int_a^t f(\tau)d\tau = f(t)$?

From first year calculus we know that the first identity is correct when f is continuously differentiable on $[a, c]$ and the second is true on intervals where f Riemann integrable and where f is continuous at t.

To shed light on these questions we need to introduce three classes of functions: the monotonic functions, functions of bounded variation, and absolutely continuous functions.

Definition 29 *Let J be a bounded or nonbounded interval of the real line (with endpoints a, b) and let f be a function on J. We say that f is* **monotone increasing** *if $f(t_1) \le f(t_2)$ for all t_1, t_2 on J such that $t_1 < t_2$. Similarly, f is* **monotone decreasing** *on J if $f(t_1) \ge f(t_2)$ for all t_1, t_2 on J such that $t_1 < t_2$. We say that f is* **monotone** *if it is either monotone increasing or monotone decreasing.*

Lemma 16 *If f is monotone on J and $c \in J$ ($c \ne b$) then f has a* **right-hand limit** *at c:*

$$f(c+) = \lim_{t \downarrow c} f(t).$$

PROOF: Suppose f is monotone increasing and set $f(c+) = \inf_{t>c} f(t)$. If $f(c+)$ is finite, then for every $\epsilon > 0$ there is a $\tau \in J$ such that $\tau > c$ and $f(c+) - f(\tau) < \epsilon$. But since f is monotone increasing we have $0 < f(c+) - f(t) < \epsilon$ for $c < t < \tau$. Thus $f(t) \to f(c+)$ as $t \downarrow c$. A similar argument works for f monotone decreasing. Q.E.D.

There is a similar result for left-hand limits.

Lemma 17 *If f is monotone on J and $c \in J$ ($c \ne a$) then f has a* **left-hand limit** *at c:*

$$f(c-) = \lim_{t \uparrow c} f(t).$$

Note that $f(c-) \leq f(c) \leq f(c+)$ if c is an interior point of J and f is monotone increasing. Similarly, $f(c-) \geq f(c) \geq f(c+)$ if c is an interior point of J and f is monotone decreasing. Thus if c is an interior point of J there are two possibilities:

1. $f(c+) = f(c-)$. In this case $f(c+) = f(c) = f(c-)$ and f is continuous at c.

2. $f(c+) \neq f(c-)$. In this case f has a *jump discontinuity* at c with jump $f(c+) - f(c-)$.

The endpoints have to be treated separately.

Lemma 18 *A monotone function has at most countably many points of discontinuity.*

PROOF: Suppose f is monotone on J. To each interior discontinuity c of f we can associate uniquely the open interval $I_c = (f(c-), f(c+))$ of the real line. The intervals I_{c_1}, I_{c_2} are non-overlapping for $c_1 \neq c_2$ and each I_c contains a rational number. Thus there is a $1-1$ mapping between the set of discontinuities $\{c\}$ and a subset of the rational numbers. Hence $\{c\}$ is countable. Q.E.D.

From this result it is not hard to verify that a bounded monotonic function on a bounded interval is Riemann integrable.

Definition 30 *Let A be a subset of the real line and let $I_1, \cdots, I_n, \cdots$ be a countable number of open intervals such that $A \subset \cup_{n=1}^{\infty} I_n$. We define the* **outer measure** *$m^*(A)$ by*

$$m^*(A) = \inf\{\sum_{n=1}^{\infty} \ell(I_n) : A \subset \cup_{n=1}^{\infty} I_n\}$$

Lemma 19 *Some simple properties of outer measure:*

1. *If $A \subset B$ then $m^*(A) \leq m^*(B)$.*

2. *A is a null set if and only if $m^*(A) = 0$.*

3. *If A is an interval of the form $[a,b], [a,b), (a,b]$ or (a,b), then $m^*(A) = b - a$.*

4. *(countable subadditivity) If $\{A_n\}$ is a countable collection of subsets of the real line, then $m^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$. (This inequality holds in the extended number system. Thus if the left-hand side is $+\infty$, so is the right-hand side.)*

PROOF: Properties 1) and 2) are easy. For 3) it is evident that $m * (A) \leq b - a$. For the rest, use the Heine-Borel theorem. We give the proof of the subadditivity. Assume that $m^*(A_n)$ is finite for all n (otherwise the result is trivial). For every $\epsilon > 0$ there is a countable collection of open intervals $\{I_{n,i}\}$ such that $A_n \subset \cup_{i=1}^{\infty} I_{n,i}$ and $\sum_{i=1}^{\infty} \ell(I_{n,i}) < m^*(A_n) + \frac{\epsilon}{2^n}$. Therefore $\cup_{n=1}^{\infty} A_n \subset \cup n, i = 1^{\infty} I_{n,i}$, and

$$m^*(\cup_{i=1}^{\infty} A_n) \leq \sum_{n,i=1}^{\infty} \ell(I_{n,i}) < \sum_{n=1}^{\infty} m^*(A_n) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \sum_{n=1}^{\infty} m^*(A_n) + \epsilon.$$

Q.E.D.

Theorem 30 *Given any subset A of the real line and any $\epsilon > 0$, there exists an open set O such that $A \subset O$ and $m^*(O) \leq m^*(A) + \epsilon$.*

PROOF: There exists a covering $\{I_n\}$ of A by open intervals such that $A \subset \cup_{n=1}^{\infty} I_n$ and $m^*(A) \geq \sum_{n=1}^{\infty} m^*(I_n) - \epsilon$. Set $O = \cup_{n=1}^{\infty} I_n$. Then $A \subset O$ and $m^*(O) \leq \sum_{n=1}^{\infty} m^*(I_n) \leq m^*(A) + \epsilon$. Q.E.D.

1.5.6 Some results on measurable sets

Note. If A is measurable then it is not difficult to show that $m^*(A) = m(A)$. However, in general A is not measurable. Recall that a set S is measurable if its characteristic function χ_S is a measurable function. Hence any interval is measurable. The measure of S is $m(S) = \int_R \chi_S(t)dt$, (which could be ∞). Every bounded measurable set has finite measure.

Lemma 20 *Let S_1, S_2 be measurable sets. Then $S_1 \cup S_2$, $S_1 \cap S_2$ and $S_1 \cup S_2 - S_1 \cap S_2$ are measurable.*

PROOF: By assumption χ_{S_1}, χ_{S_2} are measurable functions. The lemma follows from the observations that

$$\chi_{S_1 \cup S_2} = \chi_{S_1} \vee \chi_{S_2}, \quad \chi_{S_1 \cap S_2} = \chi_{S_1} \wedge \chi_{S_2}, \quad \chi_{S_1 \cup S_2 - S_1 \cap S_2} = \chi_{S_1} \vee \chi_{S_2} - \chi_{S_1} \wedge \chi_{S_2}.$$

Q.E.D.

Lemma 21 *Let S_n, $n = 1, 2, \cdots$ be measurable sets. Then $\cup_{n=1}^{\infty} S_n$ and $\cap_{n=1}^{\infty} S_n$ are measurable.*

PROOF: Let $S = \cup_{n=1}^{\infty} S_n$. Note that

$$\chi_{S_1 \cup S_2 \cup \cdots \cup S_N} = \chi_{S_1} \vee \chi_{S_2} \vee \cdots \vee \chi_{S_N}$$

is a measurable function for all N, and $\chi_{S_1 \cup S_2 \cup \cdots \cup S_N} \to \chi_S$ a.e. as $N \to \infty$. Hence S is a measurable set. Similarly, let $\tilde{S} = \cap_{n=1}^{\infty} S_n$. Note that

$$\chi_{S_1 \cap S_2 \cap \cdots \cap S_N} = \chi_{S_1} \wedge \chi_{S_2} \wedge \cdots \wedge \chi_{S_N}$$

is a measurable function for all N, and $\chi_{S_1 \cap S_2 \cap \cdots \cap S_N} \to \chi_{\tilde{S}}$ a.e. as $N \to \infty$. Hence $\tilde{S}$ is a measurable set. Q.E.D.

Theorem 31 *Every open set in R is the union of a countable number of disjoint open intervals. Thus every open set is measurable.*

PROOF: Let $\mathcal{O}$ be an open set on the real line. For each $t \in \mathcal{O}$ let $\mathcal{I}(t)$ be the maximal open interval such that $t \in \mathcal{I}(t) \subset \mathcal{O}$. Clearly $\mathcal{O} = \cup_{t \in \mathcal{O}} \mathcal{I}(t)$. However, the intervals $\mathcal{I}(t)$ aren't all distinct. The number of distinct intervals is countable, because for each $n = 1, 2, \cdots$ the bounded interval $(-n, n)$ intersects at most a finite number of distinct intervals $\mathcal{I}(t)$ of length $> 1/n$. (Note that if $\mathcal{I}(t) \cap \mathcal{I}(t') \neq \emptyset$ then $\mathcal{I}(t) = \mathcal{I}(t')$.) Q.E.D.

Note that $m(\mathcal{O}) = \sum m(\mathcal{I}(t_j)) = \sum \ell(\mathcal{I}(t_j))$ where the sum is taken over the disjoint open intervals.

Corollary 2 *Every closed set is measurable.*

PROOF: let $\mathcal{C}$ be a closed set on the real line. Then $\mathcal{O} = R - \mathcal{C}$ is open, and $\chi_{\mathcal{C}} = 1 - \chi_{calO}$. Thus $\chi_{\mathcal{C}}$ is a measurable function and $\mathcal{C}$ is a measurable set. Q.E.D.

Lemma 22 1. *If S_1, S_2 are measurable with $S_1 \subseteq S_2$, then $m(S_1) \leq m(S_2)$.*

2. *Let $S = \cup_{j=1}^{\infty} S_j$ where $S_1 \subset S_2 \subset \cdots$ is an increasing sequence of measurable sets. Then S is measurable and $m(s) = \lim_{j \to \infty} m(S_j)$.*

3. *Let $S = \cap_{j=1}^{\infty} S_j$ where $s_1 \supset S_2 \supset \cdots$ is a decreasing sequence of measurable sets. Then S is measurable and $m(s) = \lim_{j \to \infty} m(S_j)$.*

4. *Let $S = \cup_{j=1}^{\infty} S_j$ where $S_1, S_2, \cdots$ is a countable family of nonintersecting measurable sets. Then S is measurable and $m(s) = \sum_{j=1}^{\infty} m(S_j)$.*

PROOF: These results follow easily from basic results in integration theory, including the Lebesgue monotone and dominated convergence theorems.

1. $\chi_{S_1}(t) \leq \chi_{S_2}(t)$
2. $\chi_S(t) = \lim_{j\to\infty} \chi_{S_j}(t)$
3. $\chi_S(t) = \lim_{j\to\infty} \chi_{S_j}(t)$
4. $\chi_S(t) = \sum_{j=1}^{\infty} \chi_{S_j}(t)$.

Q.E.D.

Corollary 3 *For any set A on the real line, let $\sigma^*(A) = \inf\{m(B) : B \supset A,\ B \text{ measurable}\}$. Then $m^*(A) = \sigma^*(A)$.*

PROOF: If $\sigma^*(A) = \infty$ and $O \supset A$ is open, then $m(O) = \infty$ so $m^*(A) = \infty$. Now suppose $\sigma^*(A) < \infty$. It is evident that $m^*(A) \geq \sigma^*(A)$. We will show that $m^*(A) \leq \sigma^*(A)$ and the equality will follow. Given any $\epsilon > 0$ there is a measurable set $B \supset A$ such that $m(B) < \sigma^*(A) + \epsilon$. Choose $k > 1$ such that $km(B) < m(B) + \epsilon$ Then by lemma 9 there exists a sequence of step functions s_k such that

$$\sum_{k=1}^{\infty} s_k(t) = k\chi_B(t) \geq k\chi_A(t) \quad \text{a.e.},$$

and

$$\sum_{k=1}^{\infty} \int_R |s_k(\tau)| d\tau \leq \int_R k\chi_O(\tau) d\tau + \epsilon < m(B) + 2\epsilon < \sigma^*(A) + 3\epsilon.$$

By adding a sequence of nonnegative step functions with integral $< \epsilon$ and that diverges on a set of measure zero we can strengthen the above inequalities to

$$\sum_{k=1}^{\infty} s_k(t) \geq k\chi_A(t) \quad \text{everywhere},$$

and

$$\sum_{k=1}^{\infty} \int_R |s_k(\tau)| d\tau < \sigma^*(A) + 4\epsilon.$$

Now set

$$A_1 = \{t : |s_1(t)| \geq 1\}$$

$$A_n = \left\{t : \sum_{k=1}^{n-1} |s_k(t)| < 1 \leq \sum_{k=1}^{n} |s_k(t)|\right\}, \qquad n \geq 2.$$

Then each of the A_n is measurable and $A_i \cap A_j = \emptyset$ for $i \neq j$. Further,

$$\cup_{k=1}^{n} A_k = \left\{t : \sum_{k=1}^{n} |s_k(t)| \geq 1\right\},$$

so

$$\sum_{k=1}^{n} m(A_k) \leq \sum_{k=1}^{n} \int_R |s_k(t)|dt \leq \sigma^*(A) + 4\epsilon.$$

Thus

$$A \subset \cup_{k=1}^{\infty} A_k, \qquad \sum_{k=1}^{\infty} m(A_k) \leq \sigma^*(A) + 4\epsilon.$$

We can easily modify the A_k on a set of measure $< \epsilon$ to turn them into a covering of A by disjoint open intervals with total length $< \sigma^*(A)+5\epsilon$. Since ϵ is arbitrary, we have $m^*(A) \leq \sigma^*(A)$. Q.E.D.

Corollary 4 *If A is measurable then $m(A) = m^*(A)$.*

Definition 31 *A set B on the real line is a* **Borel set** *if it can be obtained by countable unions and intersections of open intervals.*

Theorem 32 *Every measurable set A can be expressed as a disjoint union $A = B \cup N$ where B is a Borel set and N is a null set.*

PROOF: Let O be an open set such that $O \supset A$. Since O is the disjoint union of a countable number of open intervals, it is easy to see that O is measurable and $m(O) = m^*(O)$. By definition, $m^*(A) = \inf_{O \text{ open},\ O \supset A} m(O)$. From theorem 30 and the preceding lemma, there exists a sequence of open sets

$$O_1 \supset O_2 \supset \cdots \supset O_n \supset \cdots \supseteq A$$

such that $m^*(A) = m(B)$ where $B = \cap_{n=1}^{\infty} O_n$ is measurable. Since A is measurable, $m(A) = m^*(A) = m(B)$. Now $B - A$ is measurable and $B = (B - A) \cup A$, a disjoint union. Hence $m(A) = m(B) = m(B - A) + m(A)$, so $m(B - A) = 0$ and $N = B - A$ is a null set. Q.E.D.

Not all sets are measurable, i.e., there are sets S such that the characteristic function χ_S is not measurable and $m(S) = \int_R \chi_S(t)dt$ has no meaning. The outer measure $m^*(S)$ still is defined but it doesn't have, in general, the additive property $m^*(S_1 \cup S_2) = m^*(S_1) + m^*(S_2)$ for disjoint sets S_1, S_1. Nonmeasurable sets and functions do not arise constructively; they will not appear in any practcal calculations because one needs the axiom of choice to show their existence. Following some preliminary lemmas, we will give a famous example.

Let f be a function on R and $a \in R$. We define the translated function f^a on R by $f^a(t) = f(t+a)$.

Lemma 23 *1. f is measurable if and only if f^a is measurable for all $a \in R$.*

2. $f \in L^1(R)$ if and only if $f^a \in L^1(R)$ and $\int_R f(t)dt = \int_R f^a(t)dt$ for all $a \in R$.

PROOF:

1. f measurable $\Longleftrightarrow$ there exists $\{s_k\} \in S(R)$ such that $\lim_{k\to\infty} s_k = f$ a.e. $\Longleftrightarrow$ there exists $\{s^a_K\} \in S(R)$ such that $\lim_{k\to\infty} s^a_k = f^a$ a.e.

2. $\int_R s_k(t)dt = \int_R s^a_k(t)dt$, for all $a \in R$, $k = 1, 2, \cdots$.

Q.E.D.

Lemma 24 *Let $S \subset R$ and $a \in R$, and define the set S^a by $S^a = \{t : t+a \in S\}$. The S is measurable if and only if S^a is measurable. χ_S is integrable if and only if χ_{S^a} is integrable. If S is measurable then $m(S) = m(S^a)$.*

PROOF: $\chi^a_S = \chi_{S^a}$. Q.E.D.

Example 2 *A nonmeasurable set. Consider the interval $[0, 2\pi)$. Think of this interval as the unit circle Γ in the complex plane. The points of Γ are $\{e^{i\theta} = \cos\theta + i\sin\theta,\ 0 \le \theta < 2\pi\}$. Set $e^{i\theta_1} \sim e^{i\theta_2}$ if $(\theta_1 - \theta_2)/2\pi$ is a rational number. Note that the relation $\sim$ divides Γ into equivalence classes. Using the axiom of choice, we select one point from each equivalence class to form a set $S \subset [0, 2\pi)$. Now let $\omega_0 = 0, \omega_1 = \frac{1}{2}(2\pi), \omega_2 = \frac{1}{3}(2\pi), \cdots$ be an enumeration of all rational multiples of 2π in the interval $[0, 2\pi)$. (Note that $\omega_j \sim 0$ for all j.) Let S_n be the set obtained by rotating S in the positive direction through the angle ω_n. Every point in S_n can be represented in the form $e^{i\omega_n}e^{i\theta_n}$ where $e^{i\theta_n}$ is a point of $S = S_0$. Then the following are true.*

- *$S_k \cap S_n = \emptyset$ if $k \neq n$. PROOF: $e^{i\omega_k}e^{i\theta_k} = e^{i\omega_n}e^{i\theta_n} \Longrightarrow e^{i(\theta_k-\theta_n)} = e^{i(\omega_n-\omega_k)} \Longrightarrow \theta_k - \theta_n$ is a rational multiple of $2\pi \Longrightarrow \theta_k = \theta_n \Longrightarrow e^{i\omega_k} = e^{i\omega_n}$. Impossible!*

- *Every point $e^{i\phi}$ of Γ belongs to some equivalence class $\Longrightarrow e^{i\phi} \in S_n$ for some n.*

Thus

$$\Gamma = S_1 \cup S_2 \cup \cdots, \qquad S_j \cap S_k = \emptyset \text{ if } j \neq k.$$

Now suppose S is measurable. Then $S \subset [0, 2\pi) \Longrightarrow m(S) \leq 2\pi$, and $m(S_j) = m(S)$ for $j = 0, 1, 2, \cdots$. Further,

$$2\pi = m(\Gamma) = \sum_{j=0}^{\infty} m(S_j) = \sum_{j=0}^{\infty} m(S).$$

If $m(S) = 0$ we find $0 = 2\pi$. Impossible! If $m(S) > 0$ we find $+\infty = 2\pi$. Impossible! Thus, S is not measurable and χ_S is not a measurable function.

1.5.7 Derivatives of monotone functions

Let $\mathcal{I}$ be a collection of closed intervals I on the real line.

Definition 32 *$\mathcal{I}$ covers a set E* **in the sense of Vitali** *if for each $\epsilon > 0$ and $t \in E$ there is an interval $I \in \mathcal{I}$ such that $t \in I$ and $\ell(I) < \epsilon$.*

Theorem 33 *Vitali covering theorem. Let E be a bounded subset of the real line and $\mathcal{I}$ a collection of closed intervals that cover E in the sense of Vitali. Then given $\epsilon > 0$ there is a finite disjoint collection $\{I_1, \cdots, I_N\} \subset \mathcal{I}$ such that $m^*(E - \cup_{N=1}^{N} I_n) < \epsilon$. Thus, there is a sequence of pairwise disjoint intervals I_n in $\mathcal{I}$ such that $E - \cup_{n=1}^{\infty} I_n$ is a null set.*

SKETCH OF PROOF: The (technical) proof can be accomplished in the following steps.

1. Since E is bounded, it is contained in a bounded interval (a, b). Then the subfamily $\mathcal{I}'$ of $\mathcal{I}$ consisting of those intervals in $\mathcal{I}$ that are contained in (a, b) also covers E in the sense of Vitali.

2. Unless E is covered by a *finite* subfamily of pairwise disjoint intervals in $\mathcal{I}'$, there exists a sequence $\{I_n\}$ of pairwise disjoint intervals in $\mathcal{I}'$ such that

$$\ell(I_{n+1}) > \frac{1}{2} \sup\{\ell(I) : I \in \mathcal{I}', I \cap \left(\cup_{j=1}^{n} I_j\right) = \emptyset\}.$$

Indeed, for each n, unless E is contained in the closed set $\cup_{j=1}^{n} I_j$, the family

$$\{I : I \in \mathcal{I}', I \cap \left(\cup_{j=1}^{n} I_j\right) = \emptyset\}$$

isn't empty.

3. Since $\sum_{n=1}^{\infty} \ell(I_n) \leq b - a$, we have $\lim_{n\to\infty} \ell(I_n) = 0$.

4. Suppose $t_0 \in E$ and t_0 belongs to some $I \in \mathcal{I}'$. If $I \cap I_j = \emptyset$ for all $j < n$ and $n > 1$ then $\ell(I) \leq 2\ell(I_n)$. This is impossible, so $I \cap I_{n_0} \neq \emptyset$ for some smallest $n_0 > 1$.

5. Let J_{n_0} be the closed interval with the same midpoint as I_{n_0} and $\ell(J_{n_0}) = 5\ell(I_{n_0})$. Then since $\ell(I) \leq 2\ell(I_{n_0})$ we must have that t_0 is at a distance of $\leq 2.5\ell(I_{n_0})$ from the midpoint of I_{n_0}. Hence $t_0 \in J_{n_0}$.

6. For each integer $N \geq 2$ and each point $t \in E$, either $t \in \cup_{n=1}^{N} I_n$ or $t \in J_n$ for some $n > N$. Hence

$$E \subset \left(\cup_{n=1}^{N} I_n\right) \cup \left(\cup_{n=N+1}^{\infty} J_n\right),$$

where

$$\sum_{n=N+1}^{\infty} \ell(J_n) = 5 \sum_{n=N+1}^{\infty} \ell(I_n) \leq 5(b - a).$$

7. Given $\epsilon > 0$ there is an integer N_ϵ so that $\sum_{n=N+1}^{\infty} \ell(J_n) < \epsilon$ for $N \geq N_\epsilon$. Hence

$$m^*(E - \cup_{n=1}^{N} I_n) \leq m^*(\cup_{n=N+1}^{\infty} J_n) \leq \sum_{n=N+1}^{\infty} \ell(J_n) < \epsilon$$

for all $N \geq N_\epsilon$.

Q.E.D.

Suppose F is defined on an interval of the real line that contains the point 0.

Definition 33

$$\overline{\lim}_{h\to 0+}F(h) = \inf_{\delta>0}\sup_{0<h<\delta} F(h), \qquad \underline{\lim}_{h\to 0+}F(h) = \sup_{\delta>0}\inf_{0<h<\delta} F(h).$$

Note that $\overline{\lim}_{h\to 0+}F(h) \geq \underline{\lim}_{h\to 0+}F(h)$ and we have equality if and only if $\lim_{h\to 0+} F(h)$ exists, in which case the three limits are the same.

Definition 34 *Let f be defined in an open interval of the real line containing t. The four* **derivates** *of f at t are given by*

$$D^+f(t) = \overline{\lim}_{h\to 0+}\frac{f(t+h)-f(t)}{h}, \quad D^-f(t) = \overline{\lim}_{h\to 0+}\frac{f(t)-f(t-h)}{h},$$

$$D_+f(t) = \underline{\lim}_{h\to 0+}\frac{f(t+h)-f(t)}{h}, \quad D_-f(t) = \underline{\lim}_{h\to 0+}\frac{f(t)-f(t-h)}{h},$$

Clearly, $D^+f(t) \geq D_+f(t)$ and $D^-f(t) \geq D_-f(t)$. If $D^+f(t) = D_+f(t) = D^-f(t) = D_-f(t) \neq \pm\infty$, we say that f is *differentiable at t* and we define $f'(t)$ to be the common value of the derivates at t.

Theorem 34 *Let f be a monotone real-valued fuction on the interval $[a, b]$. Then f' exists a.e. Furthermore, f' is Lebesgue integrable and $\int_a^b f'(t)dt \leq f(b)-f(a)$ (for f increasing) and $\int_a^b f'(t)dt \geq f(b)-f(a)$ (for f decreasing).*

PROOF: Assume that f is monotone increasing. (The results for $\tilde{f}$ monotone decreasing can be obtained by setting $\tilde{f} = -f$.) We will show that the set where any two derivates are unequal is of measure 0. Consider the set E where $D^+f(t) > D_-f(t)$, for example. We can write

$$E = \cup_{u,v \text{ rational}}E_{u,v}, \qquad E_{u,v} = \{t : D^+f(t) > u > v > D_-f(t)\}.$$

We will show that $E_{u,v}$ is a null set for all u, v, which will imply that E is a null set. Let $s = m^*(E_{u,v})$. We must show that $s = 0$.

Given $\epsilon > 0$ we can choose an open set O such that $O \supset E_{u,v}$ and $m^*(O) < s+\epsilon$. For each $t \in E_{u,v}$ there exists an arbitrarily small interval $[t-h, t] \subset O$ such that $f(t)-f(t-h) < vh$. By the Vitali covering theorem we can choose a finite collection $\{I_1, \cdots, I_N\}$ of these intervals that are pairwise disjoint and such that $m^*(A) > s-\epsilon$ where A is a subset of $E_{u,v}$ contained in the interior of $\cup_{k=1}^N I_k$.

$$\Longrightarrow \sum_{n=1}^{N}[f(t)n) - f(t_n - h_n)] < v\sum_{n=1}^{N} h_n < vm^*(O) < v(s+\epsilon).$$

Now for each point $w \in A$ there exist arbitrarily small intervals $[w, w+k]$ contained in some I_n and such that $f(w+k) - f(w) > uk \Longrightarrow$ there exists a finite collection $\{J_1, \cdots, J_M\}$ of such intervals with the property that $\cup_{n=1}^{M} J_n$ contains a subset A of outer measure greater than $s - 2\epsilon$. Thus,

$$\sum_{i=1}^{M}[f(w_i + k + i) - f(w_i)] > u \sum_{i=1}^{M} k_i > u(s - 2\epsilon).$$

For fixed n sum over those J_i contained in I_n:

$$\sum_{i}[f(w_i + k_i) - f(w_i)] \leq f(t_n) - f(t_n - h_n).$$

(Here we are using the assumption that f is increasing.) Therefore

$$v(s+\epsilon) > \sum_{n=1}^{N}[f(t_n) - f(t_n - h_n)] \geq \sum_{i=1}^{M}[f(w_i + k_i) - f(w_i)] > u(s - 2\epsilon)$$

$\Longrightarrow vs \geq us$. But $u > v \Longrightarrow s = 0$.

For the last part of the theorem, set $g(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$. Note that g is defined a.e. Let $g_n(t) = n[f(t+1/n) - f(t)]$ for $t \in [a, b]$ and note that $g_n \geq 0$. (We set $f(t) = f(b)$ for $t \geq b$.) Then $g_n(t)$ is a measurable function and $\lim_{n\to\infty} g_n(t)$ exists a.e. (but may be $\pm\infty$). By Fatou's lemma,

$$\int_a^b f'(t)dt = \int_a^b \underline{\lim}_{n\to\infty} g_n(t)dt \leq \underline{\lim}_{n\to\infty} \int_a^b g_n(t)dt$$

$$= \underline{\lim}_{n\to\infty}[n \int_a^b [f(t+\frac{1}{n}) - f(t)]dt = \underline{\lim}_{n\to\infty}[n \int_b^{b+1/n} f(t)dt - n \int_a^{a+1/n} f(t)dt$$

$$= \underline{\lim}_{n\to\infty}[f(b) - n \int_a^{a+1/n} f(t)dt] \leq f(b) - f(a) < \infty.$$

$\Longrightarrow f'(t)$ is integrable $\Longrightarrow f'$ is finite a.e. Therefore, f is differentiable a.e. and $g = f'$ a.e. Q.E.D.

Note: the inequality $\int_a^b f'(t)dt \leq f(b) - f(a)$ can't be improved.

Example 3 *The almost perfect sneak. This function is related to the Cantor set C on the interval $[0, 1]$. Recall that the Cantor set is constructed, recursively, by throwing out the (open) middle third of each interval. We construct the function f on $[0, 1]$ by first defining it on the maximal intervals of complement of the Cantor set $C' = [0, 1] - C$. The maximal intervals in C' are of* *length* $1/3, 1/9, \cdots, 1/3^n, \cdots$. *We define first* $f(t) = 1/2$ *for* $t \in (1/3, 2/3)$, *secondly* $f(t) = 1/4$ *for* $t \in (1/9, 2/9)$, $f(t) = 3/4$ *for* $t \in (7/9, 8/9)$, *etc. At step* k *we define*

$$f(t) = \frac{1}{2^k}, \cdots, \frac{2^k - 1}{2^k}$$

for t *in the respective intervals of length* $1/3^k$ *in* C'*:*

$$(\frac{1}{3^k}, \frac{2}{3^k}), \cdots, (\frac{3^k - 2}{3^k}, \frac{3^k - 1}{3^k}).$$

This defines f *as a monotone increasing function on* C'. *By construction,* C' *is dense in* C. *Thus we can define* f *on* $[0, 1]$ *by the requirement*

$$f(t) = \begin{cases} \sup_{t'<t, t' \in C'} f(t') & 0 < t \le 1 \\ 0 & t = 0. \end{cases}$$

Then f *is monotone increasing on* $[0, 1]$, *with* $f(0) = 0$ *and* $f(1) = 1$. *The range of* f *includes all values* $j/2^k$ *where* $j = 0, 1, \cdots 2^k$ *and* k *ranges over the non-negative integers. Thus the range is dense in* $[0, 1]$. *This means that* f *is continuous because it has no jump discontinuities. Since* C' *is a union of disjoint open intervals and* f *is constant on each of these intervals, it follows that* $f'(t) = 0$ *for all* $t \in C'$. *Since* C *is a null set, this means that* $f'(t) = 0$ *a.e. We see that*

$$0 = \int_0^1 f'(t)dt < f(1) - f(0) = 1,$$

so the fundamental theorem of calculus doesn't apply to f. *Note that a "sneak" moving according to the formula* $x = f(t)$ *is "almost perfect." Indeed he gets from point* 0 *to point* 1 *in one time interval but almost all of the time he isn't moving!*

1.5.8 Functions of bounded variation

Let f be a real-valued function on the closed, bounded interval $[a, b]$, and let $a = t_0 < t_1 < \cdots < t_k = b$ be a partition of $[a, b]$. For a real number r let

$$r^+ = \begin{cases} r & \text{if } r \ge 0 \\ 0 & \text{if } r \le 0 \end{cases} \qquad r^- = \begin{cases} 0 & \text{if } r \ge 0 \\ -r & \text{if } r \le 0. \end{cases}$$

We define

$$p = \sum_{i=1}^{k} [f(t_i) - f(t_{i-1})]^+, \qquad n = \sum_{i=1}^{k} [f(t_i) - f(t_{i-1})]^-.$$

Note that

$$t = p + n = \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})|, \qquad p - n = f(b) - f(a).$$

Now we set

$$P_a^b = \sup p, \qquad N_a^b = \sup n, \qquad T_a^b = \sup t,$$

where the suprema are taken over all partitions of $[a,b]$. Here P_a^b, N_a^b, T_a^b are called the **positive variation**, the *negative variation* and the **total variation**, respectively, of f on $[a,b]$. Clearly, $T_a^b \geq P_a^b, N_a^b$. If $T_a^b < \infty$ we say that f is of **bounded variation** over $[a,b]$, or $f \in BV$ for short.

Lemma 25 *If $f \in BV$ on $[a,b]$ then*

$$T_a^b = P_a^b + N_a^b \quad \text{and} \quad f(b) - f(a) = P_a^b - N_a^b.$$

PROOF: For any partition of $[a,b]$ we have $p = n + f(b) - f(a) \leq N + f(b) - f(a)$, so $P \leq N + f(b) - f(a)$. Similarly, $P \geq n + f(b) - f(a)$ so $P \geq N + f(b) - f(a)$. Hence, $P - N = f(b) - f(a)$.

Next, $t = n + p \leq N + p \leq N + P$, so $T \leq N + P$. However, $T \geq p + n = 2p - (f(b) - f(a))] = 2p + N - P$, so $T \geq P + N$. Therefore $T = P + N$. Q.E.D.

Theorem 35 *$f \in BV$ on $[a,b]$ if and only if $f = g - h$ where g, h are monotone increasing functions on $[a,b]$.*

PROOF: Set $g(t) = P_a^t$, $h(t) = N_a^t - f(a)$. Then g, h are monotone increasing on $[a,b]$ and

$$f(t) = P_a^t - N_a^t + f(a) = g(t) - h(t).$$

Conversely, suppose $f = g - h$, where g, h are monotone increasing on $[a,b]$, and let $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ be a partion. Then

$$\sum_i |f(t_i) - f(t_{i-1})| \leq \sum_i [g(t_i) - g(t_{i-1})] + \sum_i [h(t_i) - h(t_{i-1})]$$

$$< [g(b) - g(a)] + [h(b) - h(a)],$$

so $T_a^b(f) \leq g(b) - g(a) + h(b) - h(a) < \infty$ and $f \in BV$. Q.E.D.

Corollary 5 *$f \in BV$ on $[a,b] \Longrightarrow f'$ exists a.e.*

Corollary 6 *If $F(t) = \int_a^t f(\tau)d\tau + F(a)$ for $f \in L^1([a,b])$ then $F \in BV$.*

PROOF: For any partition of $[a,b]$ we have

$$t = \sum_{i=1}^{k} \left|\int_{t_{i-1}}^{t_i} f(\tau)d\tau\right| \le \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} |f(\tau)|d\tau = \int_a^b |f(\tau)d\tau,$$

so $T_a^b \le \int_a^b |f(\tau)|d\tau$. Q.E.D.

Note: With more care, it can be shown that $T_a^b = \int_a^b |f(\tau)|d\tau$.

Example 4 *A continuous function on a bounded domain that is not of bounded variation. Consider the function $y = f(x)$ on $[0,1]$ whose graph zigzags between the lines $y = x$ and $y = 0$. In particular, $f(x) = x$ for $x = 1, 1/3, 1/5, \cdots, 1/(2k+1), \cdots$ and $f(x) = 0$ for $x = 0$ and $x = 1/2, 1/4, \cdots, 1/2k, \cdots$ and the graph zigzags between these points. Now let P_n be the partition*

$$0 < \frac{1}{2n} < \frac{1}{2n-1} < \cdots < \frac{1}{2} < 1.$$

Then the variation for this partition is

$$t_n = \frac{2}{2n-1} + \frac{2}{2n-3} + \cdots + \frac{2}{3} + 1 > 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$> \int_1^n \frac{1}{n} dx = \log n$$

which is unbounded as $n \to \infty$.

1.5.9 Absolutely continuous functions and the fundamental theorem of calculus

Theorem 36 *Suppose $f(t) = \sum_{n=1}^{\infty} f_n(t)$ converges everywhere on the interval $[a,b]$ and that each f_n is monotone increasing on $[a,b]$. Then $f'(t) = \sum_{n=1}^{\infty} f_n'(t)$ a.e. in $[a,b]$.*

PROOF: f monotone increasing $\Longrightarrow$ f' exists a.e. Without loss of generality we can assume that $f(a) = f_n(a) = 0$ for all n, so

$$f_n(t) \ge 0 \text{ for } t \ge a, \qquad f_n' \ge 0, f' \ge 0.$$

Set $s_k(t) = \sum_{n=1}^{k} f_n(t)$. Then s_k is monotone increasing and $s_k \to f$ everywhere as $k \to \infty$. Now

$$s_1'(t) \leq s_2'(t) \leq \cdots \leq s_k'(t) \leq \cdots \leq f'(t),$$

except on a set of measure zero. Therefore $\lim_{k\to\infty} s_k'(t) = \sum_{n=1}^{\infty} f_n'(t)$ converges to a finite number a.e. Now choose the subsequence $\{s'_{k(\ell)}\}$ by requiring $f(b) - s'_{k(\ell)} < 1/2^\ell$, for $\ell = 1, 2, \cdots$. Then

$$0 < f(t) - s_{k(\ell)}(t) \leq f(b) - s_{k(\ell)}(b) < \frac{1}{2^\ell}$$

for all $t \in [a, b]$, because $f - s_{k(\ell)}$ is monotone increasing. From these results we see that

$$0 \leq \sum_{\ell=1}^{\infty} [f(t) - s_{k(\ell)}(t)] < \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} = 1,$$

for all $t \in [a, b]$, $(f(a) - s_{k(\ell)}(a) = 0)$. $\Longrightarrow \sum_{\ell=1}^{\infty} (f - s_{k(\ell)})'$ converges to a finite number a.e. $\Longrightarrow f'(t) - s'_{k(\ell)}(t) \to 0$ a.e. as $\ell \to \infty$. Q.E.D.

Theorem 37 *Let f be Lebesgue integrable on $[a, b]$ as set $F(t) = \int_a^t f(\tau)d\tau + c$ where $a \leq t \leq b$ and c is a constant. Then F' exists a.e. and $F'(t) = f(t)$ a.e. on $[a, b]$.*

PROOF: Without loss of generality we can assume $c = 0$. We will carry out the proof in stages.

- The theorem is true if $f = \chi_{[A,B]}$ is a characteristic function, where $a \leq A < B \leq b$. Indeed

  $$F(t) = \int_a^t \chi_{[A,B]}(\tau)d\tau = \begin{cases} 0 & a \leq t \leq A \\ t - A & A < t \leq B \\ B - A & B < t \leq b \end{cases}$$

  so $F'(t) = \chi_{[A,B]}(t)$, a.e. on $[a, b]$.

- The theorem is true if $f \in S([a, b])$, i.e., if f is a step function on $[a, b]$, since then it is a finite sum $f(t) = \sum_{i=1}^{n} c_i \chi_{[A_i,B_i]}(t)$.

- Finally, we show that the theorem is true for general $f \in L^1([a,b])$. In this case, by lemma 8, there exists a sequence $\{\phi_k\}$ in $S([a,b])$ such that $\sum_{k=1}^{\infty} \int_R |\phi_k(\tau)d\tau < \infty$, and $\sum_k \phi_k(t) = f(t)$ a.e. (Here, we assume that $\phi_k(t) = 0$ for $t \notin [a,b]$.) Setting

  $$\phi_k^+(t) = \begin{cases} \phi_k(t) & \phi_k(t) \geq 0 \\ 0 & \phi_k(t) < 0, \end{cases} \qquad \phi_k^-(t) = \begin{cases} -\phi_k(t) & \phi_k(t) \leq 0 \\ 0 & \phi_k(t) > 0, \end{cases}$$

  we see that $\phi_k(t) = \phi_k^+(t) - \phi_k^-(t)$. Clearly,

  $$\int_R \phi_k^+(\tau)d\tau \leq \int_R |\phi_k(\tau)|d\tau, \qquad \int_R \phi_k^-(\tau)d\tau \leq \int_R |\phi_k(\tau)|d\tau,$$

  so $\sum_{k=1}^{\infty} \int_R \phi_k^+(\tau)d\tau < \infty$, $\sum_{k=1}^{\infty} \int_R \phi_k^-(\tau)d\tau < \infty$. Set $f_1(t) = \sum_{k=1}^{\infty} \phi_k^+(t)$, a.e., and $f_2(t) = \sum_{k=1}^{\infty} \phi_k^-(t)$, a.e. Then f_1, f_2 are integrable and $f = f_1 - f_2$ a.e. Set

  $$F_1(t) = \int_a^t f_1(\tau)d\tau = \sum_{k=1}^{\infty} \int_a^t \phi_k^+(\tau)d\tau,$$

  $$F_2(t) = \int_a^t f_2(\tau)d\tau = \sum_{k=1}^{\infty} \int_a^t \phi_k^-(\tau)d\tau,$$

  for $a \leq t \leq b$. Since the infinite sums are monotone increasing, we see that F_1, F_2 are monotone increasing. Then by theorem 36 it follows that F_1', F_2' exist a.e. and

  $$F_1'(t) = \sum_{k=1}^{\infty} \phi_k^+(t), \qquad F_2'(t) = \sum_{k=1}^{\infty} \phi_k^-(t),$$

  these last two identities holding a.e. Thus,

  $$F'(t) = F_1'(t) - F_2'(t) = \sum_{k=1}^{\infty} (\phi_k^+(t) - \phi_k^-(t)) = \sum_{k=1}^{\infty} \phi_k(t) = f(t), \text{ a.e.}$$

Q.E.D.

Corollary 7 *Suppose f is integrable on $[a,b]$ and $\int_a^t f(\tau)d\tau = 0$ for all $t \in [a,b]$. Then $f(t) = 0$ a.e. on $[a,b]$.*

Theorem 38 *Let $f \in L^1(R)$. Given $\epsilon > 0$ there exists a $\delta > 0$ such that if the set S is measurable and $m(S) < \delta$ then $|\int_S f(\tau)d\tau| < \epsilon$.*

PROOF:

1. Suppose f is bounded, i.e., there exists a positive $M < \infty$ such that $|f| < M$ everywhere. Set $\delta = \epsilon/2M$. Then if $m(S) < \delta$ we have

$$|\int_S f(\tau)d\tau| \leq \int_S |f(\tau)|d\tau \leq \int_S M\ d\tau = Mm(S) = \frac{\epsilon M}{2M} < \epsilon.$$

2. Suppose $f \in L^1(R)$ but f is not bounded. Then by the approximation theorem (theorem 18) there exists a $\phi \in S(R)$ such that $\int_R |f(\tau) - \phi(\tau)|d\tau < \epsilon/2$. Since ϕ is bounded, there is a finite M such that $|\phi| < M$. Set $\delta = \epsilon/2M$. Then if $m(S) < \delta$ we have

$$|\int_S f(\tau)d\tau| = |\int_S [(f-\phi)+\phi]d\tau| \leq \int_S |f-\phi|d\tau + \int_S |\phi|d\tau < \frac{\epsilon}{2} + Mm(S)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Q.E.D.

Definition 35 *We say that the function f is* **absolutely continuous** *on the interval $[a, b]$ if, f is defined on $[a, b]$ and given any $\epsilon > 0$ there is a $\delta > 0$ such that $\sum_{i=1}^n |f(t'_i) - f(t_i)| < \epsilon$ for every disjoint collection $\{(t_i, t'_i)\}$ of intervals such that $\sum_{i=1}^n |t' + i - t_i| < \delta$.*

Lemma 26 *If f is absolutely continuous on $[a, b]$ then f is continuous on $[a, b]$.*

Lemma 27 *If f' exists everywhere on $[a, b]$ and is bounded, then f is absolutely continuous on $[a, b]$.*

Lemma 28 *Suppose f, g are absolutely continuous on $[a, b]$ and α, β are real scalars. Then $\alpha f + \beta g$ is absolutely continuous on $[a, b]$.*

Lemma 29 *If f is absolutely continuous on $[a, b]$ then $f \in BV$ on $[a, b]$.*

PROOF: Choose the δ corresponding to $\epsilon = 1$ in the definition of absolute continuity. Now let Δ be a partition of $[a, b]$. By adding more partition points if necessary, we can split the partition into K sets of intervals, each of total length $< \delta$, where K is the largest integer $< 1 + (b - a)/\delta$. $\Longrightarrow t \leq K$ $\Longrightarrow T \leq K$. Q.E.D.

Corollary 8 *f absolutely continuous on $[a,b] \Longrightarrow f'$ exists a.e. on $[a,b]$ and is integrable.*

Lemma 30 *If $f \in L^1([a,b])$ and $F(t) = \int_a^t f(\tau)d\tau + F(a)$ for $a \leq t \leq b$, then F is absolutely continuous on $[a,b]$.*

PROOF: let $\{(t_i, t_i')\}$ be n disjoint intervals in $[a,b]$. Then

$$\sum_{i=1}^{n} |f(t_i') - F(t_i)| = \sum_{i=1}^{n} |\int_{t_i}^{t_i'} f(\tau)d\tau|$$

$$\leq \sum_{i=1}^{n} \int_{t_i}^{t_i'} |f(\tau)|\, d\tau = \int_s |f(\tau)|\, d\tau,$$

where $S = \cup_{i=1}^{n}(t_i, t_i')$. Recall that given $\epsilon > 0$ there exists $\delta > 0$ such that $m(S) < \delta \Longrightarrow \int_S |f(\tau)|\, d\tau < \epsilon$. Q.E.D.

Lemma 31 *Suppose f is absolutely continuous on $[a,b]$ and $f'(t) = 0$ a.e. Then f is constant on $[a,b]$.*

PROOF: let $\epsilon, \delta_\epsilon$ be as in the definition of absolute continuity for f. Choose $c \in (a,b]$ and let $E = \{t \in (a,c) : f'(t) = 0\}$. Since $f' = 0$ a.e., we have $m(E) = c - a$. Now choose $\epsilon > 0, \eta > 0$. For each $t \in E$ there exists an arbitrarily small interval $[t, t+h] \subset (a,c)$ such that $|f(t+h) - f(t)| < \eta h$, $\Longrightarrow$ By the Vitali covering theorem we can find a finite number n of disjoint intervals $\{I_k = [t_k, t_k + h_k] = [t_k, \tau_k]\}$ such that $m(E - \cup_{k=1}^{n} I_k) < \delta_\epsilon$. Label the t_k so that $t_k \leq t_{k+1}$. Then defining $\tau_0 = a, t_{n+1} = c$ we have

$$\tau_0 < t_1 < \tau_1 < t_2 < \tau_2 < \cdots < t_n < \tau_n < \tau_{n+1}$$

and $\sum_{k=0}^{n} |t_{k+1} - \tau_k| < \delta$. Now

$$\sum_{k=1}^{n} |f(\tau_k) - f(t_k)| \leq \eta \sum_{k=1}^{n} (\tau_k - t_k) < \eta(c - a)$$

and $\sum_{k=0}^{n} |f(t_{k+1}) - f(\tau_k)| < \epsilon$ by the absolute continuity of f. Therefore,

$$|f(c) - f(a)| = |\sum_{k=0}^{n} [f(t_{k+1}) - f(\tau_k)]$$

$$+ \sum_{k=1}^{n} [f(\tau_k) - f(t_k)]| \leq \epsilon + \eta(c - a).$$

Since ϵ, η are arbitrary, $\Longrightarrow f(c) = f(a)$. But $c \in (a,b]$ is arbitrary, so f is constant on $[a,b]$. Q.E.D.

Definition 36 *If* $F(t) = \int_a^t f(\tau)d\tau + F(a)$ *for some* f *integrable on* $[a, b]$, *we say that* F *is an* **indefinite integral** *on* $[a, b]$.

Theorem 39 *F is an indefinite integral on $[a, b]$ if and only if F is absolutely continuous on $[a, b]$.*

PROOF: If F is an indefinite integral then F is absolutely continuous by lemma 30. Conversly, if F is absolutely continuous on $[a, b]$ then, by lemma 29, $F \in BV$ on $[a, b] \Longrightarrow F(t) = F_1(t) - F_2(t)$ where F_1, F_2 are monotone increasing, $\Longrightarrow F'(t) = F_1'(t) - F_2'(t)$ a.e., where F_1', F_2' are integrable. Therefore F' is integrable on $[a, b]$. Now set $G(t) = \int_a^t F'(\tau)d\tau$, $a \leq t \leq b$. Then $H(t) \equiv F(t) - G(t)$ is absolutely continuous and $H'(t) = F'(t) - F'(t) = 0$ a.e. $\Longrightarrow H(t)$ is constant on $[a, b] \Longrightarrow H(t) = H(a) = F(a)$. Therefore,

$$F(t) - \int_a^t F'(\tau)d\tau = F(a).$$

Q.E.D.

Corollary 9 *An absolutely continuous function is the indefinite integral of its derivative.*

Corollary 10 *(Fundamental theorem of calculus) The following are equivalent for a function F on the interval $[a, b]$:*

1. *F is absolutely continuous.*
2. *$F \in BV$ and $F(t) = \int_a^t F'(\tau)d\tau + F(a)$, $a \leq t \leq b$.*
3. *There exists a function $f \in L^1([a, b])$ such that $F(t) = \int_a^t f(\tau)d\tau + F(a)$.*

Lemma 32 *If $F(t), G(t)$ are absolutely continuous on $[a, b]$ then*

PROOF: Given $\epsilon > 0$ choose $\delta > 0$ such that for any collection of n disjoint intervals $\{(t_i, t_i')\}$ in $[a, b]$, satisfying $\sum_{i=1}^n (t_i' - t_i) < \delta$ we have

$$\sum_{i=1}^n |F(t_i') - F(t_i)| < \epsilon, \qquad \sum_{i=1}^n |G(t_i') - G(t_i)| < \epsilon.$$

Let $M = \max_{t\in[a,b]}\{|F(t)|, |G(t)|\}$. Then

$$\sum_{i=1}^{n} |F(t_i')G(t_i') - F(t_i)G(t_i)| = \sum_{i=1}^{n} |F(t_i')[G(t_i') - G(t_i)] + G(t_i)[F(t_i') - F(t_i)]|$$

$$\leq M\sum_{i=1}^{n} |G(t_i') - G(t_i)| + M\sum_{i=1}^{n} |F(t_i') - F(t_i)| < 2M\epsilon.$$

Q.E.D.

Theorem 40 *(Integration by parts) Suppose F is absolutely continuous and g is integrable on $[a,b]$. Set $G(t) = \int_a^t g(\tau)d\tau + c$, ($c$ a constant) and $f(t) = F'(t)$ a.e. Then Fg and fG are integrable on $[a,b]$ and*

$$\int_a^b F(\tau)g(\tau)d\tau = F(b)G(b) - F(a)G(a) - \int_a^b f(\tau)G(\tau)d\tau.$$

PROOF: F is measurable and bounded, g integrable $\Longrightarrow$ Fg is integrable. G is measurable and bounded, f integrable $\Longrightarrow$ fG is integrable. F, G absolutely continuous $\Longrightarrow$ FG absolutely continuous and $(f(t)G(t))' = F(t)G'(t) + F'(t)G(t)$ if F and G are differentiable at t $\Longrightarrow$ $(FG)' = Fg + fG$ a.e. $\Longrightarrow$ $F(b)G(b) = \int_a^b (Fg + fG)d\tau + F(a)G(a)$ $\Longrightarrow$

$$\int_a^b F(\tau)g(\tau)d\tau = F(b)G(b) - F(a)G(a) - \int_a^b f(\tau)G(\tau)d\tau.$$

Q.E.D.

1.6 Orthogonal projections, Gram-Schmidt orthogonalization

1.6.1 Orthogonality, Orthonormal bases

Definition 37 *Two vectors u, v in an inner product space $\mathcal{H}$ are called* **orthogonal**, *$u \perp v$, if $(u,v) = 0$. Similarly, two sets $\mathcal{M}, \mathcal{N} \subset \mathcal{H}$ are* **orthogonal**, *$\mathcal{M} \perp \mathcal{N}$, if $(u,v) = 0$ for all $u \in \mathcal{M}$, $v \in \mathcal{N}$.*

Definition 38 *Let $\mathcal{S}$ be a nonempty subset of the inner product space $\mathcal{H}$. We define $\mathcal{S}^\perp$ by $\mathcal{S}^\perp = \{u \in \mathcal{H} : u \perp \mathcal{S}\}$*

Lemma 33 *$\mathcal{S}^\perp$ is a closed subspace of $\mathcal{H}$.*

PROOF:

1. $\mathcal{S}^\perp$ is a subspace. Let $u, v \in \mathcal{S}^\perp$, $\alpha, \beta \in C$, Then $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w) = 0$ for all $w \in \mathcal{S}$, so $\alpha u + \beta v \in \mathcal{S}^\perp$.

2. $\mathcal{S}^\perp$ is closed. Suppose $\{u_n\} \subset \mathcal{S}^\perp$, $\lim_{n\to\infty} u_n = u \in \mathcal{H}$. Then $(u, v) = (\lim_{n\to\infty} u_n, v) = \lim_{n\to\infty}(u_n, v) = 0$ for all $v \in \mathcal{S} \Longrightarrow u \in \mathcal{S}^\perp$. Q.E.D.

1.6.2 Orthonormal bases for finite-dimensional inner product spaces

Let $\mathcal{H}$ be an n-dimensional inner product space, (say $\mathcal{H}_n$). A vector $u \in \mathcal{H}$ is a **unit vector** if $||u|| = 1$. The elements of a finite subset $\{u_1, \cdots, u_k\} \subset \mathcal{H}$ are **mutually orthogonal** if $u_i \perp u_j$ for $i \neq j$. The finite subset $\{u_1, \cdots, u_k\} \subset \mathcal{H}$ is **orthonormal** (ON) if $u_i \perp u_j$ for $i \neq j$, and $||u_i|| = 1$. Orthonormal bases for $\mathcal{H}$ are especially convenient because the expansion coefficients of any vector in terms of the basis can be calculated easily from the inner product.

Theorem 41 *Let $\{u_1, \cdots, u_n\}$ be an ON basis for $\mathcal{H}$. If $u \in \mathcal{H}$ then*

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

where $\alpha_i = (u, u_i)$, $i = 1, \cdots, n$.

PROOF: $(u, u_i) = (\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n, u_i) = \alpha_1$. Q.E.D.

Example 5 *Consider $\mathcal{H}_3$. The set $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ is an ON basis. The set $u_1 = (1, 0, 0), u_2 = (1, 1, 0), u_3 = (1, 1, 1)$ is a basis, but not ON. The set $v_1 = (1, 0, 0), v_2 = (0, 2, 0), v_3 = (0, 0, 3)$ is an orthogonal basis, but not ON.*

The following are very familiar results from geometry, where the inner product is the dot product, but apply generally:

Corollary 11 *For $u, v \in \mathcal{H}$:*

- $(u, v) = (\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n, \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n) = \sum_{i=1}^{n}(u, u_i)(u_i, v)$

- $||u||^2 = \sum_{i=1}^n |(u, u_i)|^2$ *Parseval's equality.*

Lemma 34 *If $u \perp v$ then $||u + v||^2 = ||u||^2 + ||v||^2$ Pythagorean Theorem*

Lemma 35 *If u, v belong to the* **real** *inner product space $\mathcal{H}$ then $||u+v||^2 = ||u||^2 + ||v||^2 + 2(u, v)$. Law of Cosines.*

Does every n-dimensional inner product space have an ON basis? Yes!

Recall that $[u_1, u_2, \cdots, u_m]$ is the subspace of $\mathcal{H}$ spanned by all linear compinations of the vectors $u_1, u_2, \cdots, u_m$.

Theorem 42 *(Gram-Schmidt) let $\{u_1, u_2, \cdots, u_n\}$ be an (ordered) basis for the inner product space $\mathcal{H}$. There exists an ON basis $\{e_1, e_2, \cdots, e_n\}$ for $\mathcal{H}$ such that*

$$[u_1, u_2, \cdots, u_m] = [e_1, e_2, \cdots, e_m]$$

for each $m = 1, 2, \cdots, n$.

PROOF: Define e_1 by $e_i = u_1/||u_1||$ This implies $||e_1|| = 1$ and $[u_1] = [e_1]$. Now set $f_2 = u_2 - \alpha e_1 \neq \Theta$. We determine the constant α by requiring that $(f_2, e_1) = 0$ But $(f_2, e_1) = (u_2, e_1) - \alpha$ so $\alpha = (u_2, e_1)$. Now define e_2 by $e_2 = f_2/||f_2||$. At this point we have $(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq 2$ and $[u_1, u_2] = [e_1, e_2]$.

We proceed by induction. Assume we have constructed an ON set $\{e_1, \cdots, e_m\}$ such that $[e_1, \cdots, e_k] = [u_1, \cdots, u_k]$ for $k = 1, 2, \cdots, m$. Set $f_{m+1} = u_{m+1} - \alpha_1 e_1 - \alpha_2 e_2 - \cdots - \alpha_m e_m \neq 0$. Determine the constants α_i by the requirement $(f_{m+1}, e_i) = 0 = (u_{m+1}, e_i) - \alpha_i$, $1 \leq i \leq m$. Set $e_{m+1} = f_{m+1}/||f_{m+1}||$. Then $\{e_1, \cdots, e_{m+1}\}$ is ON. Q.E.D.

Let $\mathcal{W}$ be a subspace of $\mathcal{H}$ and let $\{e_1, e_2, \cdots, e_m\}$ be an ON basis for $\mathcal{W}$. let $u \in \mathcal{H}$. We say that the vector $u' = \sum_{i=1}^m (u, e_i)e_i \in \mathcal{W}$ is the **projection of u on $\mathcal{W}$**.

Theorem 43 *If $u \in \mathcal{H}$ there exist unique vectors $u' \in \mathcal{W}$, $u'' \in \mathcal{W}^\perp$ such that $u = u' + u''$.*

PROOF:

1. Existence: Let $\{e_1, e_2, \cdots, e_m\}$ be an ON basis for $\mathcal{W}$, set $u' = \sum_{i=1}^m (u, e_i)e_i \in \mathcal{W}$ and $u'' = u - u'$. Now $(u'', e_i) = (u, e_i) - (u, e_i) = 0$, $1 \leq i \leq m$, so $(u'', v) = 0$ for all $v \in \mathcal{W}$. Thus $u'' \in \mathcal{W}^\perp$.

2. Uniqueness: Suppose $u = u' + u'' = v" + v''$ where $u', v' \in \mathcal{W}$, $u'', v'' \in \mathcal{W}^\perp$. Then $u' - v' = v'' - u'' \in \mathcal{W} \cap \mathcal{W}^\perp \Longrightarrow (u' - v', u' - v') = 0 = ||u' - v'||^2 \Longrightarrow u' = v', u'' = v''$. Q.E.D.

Corollary 12 *Bessel's Inequality. Let $\{e_1, \cdots, e_m\}$ be an ON set in $\mathcal{H}$. if $u \in \mathcal{H}$ then $||u||^2 \geq \sum_{i=1}^m |(u, e_i)|^2$.*

PROOF: Set $W = [e_1, \cdots, e_m]$. Then $u = u' + u''$ where $u' \in \mathcal{W}$, $u'' \in \mathcal{W}^\perp$, and $u' = \sum_{i=1}^m (u, e_i)e_i$. Therefore $||u||^2 = (u'+u'', u'+u'') = ||u'||^2 + ||u''||^2 \geq ||u'||^2 = (u', u') + \sum_{i=1}^m |(u, e_i)|^2$. Q.E.D.

Note that this inequality holds even if m is infinite.

The projection of $u \in \mathcal{H}$ onto the subspace $\mathcal{W}$ has invariant meaning, i.e., it is basis independent. Also, it solves an important minimization problem: u' is the vector in $\mathcal{W}$ that is closest to u.

Theorem 44 $\min_{v \in \mathcal{W}} ||u - v|| = ||u - u'||$ *and the minimum is achieved if and only if $v = u'$.*

PROOF: let $v \in \mathcal{W}$ and let $\{e_1, e_2, \cdots, e_m\}$ be an ON basis for $\mathcal{W}$. Then $v = \sum_{i=1}^m \alpha_i e_i$ for $\alpha_i = (v, e_i)$ and $||u - v|| = ||u - \sum_{i=1}^m \alpha_i e_i||^2 = (u - \sum_{i=1}^m \alpha_i e_i, u - \sum_{i=1}^m \alpha_i e_i) = ||u||^2 - \sum_{i=1}^m \bar{\alpha}_i (u, e_i) - \sum_{i=1}^m \alpha_i (e_i, u) + \sum_{i=1}^m |\alpha_i|^2 = ||u - \sum_{i=1}^m (u, e_i)e_i||^2 + \sum_{i=1}^m |(u, e_i) - \alpha_i|^2 \geq ||u - u'||^2$. Equality is obtained if and only if $\alpha_i = (u, e_i)$, for $1 \leq i \leq m$. Q.E.D.

1.6.3 Orthonormal systems in an infinite-dimensional separable Hilbert space

Let $\mathcal{H}$ be a separable Hilbert space. (We have in mind spaces such as ℓ^2 and $L^2[0, 2\pi]$.)

The idea of an orthogonal projection extends to infinite-dimensional inner product spaces, but here there is a problem. If the infinite-dimensional subspace $\mathcal{W}$ of $\mathcal{H}$ isn't closed, the concept may not make sense.

For example, let $\mathcal{H} = \ell^2$ and let $\mathcal{W}$ be the subspace elements of the form $(\cdots, \alpha_{-1}, \alpha_0, \alpha_1, \cdots)$ such that $\alpha_i = 0$ for $i = 1, 0, -1, -2, \cdots$ and there are a *finite* number of nonzero components α_i for $i \geq 2$. Choose $u = (\cdots, \beta_{-1}, \beta_0, \beta_1, \cdots)$ such that $\beta_i = 0$ for $i = 0, -1, -2, \cdots$ and $\beta_n = 1/n$ for $n = 1, 2, \cdots$. Then $u \in \ell^2$ but the projection of u on $\mathcal{W}$ is undefined. If $\mathcal{W}$ is closed, however, i.e., if every Cauchy sequence $\{u_n\}$ in $\mathcal{W}$ converges to an element of $\mathcal{W}$, the problem disappears.

Theorem 45 *Let $\mathcal{W}$ be a closed subspace of the inner product space $\mathcal{H}$ and let $u \in H$. Set $d = inf_{v\in\mathcal{W}}||u-v||$. Then there exists a unique $\bar{u} \in \mathcal{W}$ such that $||u-\bar{u}|| = d$, ($\bar{u}$ is called the* **projection** *of u on $\mathcal{W}$.) Furthermore $u - \bar{u} \perp \mathcal{W}$ and this characterizes $\bar{u}$.*

PROOF: Clearly there exists a sequence $\{v_n\} \in \mathcal{W}$ such that $||u - v_n|| = d_n$ with $\lim_{n\to\infty} d_n = d$. We will show that $\{v_n\}$ is Cauchy. let $v \neq \Theta$ be a vector in $\mathcal{W}$ and $\alpha \in C$. Then $d^2 \leq ||u-(v_n+\alpha v)||^2 = ||u-v_n||^2 - \alpha(v, u-v_n) - \bar{\alpha}(u-v_n, v) + |\alpha|^2||v||^2$. The right-hand side of this expression is a minimum if $\alpha = (u-v_n, v)/||v||^2$. Choosing this value we find $d^2 \leq ||u-v_n||^2 - |(u-v_n, v)|^2/||v||^2 = d_n^2 - |(u-v_n,v)|^2/||v||^2$. Thus, $|(u-v_n,v)|^2 \leq ||v||^2(d_n^2-d^2)$ for all nonzero $v \in \mathcal{W} \Longrightarrow |(v_m - v_n, v)| \leq |(v_m-u,v)| + |(u-v_n,v)| \leq ||v||(\sqrt{d_n^2-d^2} + \sqrt{d_m^2-d^2}) \to 0$ as $n, m \to \infty$. Now set $v = v_m - v_n$. Then $||v_m - v_n|| \leq \sqrt{d_n^2-d^2} + \sqrt{d_m^2-d^2} \to 0$ as $n, m \to \infty$. Thus $\{v_n\}$ is Cauchy in $\mathcal{W}$.

Since $\mathcal{W}$ is closed, there exists $\bar{u} \in \mathcal{W}$ such that $\lim_{n\to\infty} v_n = \bar{u}$. Also, $||u-\bar{u}|| = ||u - \lim_{n\to\infty} v_n|| = \lim_{n\to\infty}||u-v_n|| = \lim_{n\to\infty} d_n = d$. Furthermore, for any $v \in \mathcal{W}$, $(u-\bar{u}, v) = \lim_{n\to\infty}(u-v_n, v) = 0 \Longrightarrow u - \bar{u} \perp \mathcal{W}$.

Conversely, if $u-\bar{u} \perp \mathcal{W}$ and $v \in \mathcal{W}$ then $||u-v||^2 = ||(u-\bar{u})+(\bar{u}-v)||^2 = ||u-\bar{u}||^2 + ||\bar{u}-v||^2 = d^2 + ||\bar{u}-v||^2$. Therefore $||u-v||^2 \geq d^2$ and $= d^2$ if and only if $\bar{u} = v$. Thus $\bar{u}$ is unique. Q.E.D.

Corollary 13 *Let $\mathcal{W}$ be a closed subspace of the Hilbert space $\mathcal{H}$ and let $u \in \mathcal{H}$. Then there exist unique vectors $\bar{u} \in \mathcal{W}$, $\bar{v} \in \mathcal{W}^\perp$, such that $u = \bar{u} + \bar{v}$. We write $\mathcal{H} = \mathcal{W} \oplus \mathcal{W}^\perp$.*

Corollary 14 *A subspace $\mathcal{M} \subseteq \mathcal{H}$ is dense in $\mathcal{H}$ if and only if $u \perp \mathcal{M}$ for $u \in \mathcal{H}$ implies $u = \Theta$.*

PROOF: $\mathcal{M}$ dense in $\mathcal{H} \Longrightarrow \overline{\mathcal{M}} = \mathcal{H}$. Suppose $u \perp \mathcal{M}$. Then there exists a sequence $\{u_n\}$ in $\mathcal{M}$ such that $\lim_{n\to\infty} u_n = u$ and $(u, u_n) = 0$ for all n. Thus $(u,u) = \lim_{n\to\infty}(u, u_n) = 0 \Longrightarrow u = \Theta$.

Conversely, suppose $u \perp \mathcal{M} \Longrightarrow u = \Theta$. If $\mathcal{M}$ isn't dense in $\mathcal{H}$ then $\bar{\mathcal{M}} \neq \mathcal{H} \Longrightarrow$ there is a $u \in \mathcal{H}$ such that $u \neq \bar{\mathcal{M}}$. Therefore there exists a $\bar{u} \in \bar{\mathcal{M}}$ such that $v = u - \bar{u} \neq \Theta$ belongs to $\bar{\mathcal{M}}^\perp \Longrightarrow v \perp \mathcal{M}$. Impossible! Q.E.D.

Now we are ready to study ON systems on an infinite-dimensional (but separable) Hilbert space $\mathcal{H}$ If $\{v_n\}$ is a sequence in $\mathcal{H}$, we say that $\sum_{n=1}^{\infty} v_n =$

$v \in \mathcal{H}$ if the partial sums $\sum_{n=1}^{k} v_n = u_k$ form a Cauchy sequence and $\lim_{k\to\infty} u_k = v$. This is called **convergence in the mean** or **convergence in the norm**, as distinguished from poitwise convergence of functions. (For Hilbert spaces of functions, such as $L^2[0, 2\pi]$ we need to distinguish this mean convergence from pointwise or uniform convergence.

The following results are just slight extentions of results that we have proved for ON sets in finite-dimensional inner-product spaces. The sequence $u_1, u_2, \cdots \in \mathcal{H}$ is **orthonormal** (ON) if $(u_i, u_j) = \delta_{ij}$. (Note that an ON sequence need not be a basis for $\mathcal{H}$.) Given $u \in \mathcal{H}$, the numbers $\alpha_j = (u, u_j)$ are the **Fourier coefficients** of u with respect to this sequence.

Lemma 36 $u = \sum_{n=1}^{\infty} \alpha_n u_n \Longrightarrow \alpha_n = (u, u_n)$.

Given a fixed ON system $\{u_n\}$, a positive integer N and $u \in \mathcal{H}$ the projection theorem tells us that we can minimize the "error" $||u - \sum_{n=1}^{N} \alpha_n u_n||$ of approximating u by choosing $\alpha_n = (u, u_n)$, i.e., as the Fourier coefficients. Moreover,

Corollary 15 $\sum_{n=1}^{N} |(u, u_n)|^2 \le ||u||^2$ *for any* N.

Corollary 16 $\sum_{n=1}^{\infty} |(u, u_n)|^2 \le ||u||^2$, *Bessel's inequality.*

Theorem 46 *Given the ON system* $\{u_n\} \in \mathcal{H}$, *then* $\sum_{n=1}^{\infty} \beta_n u_n$ *converges in the norm if and only if* $\sum_{n=1}^{\infty} |\beta_n|^2 < \infty$.

PROOF: Let $v_k = \sum_{n=1}^{k} \beta_n u_n$. $\sum_{n=1}^{\infty} \beta_n u_n$ converges if and only if $\{v_k\}$ is Cauchy in $\mathcal{H}$. For $k \ge \ell$,

$$||v_k - v_\ell||^2 = ||\sum_{n=\ell+1}^{k} \beta_n u_n||^2 = \sum_{n=\ell+1}^{k} |\beta_n|^2. \tag{1.7}$$

Set $t_k = \sum_{n=1}^{k} |\beta_n|^2$. Then (1.7)$\Longrightarrow$ $\{v_k\}$ is Cauchy in $\mathcal{H}$ if and only if $\{t_k\}$ is Cauchy, if and only if $\sum_{n=1}^{\infty} |\beta_n|^2 < \infty$. Q.E.D.

Definition 39 *A subset* $\mathcal{K}$ *of* $\mathcal{H}$ *is* **complete** *if for every* $u \in \mathcal{H}$ *and* $\epsilon > 0$ *there are elements* $u_1, u_2, \cdots, u_N \in \mathcal{K}$ *and* $\alpha_1, \cdots, \alpha_N \in C$ *such that* $||u - \sum_{n=1}^{N} \alpha_n u_n|| < \epsilon$, *i.e., if the subspace* $\tilde{calK}$ *formed by taking all finite linear combinations of elements of* $\mathcal{K}$ *is dense in* $\mathcal{H}$.

Theorem 47 *The following are equivalent for any ON sequence* $\{u_n\}$ *in* $\mathcal{H}$.

1. $\{u_n\}$ *is complete* ($\{u_n\}$ *is an ON basis for* $\mathcal{H}$.)
2. *Every* $u \in \mathcal{H}$ *can be written uniquely in the form* $u = \sum_{n=1}^{\infty} \alpha_n u_n$, $\alpha_n = (u, u_n)$.
3. *For every* $u \in \mathcal{H}$, $||u||^2 = \sum_{n=1}^{\infty} |(u, u_n)|^2$. *Parseval's equality*
4. *If* $u \perp \{u_n\}$ *then* $u = \Theta$.

PROOF:

1. 1.$\Longrightarrow$ 2. $\{u_n\}$ complete $\Longrightarrow$ given $u \in \mathcal{H}$ and $\epsilon > 0$ there is an integer N and constants $\{\alpha_n\}$ such that $||u - \sum_{n=1}^{N} \alpha_n u_n|| < \epsilon \Longrightarrow ||u - \sum_{n=1}^{k} \alpha_n u_n|| < \epsilon$ for all $k \geq N$. Clearly $\sum_{n=1}^{\infty}(u, u_n)u_n \in \mathcal{H}$ since $\sum_{n=1}^{\infty} |(u, u_n)|^2 \leq ||u||^2 < \infty$. Therefore $u = \sum_{n=1}^{\infty}(u, u_n)u_n$. Uniqueness obvious.
2. 2.$\Longrightarrow$ 3. Suppose $u = \sum_{n=1}^{\infty} \alpha_n u_n$, $\alpha_n = (u, u_n)$. Therefore, $||u - \sum_{n=1}^{k} \alpha_n u_n||^2 = ||u||^2 - \sum_{n=1}^{k} |(u, u_n)|^2 \to 0$ as $k \to \infty$. Hence $||u||^2 = \sum_{n=1}^{\infty} |(u, u_n)|^2$.
3. 3.$\Longrightarrow$ 4. Suppose $u \perp \{u_n\}$. Then $||u||^2 = \sum_{n=1}^{\infty} |(u, u_n)|^2 = 0$ so $u = \Theta$.
4. 4.$\Longrightarrow$ 1. Let $\tilde{\mathcal{M}}$ be the dense subspace of $\mathcal{H}$ formed from all finite linear combinations of $u_1, u_2, \cdots$. Then given $v \in \mathcal{H}$ and $\epsilon > 0$ there exists a $\sum_{n=1}^{N} \alpha_n u_n \in \tilde{\mathcal{M}}$ such that $||v - \sum_{n=1}^{N} \alpha_n u_n|| < \epsilon$. Q.E.D.

1.7 Linear operators and matrices, Least squares approximations

Let V, W be vector spaces over F (either the real or the complex field).

Definition 40 *A* **linear transformation** *(or linear operator) from* V *to* W *is a function* $\mathbf{T} : V \to W$, *defined for all* $v \in V$ *that satisfies* $\mathbf{T}(\alpha u + \beta v) = \alpha \mathbf{T} u + \beta \mathbf{T} v$ *for all* $u, v \in V$, $\alpha, \beta \in W$. *Here, the set* $R(\mathbf{T}) = \{\mathbf{T} u : u \in V\}$ *is called the* **range** *of* $\mathbf{T}$.

Lemma 37 *$R(\mathbf{T})$ is a subspace of W.*

PROOF: Let $w = \mathbf{T}u, z = \mathbf{T}v \in R(\mathbf{T})$ and let $\alpha, \beta \in F$. Then $\alpha w + \beta z = \mathbf{T}(\alpha u + \beta v) \in R(\mathbf{T})$. Q.E.D.

If V is m-dimensional with basis $v_1, \cdots, v_m$ and W is n-dimensional with basis $w_1, \cdots, w_n$ then $\mathbf{T}$ is completely determined by its matrix representation $T = (T_{jk})$ with respect to these two bases:

$$\mathbf{T}v_k = \sum_{j=1}^{n} T_{jk} w_j, \qquad , k = 1, 2, \cdots, m.$$

If $v \in V$ and $v = \sum_{k=1}^{m} \alpha_k v_k$ then the action $\mathbf{T}v = w$ is given by

$$\mathbf{T}v = \mathbf{T}(\sum_{k=1}^{m} \alpha_k v_k) = \sum_{k=1}^{m} \alpha_k \mathbf{T}v_k = \sum_{j=1}^{n}\sum_{k=1}^{m}(T_{jk}\alpha_k)w_j = \sum_{j=1}^{n} \beta_j w_j = w$$

Thus the coefficients β_j of w are given by $\beta_j = \sum_{k=1}^{m} T_{jk}\alpha_k$, $j = 1, \cdots, n$. In matrix notation, one writes this as

$$\begin{pmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nm} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix},$$

or

$$Ta = b.$$

The matrix $T = (T_{jk})$ has n rows and m columns, i.e., it is $n \times m$, whereas the vector $a = (\alpha_k)$ is $m \times 1$ and the vector $b = (\beta_j)$ is $n \times 1$. If V and W are Hilbert spaces with ON bases, we shall sometimes represent operators $\mathbf{T}$ by matrices with an infinite number of rows and columns.

Let V, W, X be vector spaces over F, and $\mathbf{T}$, $\mathbf{U}$ be linear operators $\mathbf{T} : V \to W, \mathbf{U} : W \to X$. The **product UT** of these two operators is the composition $\mathbf{U} : V \to X$ defined by $\mathbf{UT}v = \mathbf{U}(\mathbf{T}v)$ for all $v \in V$.

Suppose V is m-dimensional with basis $v_1, \cdots, v_m$, W is n-dimensional with basis $w_1, \cdots, w_n$ and X is p-dimensional with basis $x_1, \cdots, x_p$. Then $\mathbf{T}$ has matrix representation $T = (T_{jk})$, $\mathbf{U}$ has matrix representation $U = (U_{\ell j})$,

$$\mathbf{U}w_j = \sum_{\ell=1}^{p} U_{\ell j} x_\ell, \qquad , j = 1, 2, \cdots, n,$$

and $\mathbf{Y} = \mathbf{UT}$ has matrix representation $Y = (Y_{\ell k})$ given by

$$\mathbf{Y}v_k = \mathbf{UT}v_k = \sum_{\ell=1}^{p} Y_{\ell k} x_\ell, \qquad k = 1, 2, \cdots, m,$$

A straightforward computation gives $Y_{\ell k} = \sum_{j=1}^{n} U_{\ell j} T_{jk}$, $\ell = 1, \cdots, p$, $k = 1, \cdots, m$. In matrix notation, one writes this as

$$\begin{pmatrix} U_{11} & \cdots & U_{1n} \\ \vdots & \ddots & \vdots \\ U_{p1} & \cdots & U_{pn} \end{pmatrix} \begin{pmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nm} \end{pmatrix} = \begin{pmatrix} Y_{11} & \cdots & Y_{1m} \\ \vdots & \ddots & \vdots \\ Y_{p1} & \cdots & Y_{pm} \end{pmatrix},$$

or

$$UT = Y.$$

Here, U is $p \times n$, T is $n \times m$ and Y is $p \times m$.

Now let us return to our operator $\mathbf{T} : V \to W$ and suppose that both V and W are complex inner product spaces, with inner products $(\cdot, \cdot)_V$, $(\cdot, \cdot)_W$, respectively. Then $\mathbf{T}$ induces a linear operator $\mathbf{T}^* : W \to V$ and defined by

$$(\mathbf{T}v, w)_W = (v, \mathbf{T}^* w)_V, \qquad v \in V, w \in W.$$

To show that $\mathbf{T}^*$ exists, we will compute its matrix T^*. Suppose that $v_1, \cdots, v_m$ is an ON basis for V and $w_1, \cdots, w_n$ is an ON basis for W. Then we have have

$$T_{jk} = (\mathbf{T}v_k, w_j)_W = (v_k, \mathbf{T}^* w_j)_V = \bar{T^*}_{kj}, \quad k = 1, \cdots, m, \quad j = 1, \cdots, n.$$

Thus the operator $\mathbf{T}^*$, (the **adjoint operator** to $\mathbf{T}$) has the adjoint matrix to T: $T^*_{kj} = \bar{T}_{jk}$. In matrix notation this is written $T^* = \bar{T}^{\rm tr}$ where the ${}^{\rm tr}$ stands for the matrix transpose (interchange of rows and columns). For a real inner product space the complex conjugate is dropped and the adjoint matrix is just the transpose.

There are some special operators and matrices that we will meet often in this course. Suppose that $v_1, \cdots, v_m$ is an ON basis for V. The **identity operator** $\mathbf{I} : V \to V$ is defined by $\mathbf{I}v = v$ for all $v \in V$. The matrix of $\mathbf{I}$ is $I = (\delta_{jh})$ where $\delta_{jj} = 1$ and $\delta_{jh} = 0$ if $j \neq h$, $1 \leq j, h \leq m$. The **zero operator** $\mathbf{Z} : V \to V$ is defined by $\mathbf{Z}v = \Theta$ for all $v \in V$. The $n \times n$ matrix of $\mathbf{Z}$ has all matrix elemnents 0. An operator $\mathbf{U} : V \to V$ that preserves the inner product, $(\mathbf{U}v, \mathbf{U}u) = (v, u)$ for all $u, v \in V$ is called **unitary**.

The matrix U of a unitary operator is characterized by the matrix equation $UU^* = I$. If V is a real inner product space, the operators $\mathbf{O} : V \to V$ that preserve the inner product, $(\mathbf{O}v, \mathbf{O}u) = (v, u)$ for all $u, v \in V$ are called **orthogonal**. The matrix O of an orthogonal operator is characterized by the matrix equation $OO^{tr} = I$.

1.7.1 Bounded operators on Hilbert spaces

In this section we present a few concepts and results from functional analysis that are needed for the study of wavelets.

An operator $\mathbf{T} : \mathcal{H} \to \mathcal{K}$ of the Hilbert space $\mathcal{H}$ to the Hilbert Space $\mathcal{K}$ is said to be **bounded** if it maps the unit ball $||u||_{\mathcal{H}} \le 1$ to a bounded set in $\mathcal{K}$. This means that there is a finite positive number N such that

$$||\mathbf{T}u||_{\mathcal{K}} \le N \quad \text{whenever} \quad ||u||_{\mathcal{H}} \le 1.$$

The **norm** $||\mathbf{T}||$ of a bounded operator is its least bound:

$$||\mathbf{T}|| = \sup_{||u||_{\mathcal{H}} \le 1} ||\mathbf{T}u||_{\mathcal{K}} = \sup_{||u||_{\mathcal{H}} = 1} ||\mathbf{T}u||_{\mathcal{K}}. \tag{1.8}$$

Lemma 38 *Let $\mathbf{T} : \mathcal{H} \to \mathcal{K}$ be a bounded operator.*

1. $||\mathbf{T}u||_{\mathcal{K}} \le ||\mathbf{T}|| \cdot ||u||_{\mathcal{H}}$ *for all* $u \in \mathcal{H}$.

2. *If* $\mathbf{S} : \mathcal{L} \to \mathcal{H}$ *is a bounded operator from the Hilbert space* $\mathcal{L}$ *to* $\mathcal{H}$, *then* $\mathbf{TS} : \mathcal{L} \to \mathcal{K}$ *is a bounded operator with* $||\mathbf{TS}|| \le ||\mathbf{T}|| \cdot ||\mathbf{S}||$.

PROOF:

1. The result is obvious for $u = \theta$. If u is nonzero, then $v = ||u||_{\mathcal{H}}^{-1} u$ has norm 1. Thus $||\mathbf{T}v||_{\mathcal{K}} \le ||\mathbf{T}||$. The result follows from multiplying both sides of the inequality by $||u||_{\mathcal{H}}$.

2. From part 1, $||\mathbf{TS}w||_{\mathcal{K}} = ||\mathbf{T}(\mathbf{S}w)||_{\mathcal{K}} \le ||\mathbf{T}|| \cdot ||\mathbf{S}w||_{\mathcal{H}} \le ||\mathbf{T}|| \cdot ||\mathbf{S}|| \cdot ||w||_{\mathcal{L}}$. Hence $||\mathbf{TS}|| \le ||\mathbf{T}|| \cdot ||\mathbf{S}||$.

Q.E.D.

A special bounded operator is the **bounded linear functional** $\mathbf{f} : \mathcal{H} \to C$, where C is the one-dimensional vector space of complex numbers (with the absolute value

$cdot|$ as the norm). Thus $\mathbf{f}(u)$ is a complex number for each $u \in \mathcal{H}$ and $\mathbf{f}(\alpha u + \beta v) = \alpha \mathbf{f}(u) + \beta \mathbf{f}(v)$ for all scalars α, β and $u, v \in \mathcal{H}$ The **norm** of a bounded linear functional is defined in the usual way:

$$||\mathbf{f}|| = \sup_{||u||_{\mathcal{H}}=1} |\mathbf{f}(u). \tag{1.9}$$

For fixed $v \in calH$ the inner product $\mathbf{f}(u) \equiv (u, v)$, where $(\cdot, \cdot)$ is an import example of a bounded linear functional. The linearity is obvious and the functional is bounded since $|\mathbf{f}(u)| = |(u, v)| \leq ||u|| \cdot ||v||$. Indeed it is easy to show that $||\mathbf{f}|| = ||v||$. A very useful fact is that all bounded linear functionals on Hilbert spaces can be represented as inner products. This important result, the Riesz representation theorem, relies on the fact that a Hilbert space is complete. It is an elegant application of the projection theorem.

Theorem 48 *(Riesz representation theorem) Let* $\mathbf{f}$ *be a bounded linear functional on the Hilbert space* $\mathcal{H}$. *Then there is a vector* $v \in \mathcal{H}$ *such that* $\mathbf{f}(u) = (u, v)$ *for all* $u \in \mathcal{H}$.

PROOF:

- Let $\mathcal{N} = \{w \in \mathcal{H} : \mathbf{f}(w) = \theta\}$ be the null space of $\mathbf{f}$. Then $\mathcal{N}$ is a closed linear subspace of $\mathcal{H}$. Indeed if $w_1, w_2 \in \mathcal{N}$ and $\alpha, \beta \in C$ we have $\mathbf{f}(\alpha w_1 + \beta w_2) = \alpha \mathbf{f}(w_1) + \beta \mathbf{f}(w_2) = \theta$, so $\alpha w_1 + \beta w_2 \in \mathcal{N}$. If $\{w_n\}$ is a Cauchy sequence of vectors in $\mathcal{N}$, i.e., $\mathbf{f}(w_n) = \theta$, with $w_n \to w \in \mathcal{H}$ as $n \to \infty$ then

  $$|\mathbf{f}(w)| = |\mathbf{f}(w) - \mathbf{f}(w_n)| = |\mathbf{f}(w - w_n)| \leq ||\mathbf{f}|| \cdot ||w - w_n|| \to 0$$

  as $n \to \infty$. Thus $\mathbf{f}(w) = \theta$ and $w \in \mathcal{N}$, so $\mathcal{N}$ is closed.

- If $\mathbf{f}$ is the zero functional, then the theorem holds with $v = \theta$, the zero vector. If $\mathbf{F}$ is not zero, then there is a vector $u_0 \in \mathcal{H}$ such that $\mathbf{f}(u_0) = 1$. By the projection theorem we can decompose u_0 uniquely in the form $u_0 = v_0 + w_0$ where $w_0 \in calN$ and $v_0 \perp \mathcal{N}$. Then $1 = \mathbf{f}(u_0) = \mathbf{f}(v_0) + \mathbf{f}(w_0) = \mathbf{f}(v_0)$.

- Every $u \in \mathcal{H}$ can be expressed uniquely in the form $u = \mathbf{f}(u)v_0 + w$ for $w \in \mathcal{N}$. Indeed $\mathbf{f}(u - \mathbf{f}(u)v_0) = \mathbf{f}(u) - \mathbf{f}(u)\mathbf{f}(v_0) = 0$ so $u - \mathbf{f}(u)v_0 \in \mathcal{N}$.

- Let $v = ||v_0||^{-2}v_0$. Then $v \perp \mathcal{N}$ and

$$(u, v) = (\mathbf{f}(u)v_0 + w, v) = \mathbf{f}(u)(v_0, v) = \mathbf{f}(u)||v_0||^{-2}(v_0, v_0) = \mathbf{f}(u).$$

Q.E.D.

We can define adjoints of bounded operators on general Hilbert spaces, in analogy with our constuction of adjoints of operators on finite-dimensional inner product spaces. We return to our bounded operator $\mathbf{T} : \mathcal{H} \to \mathcal{K}$. For any $v \in \mathcal{K}$ we define the linear functional $\mathbf{f}_v(u) = (\mathbf{T}u, v)_\mathcal{K}$ on $\mathcal{H}$. The functional is bounded because for $||u||_\mathcal{H} = 1$ we have

$$|\mathbf{f}_v(u)| = |(\mathbf{T}u, v)_\mathcal{K}| \le ||\mathbf{T}u||_\mathcal{K} \cdot ||v||_\mathcal{K} \le ||\mathbf{T}|| \cdot ||v||_\mathcal{K}.$$

By theorem 48 there is a unique vector $v^* \in \mathcal{H}$ such that

$$\mathbf{f}_v(u) \equiv (\mathbf{T}u, v)_\mathcal{K} = (u, v^*)_\mathcal{H},$$

for all $u \in \mathcal{H}$. We write this element as $v^* = \mathbf{T}^*v$. Thus $\mathbf{T}$ induces an operator $\mathbf{T}^* : \mathcal{K} \to \mathcal{H}$ and defined uniquely by

$$(\mathbf{T}u, v)_\mathcal{K} = (u, \mathbf{T}^*v)_\mathcal{H}, \qquad v \in \mathcal{H}, w \in \mathcal{K}.$$

Lemma 39 *1. $\mathbf{T}^*$ is a linear operator from $\mathcal{K}$ to $\mathcal{H}$.*

2. $\mathbf{T}^*$ is a bounded operator.

3. $||\mathbf{T}^*||^2 = ||\mathbf{T}||^2 = ||\mathbf{T}\mathbf{T}^*|| = ||\mathbf{T}^*\mathbf{T}||$.

PROOF:

1. Let $v \in \mathcal{K}$ and $\alpha \in C$. Then

$$(u, \mathbf{T}^*\alpha v)_\mathcal{H} = (\mathbf{T}u, \alpha v)_\mathcal{K} = \overline{\alpha}(\mathbf{T}u, v)_\mathcal{K} = \overline{\alpha}(u, \mathbf{T}^*v)_\mathcal{H}$$

so $\mathbf{T}^*(\alpha v) = \alpha \mathbf{T}^*v$. Now let $v_1, v_2 \in \mathcal{K}$. Then

$$(u, \mathbf{T}^*[v_1 + v_2])_\mathcal{H} = (\mathbf{T}u, [v_1 + v - 2])_\mathcal{K} = (\mathbf{T}u, v_1)_\mathcal{K} + (\mathbf{T}u, v_2)_\mathcal{K} = (u, \mathbf{T}^*v_+\mathbf{T}^*v_2)_\mathcal{H}$$

so $\mathbf{T}^*(v_1 + v_2) = \mathbf{T}^*v_1 + \mathbf{T}^*v_2$.

2. Set $u = \mathbf{T}^* v$ in the defining equation $(\mathbf{T}u, v)_\mathcal{K} = (u, \mathbf{T}^* v)_\mathcal{H}$. Then

   $||\mathbf{T}^*v||^2_\mathcal{H} = (\mathbf{T}^*v, \mathbf{T}^*v)_\mathcal{H} = (\mathbf{T}\mathbf{T}^*v, v)_\mathcal{K} \le ||\mathbf{T}\mathbf{T}^*v||_\mathcal{K}||v||_\mathcal{K} \le ||\mathbf{T}||\cdot||\mathbf{T}^*v||_\mathcal{H}||v||_\mathcal{K}.$

   Canceling the common factor $||\mathbf{T}^*v||_\mathcal{H}$ from the far left and far right-hand sides of these inequalities, we obtain

   $$||\mathbf{T}^*v||_\mathcal{H} \le ||\mathbf{T}|| \cdot ||v||_\mathcal{K},$$

   so $\mathbf{T}^*$ is bounded.

3. From the last inequality of the proof of 2 we have $||\mathbf{T}^*|| \le ||T||$. However, if we set $v = \mathbf{T}u$ in the defining equation $(\mathbf{T}u, v)_\mathcal{K} = (u, \mathbf{T}^*v)_\mathcal{H}$, then we obtain an analogous inequality

   $$||\mathbf{T}u||_\mathcal{K} \le ||\mathbf{T}^*|| \cdot ||u||_\mathcal{H}.$$

   This implies $||\mathbf{T}|| \le ||bfT^*||$. Thus $||\mathbf{T}|| = ||bfT^*||$. From the proof of part 2 we have

   $$||\mathbf{T}^*v||^2_\mathcal{H} = (\mathbf{T}\mathbf{T}^*v, v)_\mathcal{K}. \tag{1.10}$$

   Applying the Schwarz inequalty to the right-hand side of this identity we have

   $$||\mathbf{T}^*v||^2_\mathcal{H} \le ||\mathbf{T}\mathbf{T}^*v||_\mathcal{K}||v||_\mathcal{K} \le ||\mathbf{T}\mathbf{T}^*|| \cdot ||v||^2_\mathcal{K},$$

   so $||\mathbf{T}^*||^2 \le ||\mathbf{T}\mathbf{T}^*||$. But from lemma 38 we have $||\mathbf{T}\mathbf{T}^*|| \le ||\mathbf{T}||\cdot||\mathbf{T}^*||$, so

   $$||\mathbf{T}^*||^2 \le ||\mathbf{T}\mathbf{T}^*|| \le ||\mathbf{T}|| \cdot ||\mathbf{T}^*||| = ||\mathbf{T}^*||^2.$$

   An analogous proof, switching the roles of u and v, yields

   $$||\mathbf{T}||^2 \le ||\mathbf{T}^*\mathbf{T}|| \le ||\mathbf{T}|| \cdot ||\mathbf{T}^*||| = ||\mathbf{T}||^2.$$

Q.E.D.

1.7.2 Least squares approximations

Many applications of mathematics to statistics, image processing, nemerical analysis, global positioning systems, etc., reduce ultimately to solving a system of equations of the form $Ta = b$ or

$$\begin{pmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nm} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \tag{1.11}$$

Here $b = \{\beta_1, \cdots, \beta_n\}$ are n measured quantities, the $n \times m$ matrix $T = (T_{jk})$ is known, and we have to compute the m quantities $a = \{\alpha_1, \cdots, \alpha_m\}$. Since b is measured experimentally, there may be errors in these quantities. This will induce errors in the calculated vector a. Indeed for some measured values of b there may be no solution a.

EXAMPLE: Consider the 3×2 system

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ \beta_3 \end{pmatrix}.$$

If $\beta_3 = 5$ then this system has the unique solution $\alpha_1 = -1, \alpha_2 = 3$. However, if $\beta_3 = 5 + \epsilon$ for ϵ small but nonzero, then there is no solution!

We want to guarantee an (approximate) solution of (1.11) for all vectors b and matrices T. We adopt a least squares approach. let's embed our problem into the inner product spaces V and W above. That is T is the matrix of the operator $\mathbf{T} : V \to W$, b is the component vector of a given $w \in W$ (with respect to the $\{w_j\}$ basis), and a is the component vector of $v \in V$ (with respect to the $\{v_k\}$ basis), which is to be computed. Now the original equation $Ta = b$ becomes $\mathbf{T}v = w$.

Let us try to find an approximate solution v of the equation $\mathbf{T}v = w$ such that the norm of the error $||w - \mathbf{T}v||_W$ is minimized. If the orginal problem has an exact solution then the error will be zero; otherwise we will find a solution v_0 with minimum (least squares) error. The square of the error will be

$$\epsilon^2 = \min_{v \in V} ||w - \mathbf{T}v||_W^2 = ||w - \mathbf{T}v_0||_W^2$$

This may not determine v_0 uniquely, but it will uniquely determine $bfTv_0$.

We can easily solve this problem via the projection theorem. recall that the range of $\mathbf{T}$, $R(\mathbf{T}) = \{\mathbf{T}u : u \in V\}$ is a subspace of W. We need to find the point on $R(\mathbf{T})$ that is closest in norm to w. By the projection theorem, that point is just the projection of w on $R(\mathbf{T})$, i.e., the point $\mathbf{T}v_0 \in R(\mathbf{T})$ such that $w - \mathbf{T}v_0 \perp R(\mathbf{T})$. This means that

$$(w - \mathbf{T}v_0, \mathbf{T}v)_W = 0$$

for all $v \in V$. Now, using the adjoint operator, we have

$$(w - \mathbf{T}v_0, \mathbf{T}v)_W = (\mathbf{T}^*[w - \mathbf{T}v_0], v)_V = (\mathbf{T}^*w - \mathbf{T}^*\mathbf{T}v_0, v)_V = 0$$

for all $v \in V$. This is possible if and only if

$$\mathbf{T}^*\mathbf{T}v_0 = \mathbf{T}^*w.$$

In matrix notation, our equation for the least squares solution a_0 is

$$T^*Ta_0 = T^*b. \tag{1.12}$$

The original system was rectangular; it involved m equations for n unknowns. Here however, the $n \times n$ matrix T^*T is square and the are n equations for the n unknowns $a_0 = \{\alpha_1, \cdots, \alpha_n\}$. If the matrix T is real, then equations (1.12) become $T^{\mathrm{tr}}Ta_0 = T^{\mathrm{tr}}b$.

Chapter 2

Contraction Mappings and Fixed Points

Let V be a normed linear space with norm $||\cdot||$. Let X be a *subset* of this space and T a mapping $T : X \longrightarrow X$. (We are not requiring that X be a subspace or the T be a linear mapping, only that T map X into X.) Many important applications of functional analysis are concerned with showing the existence of (and finding) *fixed points* of such mappings, i.e., points $x_0 \in X$ such that $Tx_0 = x_0$. A standard method for finding such points in practice is to start with a guess $x_0 \in X$ and then to compute recursively the iterates $x_1 = Tx, x_2 = T^2x = T(Tx), x_3 = T^3x = T(T^2x)\cdots, x_n = T^nx, \cdots$. If $\lim_{n\to\infty} T^nx = x_\infty$, where the convergence is in the norm, then it is easy to show that $Tx_\infty = x_\infty$ and x_∞ is a fixed point of T. This procedure will be an important tool for us, particularly in the study of wavelets and of fractals. However, in general the method won't work; there is no reason in general that the sequence of iterates will converge. This chapter is devoted to the study of a family of mappings (the contraction mappings) where convergence of the iterates is guaranteed. Most of the results are meant for use later in the course, but we will give some practical examples.

Definition 41 *T is called a contraction mapping on X if there is a constant c such that $0 < c < 1$ and $||Tx - Ty|| \le c||x - y||$ for all $x, y \in X$.*

Example 6 *let $V = X$ be the real line, with the absolute value as norm, and let T be the mapping $Tx = cx + b$, where $x \in R$ and b, c are real constants with $c > 0$. Now*

$$|Tx - Ty| = |(cx + b) - (cy + b)| = |c(x - y)| = c|x - y|$$

for x, y real, so T is a contraction mapping provided $0 < c < 1$. Since the form of T is so simple, we can find the fixed points directly. Here $Ty_0 = y_0$ means $cy_0 + b = y_0$ or $y_0 = \frac{b}{1-c}$ if $c \neq 1$. If $c = 1$ tnen there is no fixed point unless $b = 0$, in which case every point $x \in R$ is fixed.

If T is a contraction mapping $(0 < c < 1)$ and x_0 is an initial guess for the fixed point $y_0 = \frac{b}{1-c}$ then the error is $|x_0 - y_0|$. Then after one iteration the error is

$$|x_1 - y_0| = |Tx_0 - y_0| = |Tx_0 - Ty_0| = c|x_0 - y_0| < |x_0 - y_0|,$$

so the error has decreased by the factor $c < 1$. Similarly at the nth step the error is $|x_n - y_0| = c^n|x_0 - y_0|$. It follows that $\lim_{n\to\infty} x_n = y_0$ and the iterates converge to the fixed point, now matter what was the initial guess. We say that the fixed point is attractive in this case. On the other hand, if $c > 1$ then the equality $|x_n - y_0| = c^n|x_0 - y_0|$ shows that the iterates diverge and $|x_n| \to \infty$, unless $x_0 = y_0$. However, even in the latter case where one has started with the fixed point, the slightest error (say due to roundoff) will cause $|x_n|$ to diverge to ∞. We say that the fixed point is repelling in this case.

Example 7 *Again let V be the real line with the absolute value as norm. Let $Tx = T(x) = \frac{3}{4}x + x^3$. Again this mapping is simpole enoughthat we can find the fixed points directly. They are $x_\infty = 0, \pm\frac{1}{2}$. Now $T'(x) = \frac{3}{4} + 3x^2$ and, by the mean value theorem of calculus, if $x < y$ then $T(x) = T(y) + T'(\tilde{x})(x - y)$ where $\tilde{x}$ lies in the the interval (x, y). Let $X = (-\frac{1}{4}, \frac{1}{4})$. Then $|T'(\tilde{x})| < \frac{15}{16} = c < 1$ for $\tilde{x} \in X$, and $|Tx - Ty| < c|x - y|$ for $x, y \in X$. Further $|Tx| < c|x|$ so $Tx \in X$ for $x \in X$, and $x_\infty = 0$ is an attractive fixed point. That is, if $x_0 \in X$ then all $x_n \in X$ and $x_n \to x_\infty = 0$ as $n \to \infty$. A similar analysis shows that $\pm\frac{1}{2}$ are repelling fixed points.*

Theorem 49 *(Banach contraction principle) Let X be a closed subset of the Banach space V and suppose that $T : X \to X$ is a contractive mapping with constant c. Then T has a unique fixed point $x_\infty \in X$. If $x \in X$ then $x_\infty = \lim_{n\to\infty} T^n x$ and*

$$||T^n x - x_\infty|| \leq c^n ||x - x_\infty|| \leq \frac{c^n}{1-c} ||Tx - x||.$$

PROOF: Let $x \in X$ and set $x_{n+1} = Tx_n = T^{n+1}x$, $n = 0, 1, \cdots$. Now

$$||x_{n+1} - x_n|| = ||Tx_n - Tx_{n-1}|| \leq c||x_n - x_{n-1}|| \leq \cdots \leq c^n ||x_1 - x_0||.$$

From the triangle inequality we have

$$
\begin{aligned}
||x_{n+m} - x_n|| = ||(x_{n+m} - x_{n+m-1}) + (x_{n+m-1} - x_{n+m-2}) + \cdots + (x_{n+1} - x_n)|| \\
\leq \sum_{j=0}^{m-1} ||x_{n+j+1} - x_{n+j}|| \leq \sum_{j=0}^{m-1} c^{n+j} ||x_1 - x_0|| \\
< \frac{c^n}{1-c} ||x_1 - x_0||,
\end{aligned}
$$

or

$$||x_{n+m} - x_n|| < \frac{c^n}{1-c} ||x_1 - x_0||. \tag{2.1}$$

Thus given $\epsilon > 0$ one can chose an integer N_ϵ such that $||x_{n+m} - x_n|| < \epsilon$ for $n \geq N_\epsilon$ and all $m \geq 0$. This means that the sequence $\{x_n\}$ in X is Cauchy in the norm. Since V is a Banach space, i.e., is complete in the norm, and X is closed, there is a vector $x_\infty \in X \cap V$ such that $\lim_{n\to\infty} x_n = x_\infty$.. Now $||Tx_\infty - Tx_n|| \leq c||x_\infty - x_{n-1}||$. Since $x_n \to x_\infty$ it follows that $Tx_n \to Tx_\infty$ as $n \to \infty$. But $Tx_n = x_{n+1} \to x_\infty$ so $Tx_\infty - x_\infty$ and x_∞ is a fixed point. if we let $m \to \infty$ on the left-hand side of (2.1) we get the estimate

$$||x_\infty - x_n|| < \frac{c^n}{1-c} ||x_1 - x_0||.$$

The fixed point is unique, because if $\{x_n\}, \{x'_n\}$ are sequences converging to fixed points x_∞, x'_∞, respectively, then

$$||x_\infty - x'_\infty|| = ||Tx_\infty - Tx'_\infty|| \leq c||x_\infty - x'_\infty||$$

with $0 < c < 1$. This is possible only if $x_\infty = x'_\infty$.

Finally we have

$$
\begin{aligned}
||T^n x - x_\infty|| = ||T(T^{n-1}x) - Tx_\infty|| \leq c||T^{n-1}x - x_\infty|| \\
\leq \cdots \leq c^n ||x - x_\infty||.
\end{aligned}
$$

Q.E.D.

Note that though we have stated and proved the contraction principle for Banach spaces, the same proof carries over for complete metric spaces.

Corollary 17 *Let $\mathcal{X}$ be a closed subset of the complete metric space $\mathcal{M}$ and suppose that $T : \mathcal{X} \to \mathcal{X}$ is a contractive mapping with constant c. (That is,*

$$\rho(Tu, Tv) < c\rho(u, v), \qquad 0 < c < 1$$

for all $u, v \in \mathcal{X}$, where $\rho(\cdot, \cdot)$ is the metric on $\mathcal{M}$.) Then T has a unique fixed point $u_\infty \in \mathcal{X}$. If $u \in \mathcal{X}$ then $u_\infty = \lim_{n\to\infty} T^{(n)}u$ and

$$\rho(T^{(n)}u, u_\infty) \le c^n \rho(u, u_\infty) \le \frac{c^n}{1-c}\rho(Tu, u).$$

Here,

$$T^{(1)u} = Tu, \quad T^{(n+1)}u = T(T^{(n)}u), \qquad n = 1, 2, \cdots$$

Corollary 18 *(Collage theorem) Let $\mathcal{X}$ be a closed subset of the complete metric space $\mathcal{M}$ and $T : \mathcal{X} \to \mathcal{X}$ a contractive mapping with constant c and unique fixed point $u_\infty \in \mathcal{X}$. Then for any $u \in \mathcal{X}$,*

$$\rho(u, u_\infty) \le \frac{1}{1-c}\rho(u, Tu).$$

PROOF: By the triangle inequality,

$$\rho(u, u_\infty) \le \rho(u, Tu) + \rho(Tu, u_\infty) = \rho(u, Tu) + \rho(Tu, Tu_\infty)$$

$$\le \rho(u, Tu) + c\rho(u, u_\infty),$$

so $(1 - c)\rho(u, u_\infty) \le \rho(u, Tu)$. Q.E.D.

This simple consequence of the Banach contraction principle has imortant applications in fractal image compression. We shall explain this (and the term "collage") later.

2.1 Newton's method and the contraction principle

Newton's method for computing the zeros of functions is a good example of the contraction principle. Let $f(x)$ be a real-valued function on the real line that has two continuous derivatives. We are looking for a *root* of f, i.e., a point $\hat{x}$ such that $f(\hat{x}) = 0$. In Newton's method, which is geometrical, we consider the curve $y = f(x)$. Then the curve crosses the x-axis at the point

$(\hat{x}, f(\hat{x})$. Let x_0 be an initial guess for the root. To improve on the guess we construct the tangent line to the curve $y = f(x)$ that passes through the point $(x_0, f(x_0))$ on the curve. This tangent line satisfies the equation

$$y - f(x_0) = f'(x_0)(x - x_0).$$

The tangent line crosses the x-axis at the point

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},$$

and we take x_1 as our improved estimate of the root $\hat{x}$. Now we repeat this procedure with x_1 to get an improved estimate x_2, and so on. Thus we have a sequence $\{x_n\}$ such that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, \cdots.$$

We need to give conditions that will guarantee that the sequence will converge to a root of $f(x)$, and will provide information about the rate of convergence.

We cast this as a fixed point problem and apply the Banach contraction principle. Choose the Banach space to be the real numbers with the absolute value as norm, and define the operator T by

$$Tx = T(x) = x - \frac{f(x)}{f'(x)}.$$

We will not yet fix the domain X, but it is clear that we must require $f'(x) \neq 0$ for all $x \in X$. Then $\hat{x}$ will be a fixed point of T, $(T\hat{x} = \hat{x})$ if and only if $f(\hat{x}) = 0$. To get the growth rate for the iteration we compute the derivative of $T(x)$:

$$T'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

Since $T'(\hat{x}) = 0$, in the neighborhood of the root we will be able to select a decay constant $c < 1$, so the root is an attractive fixed point. In particular let $X = [\hat{x} - r, \hat{x} + r]$ where $|T'(x)| \leq c < 1$ for all $x \in X$. (If $f'(\hat{x}) \neq 0$ we can always find an r such that the inequality holds for the given constant c.) Then if $u, v \in X$, the mean value theorem gives $T(u) - T(v) = T'(\tilde{u})(u - v)$ for some $\tilde{u} \in X$ between u and v. Thus $|T(u) - T(v)| \leq c|u - v|$ for all $u, v \in X$. In particular

$$|x_n - \hat{x}| \leq c|x_{n-1} - \hat{x}| \leq \cdots \leq c^n|x_0 - \hat{x}| \leq \frac{c^n}{1-c}|x_0 - x_1|.$$

Thus if $x_0 \in X$ then so are all of the $x_n \in X$ and $x_n \to \hat{x}$ as $n \to \infty$.

The convergence of the Newton algorithm is actually much faster than indicated from the contraction principle. This is due to the fact that $T'(\hat{x}) = 0$. We can, indeed, prove quadratic convergence. Let $C = sup_{x \in X} |\frac{f''(x)}{f'(x)}|$ and assume that C is finite. By the mean value theorem there is a point $\tilde{x}_n \in X$, between $\hat{x}$ and x_n, such that

$$f(x_n) = f(x_n) - f(\hat{x}) = f'(\tilde{x}_n)(x_n - \hat{x}),$$

so $x_n - \hat{x} = f(x_n)/f'(\tilde{x}_n)$. Furthermore, the mean value theorem applied to $f'(x)$ yields a point $\breve{x}_n$ between x_n and $\tilde{x}_n$ such that

$$f'(x_n) - f'(\tilde{x}_n) = f''(\breve{x}_n)(x_n - \tilde{x}_n).$$

Then

$$|x_{n+1} - \hat{x}| = |(x_{n+1} - x_n) + (x_n - \hat{x})| = |\frac{f(x_n)}{f'(\tilde{x}_n)} - \frac{f(x_n)}{f'(x_n)}|$$

$$= |\frac{f(x_n)}{f'(x_n)f'(\tilde{x}_n)}(f'(x_n) - f'(\tilde{x}_n))| = |\frac{x_n - \hat{x}}{f'(x_n)}(f'(x_n) - f'(\tilde{x}_n))|$$

$$= |(x_n - \tilde{x}_n)(x_n - \hat{x})\frac{f''(\breve{x}_n)}{f'(x_n)}| \leq C|x_n - \hat{x}|^2.$$

Thus $|x_{n+1} - \hat{x}| \leq C|x_n - \hat{x}|^2$ and the convergence is quadratic. This means that the number of digits of accuracy in our approximation roughly doubles with each iteration.

2.2 Contractions and iterated function systems

Contraction mappings are very useful tools, but their dynamics appears to be boring. Starting with any point x in the basin of contraction X, repeated iteration with the contraction mapping T inevitably leads to the unique limit point $x_\infty \in X$. If we were to consider points x outside of X we could find all kinds of interesting dynamical behavior: periodic systems, chaotic systems (sensitive dependence on initial conditions) etc. However, in this course we shall not go that route. We shall stick with contractions but consider a *system* of contraction mappings where a rich dynamical behavior (with lots of applications) emerges. The individual contractions can be very simple (we shall usually restrict to linear affine transformations); the richness of the approach emerges from forming systems of these simple maps.

Let $V_n \equiv R_n$ be the space of real n-tuples $v = (v_1, \cdots, v_n)$ with inner product $(u, v) = \sum_{j=1}^n u_n v_n$, the usual dot product. Let X be a closed subset of V_n and $\mathcal{T} = \{T_1, \cdots T_r\}$ be a finite set of contraction mappings $T_i : X \to X$. Starting with any $x \in X$ we can apply the maps in $\mathcal{T}$ to x, in arbitrary orders and including repetition, to obtain points of the form $T_{i_1} T_{i_2} \cdots T_{i_k} x$. We can associate a *word* $i_1 i_2 \cdots i_k$ to each individual mapping of this form. Each letter i_ℓ in a word is taken from the alphabet $\{1, 2, \cdots r\}$. The set of all possible points $T_{i_1} T_{i_2} \cdots T_{i_k} x$ as $i_1 i_2 \cdots i_k$ runs over all words in the alphabet is the *orbit* $\mathcal{O}(x)$. What we are desribing is an *iterated function system* or IFS. Our main interest is in the structure of the orbits of the IFS.

The contraction mappings of an IFS can be arbitrary; in practice most examples are linear affine transformations. A *linear affine transformation* $T : V_n \to V_n$ is defined by $Tv = Av + b$ where A is a real $n \times n$ matrix and b is a real n-tuple. In components, $(Tv)_j = \sum_{k=1}^n A_{jk} v_k + b_j$, $j = 1, 2, \cdots, n$. Then $||Tv - Tu|| = ||A(v - u)|| \le ||A|| \cdot ||v - u||$, so T is a contraction mapping if the operator norm of A is < 1. Thus if $||A|| < 1$ the fixed point equation $(I - A)x = b$ has a unique solution $x = x_b$, i.e., for fixed A there is a unique solution x_b for every b. This means that the rank of the $n \times n$ matrix $I - A$ is n, so $(I - A)^{-1}$ exists and $x = (I - A)^{-1} b$.

Among the linear affine transformations there are two classes that will be of special interest to us. The first class consists of *isometries*. An isometry $Tv = Ov + b$ has the property that $||Tu - Tv|| = ||u - v||$ for all vectors $u, v \in V_n$, i.e., T is length preserving and $||Ov|| = ||v||$. Since

$$(u, v) = \frac{1}{4}(||u + v||^2 - ||u - v||^2)$$

it follows that $(Ou, Ov) = (u, v)$. Thus O preserves length and inner products. But the dot product $(u, v) = u \cdot v = ||u||\ ||v|| \cos\theta$ where θ is the angle between the vectors u and v. Thus, isometries preserve both length and angle. (Since isometries preseve length they are *not* contractions.) it follows directly form the property that $(Ou, Ov) = (u, v)$ for all $u, v \in V_n$ that the matrix elements of the matrix O are characterized by the equation $\sum_{k=1}^n O_{kj} O_{k\ell} = \delta_{j\ell}$ for $j, \ell = 1, 2, \cdots, n$, i.e., O is an *orthogonal* matrix.

The second class of linear affine transformations of special interest in this course is the class of *similitudes.* A similitude $Tv = Av + b$ has the property that there is a fixed positive constant r such that $||Tu - Tv|| = r||u - v||$ for all $u, v \in V_n$. It is easy to show that if T is a similitude then $Tv = rOv + b$, where O is an orthogonal matrix. It follows that T is a similitude with scaling r if and only if $T = r\tilde{T}$ where $\tilde{T}$ is an isometry. (A similitude with scaling factor $r < 1$ is a contraction.) Thus similitudes preserve angle and the ratios of lengths. They are the similarity transformations of Euclidean geometry, i.e., two geometrical objects are similar if and only if one object can be mapped $1 - 1$ onto the other by a similitude.

A general linear affine transformation $T : V_n \to V_n$ given by $Tv = Av + b$ is uniquely determined by its action on any given set of $n + 1$ points $v_j = (v_{j1}, \cdots, v_{jn})$, $j = 1, \cdots, n + 1$ such that the points don't all lie on the same hyperplane $\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta = 0$, i.e., such that $\det E \neq 0$ where E is the $(n + 1) \times (n + 1)$ matrix

$$\begin{pmatrix} v_{11} & \cdots & v_{1n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n+1,1} & \cdots & v_{n+1,n} & 1 \end{pmatrix}.$$

If we are given that $Tv_j = w_j$ for $j = 1, \cdots, n + 1$, then $Av_j + b = w_j$. Defining the $(n + 1) \times n + 1$ augmented matrix

$$A' = \begin{pmatrix} A \\ b \end{pmatrix}$$

and the $(n + 1) \times n$ matrix $W = (w_{jk})$ where $1 \leq j \leq n + 1$, $1 \leq k \leq n$, we see that the conditions on A and b can be written in the form $EA' = W$ so, since E is nonsingular, $A' = E^{-1}W$.

In two dimensions, where most of our examples occur, an affine transformation is uniquely determined by its action on 3 noncolinear points: $v_1, v_2, v_3 \to w_1, w_2, w_3$. Thus T maps the triangle $\Delta v_1 v_2 v_2$ onto the triangle $\Delta w_1 w_2 w_3$ with vertex v_j going to vertex w_j, and T is uniquely determined by that action. If T is a similitude in two dimensions then, it is almost uniquely detrmined by its action on 2 distinct points v_1, v_2. Indeed suppose $Tv_1 = w_1, Tv_2 = w_2$. Then for any $v \in V_2$ we must have $\Delta v_1 v_2 v \sim \Delta w_1 w_2 Tv$. There are just 2 possibilities for Tv. In one case T is orientation preserving ($\det A > 0$, i.e., obtained by translation, rotation, dilation alone) and in the other case T is orientation reversing ($\det A < 0$, i.e., involves a reflection).

Example 8 *The Cantor set. Here we choose* $X = [0,1]$ *with the usual Euclidean metric, and* $r = 2$*. The first contraction mapping is* $T_1(t) = t/3$ *for* $t \in [0,1]$*. Here the fixed point is* $t_1 = 0$ *and the contraction factor is* $\frac{1}{3}$*. The second contraction mapping is* $T_2(t) = t/3 + 2/3$ *for* $t \in [0,1]$*. Now the fixed point is* $t_2 = 1$ *and the contraction factor is again* $\frac{1}{3}$*. We start with the set* $A_0 = X = [0,1]$*. Note that* $T_1A_0 = [0, \frac{1}{3}], T_2A_0 = [\frac{2}{3}, 1]$*, so that the affine transformation* T_1 *maps the interval* $[0,1]$ *to the interval* $[0, \frac{1}{3}]$*, whereas* T_2 *maps the interval* $[0,1]$ *to the interval* $[\frac{2}{3}, 1]$*. For any set* $A \subset X$ *we define the set* TA *by* $TA = T_1A \cup T_2A$*. Then*

$$A_1 = TA_0 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1],$$

$$A_2 = TA_1 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1],$$

and so on. Note that we are, in effect, repeating the "throw out the middle third" construction of the Cantor set C*. Indeed it isn't difficult to show that* C *is the unique fixed point of the contraction mapping* T*:*

$$C = TC = T_1C \cup T_2C$$

and that $A_n \to C$ *in the Hausdorff metric as* $n \to \infty$*. This expression exhibits the self-similarity of* C *under each of the contraction mappings* T_1 *and* T_2*.*

Example 9 *The Sierpinski gasket. Here we choose* $X = [0,1]^2$ *(i.e.,* $X = \{(x,y) : 0 \le x, y \le 1\}$*, with the usual Euclidean metric, and* $r = 3$*. With* $v = (x,y)$*, the contraction mappings are*

$$T_1v = (\frac{1}{2}x, \frac{1}{2}y), \qquad T_2v = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y),$$

$$T_3v = (\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{1}{4}\sqrt{3}).$$

The contraction factor for each map is $\frac{1}{2}$ *and the fixed points are* $(0,0)$*,* $(1,0)$*,* $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$*, respectively. Notice that these fixed points lie on the vertices of an equilateral triangle* A_0*. Starting with the set* A_0*, note that the equilateral triangle* T_1A_0 *lies within* A_0*, has sides parallel to those of* A_0*, but with half the length, and shares the vertex* $(0,0)$*. The same can be said of the equilateral triangles* T_2A_0 *and* T_2A_0*, except that the shared vertices are* $(1,0), (\frac{1}{2}, \frac{1}{2}\sqrt{3})$*,* *respectively. Here, A_0 can be decomposed into 4 congruent equilateral triangles. It follows that $A_1 = TA_0 = T_1A_0 \cup T_2A_) \cup T_3A_0$ is the original figure with the the middle triangle removed. Iterating this map we get convergence in the Hausdorff metric to a unique fixed point $\breve{A}$ called the Sierpinski gasket (see the figure on my web site). Then*

$$\breve{A} = T\breve{A} = T_1\breve{A} \cup T_2\breve{A} \cup T_3\breve{A}$$

Again this expression exhibits the self-similarity of $\breve{A}$ under each of the contraction mappings T_j, $j = 1, 2, 3$.

Example 10 *Spleenwort fern. Here we take $X = \{(x, y) : -3 \leq x \leq 3, 0 \leq y \leq 10\}$, with the usual Euclidean metric, and $r = 4$. The contraction mappings are*

$$T_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.00 \\ 1.60 \end{pmatrix} \tag{2.2}$$

$$T_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.44 \end{pmatrix} \tag{2.3}$$

$$T_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.00 \\ 1.60 \end{pmatrix} \tag{2.4}$$

$$T_4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.00 & 0.00 \\ 0.00 & 0.16 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.5}$$

The fixed point looks very much like the black spleenwort fern. (See the image on my website.) This IFS needs some explanation. The fixed point of T_1, $(x, y) = (640/241, 2400/241) \sim (2.6556, 9.9585)$ is the tip of the fern. T_1 is a similitude that maps the entire fern to the part of the fern starting at $(0, 1.6)$, i.e., the portion that omits the bottom leaves on the left and right hand sides. T_2 maps the fern onto the bottom right hand leaf, reversing orientation. T_3 maps the fern onto the bottom left hand leaf. The degenerate affine transformation T_4 maps the entire leaf onto the stem from $(0, 0)$ to $\sim (0, 1.5934) \sim (0, 1.6)$. The fixed point $(0, 0)$ of T_4 is the base of the stem.

This example, produced by Barnsley in 1984, produced some excitement, for it held out the promise that IFS could play an effective role in image compression. The IFS in this example is completely determined by only 24 numbers. Yet when the algorithm is iterated it yields a fixed point of great complexity that is a realistic representation of the black spleenwort fern. (Even *better in some sense because the fractal has self-simularity.) In contrast a graphics file depicting a black spleenwork fern would be of the order of a megabyte and would not display self-similarity at all resolutions. We shall see that wavelets are a better* **general purpose** *tool for image compression than are fractals. However, for certain special purposes fractal image compression is astounding.*

Now let us return to a general iterated function system $\mathcal{T} = \{T_1, \cdots T_r\}$ acting on a closed subset X of V_n. Rather looking at the orbit of each point $x \in X$ separately, we will start with a compact set $A \in K(X)$ and construct the set TA where

$$TA = T_1 A \cup T_2 A \cup \cdots \cup T_r A.$$

Our strategy will be as follows:

1. We will show that the metric space $K(X)$ with the Hausdorff metric d_H is complete.

2. We will show that TA is a compact set, i.e., $TA \in K(X)$. Thus we can consider T as a mapping from $K(X)$ into itself – $T : K(X) \to K(X)$.

3. We will then show that T is a contraction mapping on $K(X)$, with respect to the Hausdorff metric d_H. Here the points in this metric space are compact sets.

4. We can then iterate the mapping T and investigate the behavior of the iterate $T^n A$ as $n \to \infty$. By the contraction mapping principle T has a fixed point $\breve{A}$, a unique nonempty compact set to which all the iterates converge.

5. Since $\breve{A}$ is a fixed point of T we have

$$T\breve{A} = T_1\breve{A} \cup T_2\breve{A} \cup \cdots \cup T_r\breve{A}. \tag{2.6}$$

6. The limit set $\breve{A}$ with the property (2.6) is called a *fractal*. Note that if the contractions T_ℓ are similitudes then (2.6) shows that each component $T_\ell\breve{A}$ of $\breve{A}$ is similar to $\breve{A}$ itself. This self-similarity is particularly striking when the components $T_\ell\breve{A}$ are mutually nonintersecting.

We begin our theoretical development by pointing out the following (easily derived) relation between the Hausdorff metric and the distance between a point and a set:

Lemma 40 *Let $A, B \in K(X)$. For $\epsilon > 0$ let*

$$A_\epsilon = \{x \in X : \mathrm{dist}(x, A) \leq \epsilon\}.$$

Then $d_H(A, B) < \epsilon$ if and only if $A \subset B_\epsilon$ **and** *$B \subset A_\epsilon$.*

Theorem 50 *(Completeness of Hausdorff metric) If X is closed, the metric space $K(X)$ is complete.*

PROOF: Let A_n be a Cauchy sequence of nonempty compact sets in $K(X)$. We must show that there is a nonempty compact set $A \in K(X)$ such that $d_H(A_n, A) \to 0$ as $n \to \infty$. In analogy with Fatou's lemma of integration theory, we define

$$A = \cap_{k\geq 1}\overline{\cup_{i\geq k} A_i} = \cap_{k\geq 1} D_k.$$

- A is nonempty and compact: The closed sets $D_1 \supset D_2 \supset \cdots$ are monotone decreasing. For anu $\epsilon > 0$ there is a postive integer N so that $d_H(A_i, A_j) < \epsilon$ for $i, j \geq N$. Thus $A_i \subset (A_N)_\epsilon$, for $i \geq N$. Thus $A \subset D_N \subset (A_N)_\epsilon$ and since D_N is bounded. It follows that D_i is compact for all $i \geq N$, ao A (the inersection of a decreasing sequence of nonempty compact sets) is compact and nonempty.

- $d_H(a_n, A) \to 0$ as $n \to \infty$: Since $d_H(A_N, A_i) < \epsilon$ for $i \geq N$ we have not only $A_i \subset (A_N)_\epsilon$ but also $A_N \subset (A_i)_\epsilon$, so

  $$A \subset (AQ_N)_\epsilon \subset ((A_i)_\epsilon)_\epsilon = (A_i)_{2\epsilon}.$$

  If we can show that $A_i \subset (A)_{2\epsilon}$ then it will follow that $d_H(A, A_i) < 2\epsilon$. If we choose $a_i \in A_i$ for fixed i, and $j > i$ we have $d_H(A_i, A_j) < \epsilon$ so we can find $a_j \in A_j$ such that $||a_i - a_j|| < \epsilon$. Since all these points $\{a_j\}$ lie in the bounded set $(A_N)_\epsilon$ there must be a convergent subsequence a_{j_k} such that $a_{j_k} \to a$ as $k \to \infty$. Then $||a_j - a|| \leq \epsilon$. Now $a_{j_k} \in D_n$ for all $j_k \geq n$, and since D_n is closed we have $a \in D_n$ for all n, or $a \in A$. This means that $\mathrm{dist}(a_j, A) \leq \epsilon$ for any $a_j \in A_j$. Thus $A_j \subset \overline{A_\epsilon} \subset A_{2\epsilon}$, so $d_H(A_j, A) < 2\epsilon$ for $j > N$, and $A_j \to A$ as $j \to \infty$.

Q.E.D.

Now consider again a general iterated function system $\mathcal{T} = \{T_1, \cdots T_r\}$ acting on a closed subset X of V_n. For any compact set $A \in K(X)$ we define the map $T : K(X) \to K(X)$ by

$$TA = T_1 A \cup T_2 A \cup \cdots \cup T_r A.$$

In particular from the results of section 1.3.1 se see that T_jA is a compact set since T_j is a continuous function and then TA is compact since it is a finite union of compact sets.

Lemma 41 *If $A_1, \cdots, A_r$ and $B_1, \cdots, B_r$ belong to $K(X)$*

$$d_H(A_1 \cup \cdots \cup A_r, B_1 \cup \cdots \cup B_r) \le \max\{d_H(A_1, B_1), \cdots d_H(A_r, B_r)\}.$$

PROOF: Since $A = A_1 \cup \cdots \cup A_r$ and $B = B_1 \cup \cdots \cup B_r$ are compact, there exists either an $a \in A$ such that $\mathrm{dist}(a, B) = d_H(A, B)$ or a $b \in B$ such that $\mathrm{dist}(b, A) = d_H(A, B)$. Without loss of generality we can assume the former is true. Then $a = a_j \in A_j$ for some $j = 1, \cdots, r$, and we have

$$\begin{aligned} d_H(A, B) = \mathrm{dist}(a_j, B) = \mathrm{dist}(a_j, B_1 \cup \cdots \cup B_r) \le \mathrm{dist}(a_j, B_j) \\ \le \max\{\mathrm{dist}_{a_1 \in A_1}(a_1, B_1), \cdots, \mathrm{dist}_{a_r \in A_r}(a_r, B_r)\} \\ \le \max\{d_H(A_1, B_1), \cdots d_H(A_r, B_r)\}. \end{aligned}$$

Q.E.D

Theorem 51 *Let $K(X)$ and T be defined as above where the $T_1, \cdots, T_r$ are contraction mappings on X with contraction constants $c_1, \cdots, c_r$ respectively. Then T is a contraction mapping with respect to the Hausdorf metric on $K(X)$, with contraction constant $c = \max\{c_1, \cdots, c_r\}$, and there is a unique compact set $\breve{A}$ (fixed point) such that*

$$\breve{A} = T\breve{A} = T_1\breve{A} \cup T_2\breve{A} \cup \cdots \cup T_r\breve{A}.$$

For any $B \in K(X)$ we have the convergence estimates

$$d_H(T^k B, A) \le c^k d_H(B, A) \le \frac{c^k}{1-c} d_H(TB, B).$$

PROOF: Once we demonstrate that T is a contraction mapping with contraction constant c, the convergence estimates will follow immediately from the Banach contraction mapping theorem (theorem 49). Let $A, B \in K(X)$. From the supremum property of the Hausdorff metric, given any $a \in A$ we can find a $b \in B$ such that $||a - b|| \le d_H(A, B)$, hence

$$||T_j a - T_j b|| \le c_j ||a - b|| \le c_j d_H(A, B).$$

This shows that $\sup_{a\in A} \mathrm{dist}(T_j a, T_j B) \le c_j d_H(A,B)$. A similar argument, starting from a given $b \in B$ shows that $\sup_{b\in B} \mathrm{dist}(T_j b, T_j A) \le c_j d_H(B,A) = c_j d_H(A,B)$. Thus

$$d_H(T_jA, T_jB) \le c_j d_H(A,B).$$

Then from the preceding lemma we have

$$d_H(TA,TB) = d_H(T_1A \cup \cdots \cup T_rA, T_1B \cup \cdots \cup T_rB)$$

$$\le \max\{d_H(T_1A,T_1B), \cdots, d_H(T_rA,T_rB)\}$$

$$\le \max\{c_1 d_H(A,B), \cdots, c_r d_H(A,B)\}$$

$$= \max\{c_1, \cdots, c_r\} d_H(A,B) = c d_H(A,B).$$

Q.E.D.

Given a IFS T, we know that from any initial compact set the iterates of the system will eventually converge to a unique target set. Now we look at some of the details of that convergence and consider the effect of chosing different initial sets. One way to proceed is to start with an initial set A such that $TA \subset A$. (This was the case with our Cantor set and Sierpinski gasket examples.) Then each of the iterates A_j will contain the target set $\breve{A}$ and the target will gradually emrge by subtraction of points.

Lemma 42 *If T is a contraction of $K(X)$ into itself and $A \in K(X)$ is an initial set such that $TA \subset A$, then the fixed point of T is the set $\breve{A} = \cap_{k\ge 0} T^k A$.*

PROOF: A simple induction argument shows that $T^{k+1}A \subset T^k A$ for all $k \ge 0$. Then by theorem 51 the sequence $\{T^k A\}$ is Cauchy with respect to the Hausdorff metric and $T^k A \to \breve{A}$, the uniqued fixed point as $k \to \infty$. Further from the proof of the Hausdorff completeness result (theorem 50) with $A_k = T^k A$ we have

$$\breve{A} = \cap_{k\ge 1} \overline{\cup_{j\ge k} T^j A} = \cap_{k\ge 0} T^k A.$$

Q.E.D.

Of course, we can start with *any* nonempty compact set A and end up with the unique fixed point $\breve{A}$. However, the rate of convergence will be affected by our initial choice. Note also that if y is a fixed point of one of the contractions T_j, i.e. $T_j y = y$, then $y \in \breve{A}$. Thus if $v \in A$ then it will remain in $T^k A$ for all iterations. To strengthen this result, recall that starting with any $x \in X$ we can apply the maps in $\mathcal{T}$ to x, in arbitrary orders and including repetition, to obtain points of the form $T_{i_1}T_{i_2}\cdots T_{i_k}x$. We associate a *word* $w = i_1 i_2 \cdots i_k$ to each such mapping. Each letter i_ℓ in a word is taken from the alphabet $\{1, 2, \cdots r\}$. The set of all possible points $T_{i_1}T_{i_2}\cdots T_{i_k}x$ as $i_1 i_2 \cdots i_k$ runs over all words in the alphabet is the *orbit* $\mathcal{O}(x)$. Now we can state the further result

Theorem 52 *For each word $w = i_1 i_2 \cdots i_k$ there is a unique fixed point x_w of the mapping $T_w = T_{i_1}T_{i_2}\cdots T_{i_k}$. Here $x_w \in \breve{A}$ and the set of all fixed points x_w, as w runs over all words, is dense in $\breve{A}$. Further, if $a \in \breve{A}$, then $\mathcal{O}(a)$ is dense in $\breve{A}$.*

PROOF: We carry out the proof in a series of simple steps:

- T_w has a unique fixed point x_w. Indeed is easy to show that a k-fold product of contractions is a contraction, and if T has contraction factor c then T_w must have a contraction factor no greater than c^k.

- $x_w \in \breve{A}$. Note that for any $x \in X$ we have $T_w^k \to x_w$ as $k \to \infty$. Choose $x \in \breve{A}$. Then $T_w^k \in \breve{A}$ for all k and $T_w^k \to x_w$. Since $\breve{A}$ is closed it follows that $x_w \in \breve{A}$.

- The set $B = \{x_w\}$, where w ranges over all words, is dense in $\breve{A}$. Note that $T^k\breve{A} = \breve{A}$ for all $k \geq 0$, so

$$\breve{A} = T\breve{A} = \cup_{i=1,\cdots,r}T^i\breve{A}$$

$$\breve{A} = T^2\breve{A} = \cup_{i,j=1,\cdots,r}T^iT^j\breve{A}.$$

  etc. Hence

$$\breve{A} = T^k\breve{A} = \cup_{i_1,i_2,\cdots,i_k=1,\cdots,r}T^{i_1}T^{i_2}\cdots T^{i_k}\breve{A}.$$

  Thus for any $a \in \breve{A}$ and any $k \geq 0$ we can always find a word w_k of length k and an $a_k \in \breve{A}$ such that $a = T_{w_k}a_k$. Since $\breve{A} \subset V_n$ is compact, it is also bounded: there is an $M > 0$ such that $||x - y|| < M$ for all $x, y \in \breve{A}$. Now, given $\epsilon > 0$ choose the integer N so large that $c^N M < \epsilon$. Then for any $a \in \breve{A}$ we can find a word w of length N such that $a = T_w a'$ for some $a' \in \breve{A}$. Let $x_w \in \breve{A}$ be the fixed point of T_w, so $x_w \in B$. Then

$$||a - x_w|| = ||T_w a' - T_w x_w|| \leq c^N ||a' - x_w|| < \epsilon,$$

and B is dense in $\breve{A}$.

- If $a \in \breve{A}$, then $\mathcal{O}(a)$ is dense in $\breve{A}$. Indeed, it is elementary that 1) $T\mathcal{O}(a) \subset \mathcal{O}(a)$, 2) $\mathcal{O}(a) \subset \breve{A}$ and that $B = \overline{\mathcal{O}(a)}$ is compact. By continuity, $TB \subset B$. Thus we can apply lemma 42 to obtain $\breve{A} = \cap_{k \geq 1} T^k B \subset B$. Hence $B = A$.

Q.E.D.

We see from this result that if we start out with an intial set with only one point $A = \{a\}$ where $a \in \breve{A}$, say that a is a fixed point for one of the T_j, then all of the points in $\mathcal{O}(a)$ belong to $\breve{A}$ and $\overline{\mathcal{O}(a)} = \breve{A}$.

Clearly, the IFS process can be used for image compression of very special images that are obvously the fixed points of such processes. How do we recognise these fixed points? Given a general image, can we approximate it by the fixed point of a some iterated function system? Can one do this on the fly, by an algorithm, to obtain practical real-time image compression? We shall obtain partial answers to these questions.

First of all, we will use the Hausdorff metric to decide when one set or image is "close" to another. In practice one might wish to employ other metrics, but that will not concern us at this point. Our fractal image compression problem can initially be stated as follows: Given a compact set C and a tolerance $\epsilon > 0$, find an IFS $\mathcal{T}$ such that the fixed point $\breve{A}_T$ of T approximates C to within tolerance ϵ, $d_H(C, \breve{A}_T) < \epsilon$. We can solve this problem theoretically. Among the practical issues that arise are 1) how easy is it to automate the solution, 2) how fast is it, and 3) how much compression is achieved, i.e., how many paramters are needed to define the IFS.

The collage theorem (corollary 18) sheds some light on this problem. It says that $d_H(C, \breve{A}_T) \leq \frac{1}{1-c_T} d_H(C, TC)$ where c_T is the contraction constant of T. Thus if we can assure that c_T is always bounded away from 1, say $c_T \leq \frac{1}{2}$, then if we choose T such that $d_H(C, TC) < \frac{\epsilon}{2}$ we will have $d_H(C, \breve{A}_T) < \epsilon$ and a solution to our problem. Thus we can focus on finding contractions T such that $TC \sim C$. (Note: $d_H(C, TC)$ is called the *collage distance.*)

The following result shows that we can't expect miracles to occur using this procedure, i.e., we can't expect that $d_H(C, \breve{A}_T)$ will turn out to be much less than $d_H(C, TC)$.

Theorem 53 *(Anti-collage theorem) Let $\mathcal{X}$ be a closed subset of the complete metric space $\mathcal{M}$ and $T : \mathcal{X} \to \mathcal{X}$ a contractive mapping with constant c and unique fixed point $u_\infty \in \mathcal{X}$. Then for any $u \in \mathcal{X}$,*

$$\rho(u, u_\infty) \geq \frac{1}{1+c}\rho(u, Tu).$$

PROOF: By the triangle inequality,

$$\rho(u, Tu) \leq \rho(u, u_\infty) + \rho(u_\infty, Tu) = \rho(u, u_\infty) + \rho(Tu_\infty, Tu)$$

$$\leq \rho(u, u_\infty) + c\rho(u_\infty, u) = (1+c)\rho(u_\infty, u)$$

so $(1+c)\rho(u, u_\infty) \geq \rho(u, Tu)$. Q.E.D.

Thus for fractal image compression we have the estimate

$$\frac{1}{1+c_T} d_H(C, TC) \leq d_H(C, \breve{A}_T) \leq \frac{1}{1-c_T} d_H(C, TC).$$

Now how do we choose an IFS $\mathcal{T}$ to make $d_H(C, TC)$ small? Here is a simple approach that will work. The set TC in this construction is an example of a "collage" of C. It is the union of shrunken copies of C.

Theorem 54 *Let C be a compact set in V_n and let $\epsilon > 0$. Then there is an IFS $\mathcal{T} = \{T_1, \cdots, T_r\}$ such that $d_H(C, \breve{A}_T) < \epsilon$.*

PROOF: let $\epsilon' = \epsilon/2$. Since C is compact we can find a finite set of points $C_0 = \{c_1, \cdots, c_r\} \subset C$ such that the balls

$$B_{\epsilon'}(c_j) = \{u \in V_n : ||u - c_j|| < \epsilon', \quad j = 1, \cdots, r\}$$

cover C:

$$C \subset B = B_{\epsilon'}(c_1) \cup B_{\epsilon'}(c_2) \cup \cdots \cup B_{\epsilon'}(c_r).$$

Then we can find an $R > 2\epsilon'$ such that $B \subset B_R(\theta)$ where θ is the zero vector. Define the IFS by

$$T_j u = \frac{\epsilon'}{2R}(u - c_j) + c_j, \qquad j = 1 \cdots, r.$$

Note that T_j is a contraction, and a similitude, that shrinks the distance between any point u and $c_j \in C$ uniformly by the factor $c_T = \frac{\epsilon'}{2R} < \frac{1}{4}$.

Furthermore $Tc_j = c_j$. Now if $u \in C$ then u lies in one of the balls $B'_\epsilon(c_j)$, so $\mathrm{dist}(u, TC) < \epsilon'$. Further, if $v \in TC$ then $v = T_j u$ for some j and some $u \in C$, so

$$\mathrm{dist}(v, C) \leq ||T_j u - c_j|| = \frac{\epsilon'}{2R}||u - c_j||$$

$$\leq \frac{\epsilon'}{2R}(||u|| + ||c_j||) < \epsilon'.$$

Thus $d_H(C, TC) < \epsilon'$, so $d_H(C, \breve{A}_T) < \frac{1}{1-c_T} d_H(C, TC) < \frac{4}{3}\epsilon' < \epsilon$. Q.E.D.

Our approach to generating the fixed point, i.e., set, of an IFS has been deterministic. It is somewhat awkward in that for each iteration of the contraction algorithm one must keep track of many points. There is an alternative probabilistic approach, Barnsley's so-called "chaos game," in which one needs only to follow a single point in each iteration. Given an IFS $\mathcal{T} = \{T_1, \cdots, T_r\}$ on the space X start with any $v_0 \in X$. Then pick a number i_1 from the set $\{1, 2, \cdots, r\}$ randomly and compute $v_1 = T_{i_1} v_0$. Iterating this process, for $N = 1, 2, \cdots$ let $v_{n+1} = T_{i_{n+1}} v_n$ where i_{n+1} is chosen randomly from the set $\{1, 2, \cdots, r\}$. If v_0 is chosen so that $A = \{v_0\}$ belongs to the attractor $\breve{A}$, then the points v_{i_k} perform a random walk on the attractor, and with probabilty one, form a dense subset of the attractor. Even if v_0 doesn't belong to the attractor, the points v_{i_k} will come arbitrarily close to any point of the attractor, with probability one. We shall not pursue this approach because it would take us into realms of ergodic theory that are beyond the scope of this course.

2.2.1 Fractal image compression and IFSM

The preceding section suggests a fixed point method whereby images can be compressed via computer, but it is oversimplified in that it represents images via shapes alone. A more realistic model of a image would be to think of it as a photograph made up of pixels. Each point (x, y) in the photograph lies in the range $0 \leq x, y \leq 1$ so the space is $X = [0, 1]^2$. Let's assume that the pixels are laid out on a 512×512 array ($512 = 2^9$), and that each pixel has an associated discrete grey scale level chosen from among $256 = 2^8$ values $u(x, y) = 0, 1, \cdots, 255$, where, say, 0 is white and 255 is black. Thus the image is represented by the piecewise constant function $u(x, y)$ on X. One can think of $z = u(x, y)$ as defining a surface over X in three space. The problem of approximating an image becomes the problem of approximating the piecewise constant function u which takes integer values from 0 to 255. To measure the accuracy of the approximation we need to choose a norm on the function space. One possible choice is the L^1 norm for functions on X. With this norm the error made in approximating the image u by the image f would be $||u - f|| = \int_X |u(x, y) - f(x, y)| dx\ dy$. The original image u is defined by $2^9 \times 2^9$ integers, each integer chosen from 2^8 values. One wants to approximate u by f, to within a given tolerance ϵ, such that the amount of data needed to construct f is, say, 10% of the amount of data needed to define u.

Let $\mathcal{T} = \{T_1, \cdots, T_r\}$ be a set of one-to-one affine contraction maps on X, and let $|J_k|$ be the Jacobian of the contraction map $(z, w) = T_k(x, y)$. A new feature needed for fractal image compression is a set of grey level maps $\Phi = \{\phi_1, \phi_2, \cdots, \phi_r\}$. Each ϕ_j takes the affine form $\phi_j(t) = a_j t + b_j$, $a_j, b_j \in R^+$ and maps the positive real line R^+ to R^+. The IFS plus the grey level maps constitute an *iterated function system with grey level maps* or IFSM. Associated with this system is a *fractal transform* operator $T : L^1(X) \to L^1(X)$ and defined for each $u \in L^1(X)$ by

$$(Tu)(z, w)) = \sum_{j=1}^{r} f_j(z, w),$$

where the *fractal components* $f_j(z, w)$ are

$$f_j(z, w) = \begin{cases} \phi_j[u(T_j^{-1}(z, w)] & (z, w) \in T_j(X) \\ 0, & (z, w) \notin T_j(X). \end{cases}$$

Thus, if (z, w) is in the range of T_j, i.e., if $(z, w) = T_j(x, y)$ for some $(x, y) \in X$ (unique because T_j is one-to-one), then $f_j(z, w)$ is $u(x, y)$ filtered by the grey level map ϕ_k. If (z, w) is not in the range of T_j then $f_j(z, w) = 0$. Now a straight-forward computation in multivariable calculus gives the estimate

$$||Tu - Tv|| \leq C||u - v||, \qquad C = \sum_{j=1}^{r} |J_j| a_j.$$

Thus if $C < 1$ then T is a contractive mapping on the Banach space $L^1(X)$. The Banach contraction theorem implies that there exists a unique fixed point $\overline{u} \in L^1(X)$. Indeed, for any $u \in L^1(X)$ we have $||T^n u - \overline{u}|| \to 0$ as $n \to \infty$. This is the basis for a deterministic algorithm to generate approximations to $\overline{u}$.

We conclude that the fixed point $\overline{u}$ satisfies the equation $\overline{u} = T\overline{u}$, or

$$\overline{u}(z,w) = \sum_j{}' \phi_j[u(T_j^{-1}(z,w)]$$

where the prime denotes that for each point (z,w) the sum is taken over all values of j such that (z,w) is in the range of T_j. We see that the graph of $\overline{u}$ satisfies a self-tiling property in the sense that the graph is a sum of distorted copies of itself.

Note: In order that $\mathcal{T}$ be useful for image compression of an image defined on X, we want $(z,w) \in X$ to lie in the range R_j of some mapping T_j. Thus we require $\cup_j R_j \supset X$. On the other hand for simplest analysis of the image it is to our advantage to make the overlaps between pairs of sets R_j as small as possible. It is usual to require that the sets overlap only at their boundaries, i.e., only on sets of measure zero. Then the fixed point function $\overline{u}$ will lie in the same $L^1(X)$ equivalence class as a function for which there is no overlap.

Example 11 *We revisit a simple example in one space dimension: the almost perfect sneak, see Example 3. This is a three-map IFSM with $X = [0,1]$.*

$$\begin{array}{ll} T_1(x) = \frac{1}{3}x, & \phi_1(y) = \frac{1}{2}y \\ T_2(x) = \frac{1}{3}x + \frac{1}{3}, & \phi_2(y) = \frac{1}{2} \\ T_3(x) = \frac{1}{3}x + \frac{2}{3}, & \phi_3(y) = \frac{1}{2}y + \frac{1}{2}. \end{array}$$

Now $T_1(X) = [0, \frac{1}{3}]$ and $T_2(X) = [\frac{1}{3}, \frac{2}{3}]$, so these two sets overlap at the single point $\frac{1}{3}$. Similarly $T_2(X)$ and $T_3(X) = [\frac{2}{3}, 1]$ overlap at the single point $\frac{2}{3}$. [In fact, due to the overlap at the points $\frac{1}{3}$ and $\frac{2}{3}$ and the iterations of the T map, the fixed point $\overline{u}(x)$ will differ from the almost perfect sneak at a countable number of points, a set of measure zero. However, as an $L^1(X)$ function, $\overline{u}$ lies in the same equivalence class as the almost perfect sneak.

How can one make this procedure into an effective tool for approximating a given image u on X by the fixed point $\overline{u}$ of T? Note first that by the Collage Theorem, corollary 18, if we can find a fractal transform operator T such that the collage distance $||u - Tu||$ is small, then $||u - \overline{u}||$ will be small, where $\overline{u} = T\overline{u}$ is the unique fixed point of T. Thus we need to choose T such that the collage distance $||u - Tu||$ is small.

Assuming no overlap for any point (z,w) in the image there will be a unique j such that $(z,w) \in R_j$. In order that $u \sim Tu$ we want for any $(z_j, w_j) \in R_j$,

$$u(z_j, w_j) \sim \phi_j \left(u(T_j^{-1}(z_j, w_j)) \right)$$

where $\phi_j : R \to R$ is the greyscale map associated with T_j. The basic strategy is to fix the mappings T_j and associated sets R_j for all of our images, and to use 2.2.1 to determine the greyscale maps ϕ_j appropriate to the image u. Under favorable circumstances we could compute ϕ_j exactly by requiring 2.2.1 to be an equality. However, this isn't possible in general. Indeed there could be points $(z_j, w_j), (z'_j, w'_j)$ such that $u(T_j^{-1}(z_j, w_j) = u(T_j^{-1}(z'_j, w'_j)$ but $u(z_j, w_j) \neq u(z'_j, w'_j)$. In that case $\phi(u(T_j^{-1}(z_j, w_j))$ is undefinable. In practice, one requires that the greyscale maps belong to a restricted, parametrized family, such as the affine maps $\phi_j(t) = \alpha_j t + \beta_j$ and use least squares to determine the parameters α_j, β_j from the given image u. If the procedure is well designed, this will give us greyscale maps such that the collage distance is "small," in a sense that can be made precise. Then the distance $||u - \overline{u}||$ will also be small. We need to transmit only the parameters α_j, β_j of the greyscale maps. Then the synthesis algorithm will reconstruct the fixed point $\overline{u}$, a good approximation of the original image.

Chapter 3

The Fourier Transform

3.1 The transform as a limit of Fourier series

We start by constructing the Fourier series (complex form) for functions on an interval $[-\pi L, \pi L]$. The ON basis functions are

$$e_n(t) = \frac{1}{\sqrt{2\pi L}} e^{\frac{int}{L}}, \qquad n = 0, \pm 1, \cdots,$$

and a sufficiently smooth function f of period $2\pi L$ can be expanded as

$$f(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi L} \int_{-\pi L}^{\pi L} f(x) e^{-\frac{inx}{L}} dx \right) e^{\frac{int}{L}}.$$

For purposes of motivation let us abandon periodicity and think of the functions f as differentiable everywhere, vanishing at $t = \pm \pi L$ and identically zero outside $[-\pi L, \pi L]$. We rewrite this as

$$f(t) = \sum_{n=-\infty}^{\infty} e^{\frac{int}{L}} \frac{1}{2\pi L} \hat{f}(\frac{n}{L})$$

which looks like a Riemann sum approximation to the integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda t} d\lambda \qquad (3.1)$$

to which it would converge as $L \to \infty$. Here,

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt. \qquad (3.2)$$

Similarly the Parseval formula for f on $[-\pi L, \pi L]$,

$$\int_{-\pi L}^{\pi L} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi L} |\hat{f}(\frac{n}{L})|^2$$

goes in the limit as $L \to \infty$ to the *Plancherel identity*

$$2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda. \tag{3.3}$$

Expression (3.2) is called the *Fourier integral* or *Fourier transform* of f. Expression (3.1) is called the *inverse Fourier integral* for f. The Plancherel identity suggests that the Fourier transform is a $1-1$ norm preserving of the Hilbert space $L^2[-\infty, \infty]$ onto itself (or to another copy of itself). We shall show that this is the case. Forthermore we shall show that the poinwise convergence properties of the inverse Fourier transform are somewhat similar to those of the Fourier series. Although we could make a rigorous justification of the the steps in the Riemann sum approximation above, we will follow a different course and treat the convergence in the mean and pointwise convergence issues separately.

A second notation that we shall use is

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt = \frac{1}{\sqrt{2\pi}} \hat{f}(\lambda) \tag{3.4}$$

$$\mathcal{F}^*[g](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\lambda) e^{i\lambda t} d\lambda \tag{3.5}$$

Note that, formally, $\mathcal{F}^*[\hat{f}](t) = \sqrt{2\pi} f(t)$. The first notation is used more often in the engineering literature. The second notation makes clear that $\mathcal{F}$ and $\mathcal{F}^*$ are linear operators mapping $L^2[-\infty, \infty]$ onto itself in one view [and $\mathcal{F}$ mapping the *signal space* onto the *frequency space* with $\mathcal{F}^*$ mapping the frequency space onto the signal space in the other view. In this notation the Plancherel theorem takes the more symmetric form

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}[f](\lambda)|^2 d\lambda.$$

EXAMPLES:

1. The box function (or rectangular wave)

$$\Pi(t) = \begin{cases} 1 & \text{if } -\pi < t < \pi \\ \frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Then, since $\Pi(t)$ is an even function, we have

$$\hat{\Pi}(\lambda) = \sqrt{2\pi}\mathcal{F}[\Pi](\lambda) = \int_{-\infty}^{\infty} \Pi(t)e^{-i\lambda t}dt = \int_{-\infty}^{\infty} \Pi(t)\cos(\lambda t)dt$$

$$= \int_{-\pi}^{\pi} \cos(\lambda t)dt = \frac{2\sin(\pi\lambda)}{\lambda} = 2\pi \text{ sinc } \lambda.$$

Thus sinc λ is the Fourier transform of the box function. The inverse Fourier transform is

$$\int_{-\infty}^{\infty} \text{sinc}(\lambda)e^{i\lambda t}d\lambda = \Pi(t),$$

as follows from complex variable theory (or my wavelets notes). Furthermore, we have

$$\int_{-\infty}^{\infty} |\Pi(t)|^2 dt = 2\pi$$

and

$$\int_{-\infty}^{\infty} |\text{ sinc } (\lambda)|^2 d\lambda = 1$$

(from complex variable theory or my wavelets notes), so the Plancherel equality is verified in this case. Note that the inverse Fourier transform converged to the midpoint of the discontinuity, just as for Fourier series.

2. A truncated cosine wave.

$$f(t) = \begin{cases} \cos 3t & \text{if } -\pi < t < \pi \\ -\frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases}$$

Then, since the cosine is an even function, we have

$$\hat{f}(\lambda) = \sqrt{2\pi}\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{i\lambda t}dt = \int_{-\pi}^{\pi} \cos(3t)\cos(\lambda t)dt$$

$$= \frac{2\lambda\sin(\pi\lambda)}{(9-\lambda^2)}.$$

3. A truncated sine wave.

$$f(t) = \begin{cases} \sin 3\pi t & \text{if } -\pi \le t \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

Then, since the sine is an odd function, we have

$$\hat{f}(\lambda) = \sqrt{2\pi}\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = -i\int_{-\pi}^{\pi} \sin(3t)\sin(\lambda t)dt$$
$$= \frac{-6i\sin(\lambda)}{(9-4\lambda^2)}.$$

4. A triangular wave.

$$f(t) = \begin{cases} \pi + t & \text{if } -\pi \le t \le 0 \\ \pi - t & \text{if } 0 \le t \le \pi \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Then, since f is an even function, we have

$$\hat{f}(\lambda) = \sqrt{2\pi}\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = 2\int_{0}^{\pi} (\pi - t)\cos(\lambda t)dt$$
$$= \frac{2-2\cos\lambda}{\lambda^2}.$$

NOTE: The Fourier transforms of the discontinuous functions above decay as $\frac{1}{\lambda}$ for $|\lambda| \to \infty$ whereas the Fourier transforms of the continuous functions decay as $\frac{1}{\lambda^2}$. The coefficients in the Fourier series of the analogous functions decay as $\frac{1}{n}$, $\frac{1}{n^2}$, respectively, as $|n| \to \infty$.

3.1.1 Properties of the Fourier transform

Recall that

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = \frac{1}{\sqrt{2\pi}}\hat{f}(\lambda)$$
$$\mathcal{F}^*[g](t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\lambda)e^{i\lambda t}d\lambda$$

We list some properties of the Fourier transform that will enable us to build a repertoire transform from a few basic examples. Suppose that f, g belong to $L^1[-\infty,\infty]$, i.e., $\int_{-\infty}^{\infty}|f(t)|dt < \infty$ with a similar statement for g. We can state the following (whose straightforward proofs are left to the student):

1. $\mathcal{F}$ and $\mathcal{F}^*$ are linear operators. For $a, b \in C$ we have

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g], \quad \mathcal{F}^*[af + bg] = a\mathcal{F}^*[f] + b\mathcal{F}^*[g].$$

2. Suppose $t^n f(t) \in L^1[-\infty, \infty]$ for some positive integer n. Then

$$\mathcal{F}[t^n f(t)](\lambda) = i^n \frac{d^n}{d\lambda^n}\{\mathcal{F}[f](\lambda)\}.$$

3. Suppose $\lambda^n f(\lambda) \in L^1[-\infty, \infty]$ for some positive integer n. Then

$$\mathcal{F}^*[\lambda^n f(\lambda)](t) = i^n \frac{d^n}{dt^n}\{\mathcal{F}^*[f](t)\}.$$

4. Suppose the nth derivative $f^{(n)}(t) \in L^1[-\infty, \infty]$ and piecewise continuous for some positive integer n, and f and the lower derivatives are all continuous in $(-\infty, \infty)$. Then

$$\mathcal{F}[f^{(n)}](\lambda) = (i\lambda)^n \mathcal{F}[f](\lambda)\}.$$

5. Suppose nth derivative $f^{(n)}(\lambda) \in L^1[-\infty, \infty]$ for some positive integer n and piecewise continuous for some positive integer n, and f and the lower derivatives are all continuous in $(-\infty, \infty)$. Then

$$\mathcal{F}^*[f^{(n)}](t) = (-it)^n \mathcal{F}^*[f](t).$$

6. The Fourier transform of a translation by real number a is given by

$$\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a}\mathcal{F}[f](\lambda).$$

7. The Fourier transform of a scaling by positive number b is given by

$$\mathcal{F}[f(bt)](\lambda) = \frac{1}{b}\mathcal{F}[f](\frac{\lambda}{b}).$$

8. The Fourier transform of a translated and scaled function is given by

$$\mathcal{F}[f(bt-a)](\lambda) = \frac{1}{b}e^{-i\lambda a/b}\mathcal{F}[f](\frac{\lambda}{b}).$$

EXAMPLES

- We want to compute the Fourier transform of the rectangular box function with support on $[c, d]$:

$$R(t) = \begin{cases} 1 & \text{if } c < t < d \\ \frac{1}{2} & \text{if } t = c, d \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the box function

$$\Pi(t) = \begin{cases} 1 & \text{if } -\pi < t < \pi \\ \frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases}$$

has the Fourier transform $\hat{\Pi}(\lambda) = 2\pi \text{ sinc } \lambda$. but we can obtain R from Π by first translating $t \to s = t - \frac{(c+d)}{2}$ and then rescaling $s \to \frac{2\pi}{d-c} s$:

$$R(t) = \Pi(\frac{2\pi}{d-c} t - \pi \frac{c+d}{d-c}).$$

$$\hat{R}(\lambda) = \frac{4\pi^2}{d-c} e^{i\pi\lambda(c+d)/(d-c)} \text{sinc}(\frac{2\pi\lambda}{d-c}). \tag{3.8}$$

Furthermore we can check that the inverse Fourier transform of $\hat{R}$ is R, i.e., $\mathcal{F}^*(\mathcal{F})R(t) = R(t)$.

- Consider the truncated sine wave

$$f(t) = \begin{cases} \sin 3\pi t & \text{if } -\pi \leq t \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

with

$$\hat{f}(\lambda) = \frac{-6i\sin(\lambda)}{(9 - 4\lambda^2)}.$$

Note that the derivative f' of $f(t)$ is just $3g(t)$ (except at 2 points) where $g(t)$ is the truncated cosine wave

$$g(t) = \begin{cases} \cos 3t & \text{if } -\pi < t < \pi \\ -\frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases}$$

We have computed

$$\hat{g}(\lambda) = \frac{2\lambda \sin(\pi\lambda)}{(9 - 4\lambda^2)}.$$

so $3\hat{g}(\lambda) = (i\lambda)\hat{f}(\lambda)$, as predicted.

- Reversing the example above we can differentiate the truncated cosine wave to get the truncated sine wave. The prediction for the Fourier transform doesn't work! Why not?

3.1.2 Fourier transform of a convolution

There is one property of the Fourier transform that is of particular importance in this course. Suppose f, g belong to $L^1[-\infty, \infty]$.

Definition 42 *The convolution of f and g is the function $f * g$ defined by*

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t - x)g(x)dx.$$

Note also that $(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t - x)dx$, as can be shown by a change of variable.

Lemma 43 *$f * g \in L^1[-\infty, \infty]$ and*

$$\int_{-\infty}^{\infty} |f * g(t)|dt = \int_{-\infty}^{\infty} |f(x)|dx \int_{-\infty}^{\infty} |g(t)|dt.$$

SKETCH OF PROOF:

$$\int_{-\infty}^{\infty} |f * g(t)|dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x)g(t - x)|dx \right) dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |g(t - x)|dt \right) |f(x)|dx = \int_{-\infty}^{\infty} |g(t)|dt \int_{-\infty}^{\infty} |f(x)|dx.$$

Q.E.D.

Theorem 55 *Let $h = f * g$. Then*

$$\hat{h}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda).$$

SKETCH OF PROOF:

$$\hat{h}(\lambda) = \int_{-\infty}^{\infty} f * g(t)e^{-i\lambda t}dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)g(t-x)dx\right)e^{-i\lambda t}dt$$

$$= \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\left(\int_{-\infty}^{\infty} g(t-x)e^{-i\lambda(t-x)}dt\right)dx = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx\hat{g}(\lambda)$$

$$= \hat{f}(\lambda)\hat{g}(\lambda).$$

Q.E.D.

3.2 L^2 convergence of the Fourier transform

In this course our primary interest is in Fourier transforms of functions in the Hilbert space $L^2[-\infty,\infty]$. However, the formal definition of the Fourier integral transform,

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt \tag{3.9}$$

doesn't make sense for a general $f \in L^2[-\infty,\infty]$. If $f \in L^1[-\infty,\infty]$ then f is absolutely integrable and the integral (3.9) converges. However, there are square integrable functions that are not integrable. (Example: $f(t) = \frac{1}{1+|t|}$.) How do we define the transform for such functions?

We will proceed by defining $\mathcal{F}$ on a dense subspace of $f \in L^2[-\infty,\infty]$ where the integral makes sense and then take Cauchy sequences of functions in the subspace to define $\mathcal{F}$ on the closure. Since $\mathcal{F}$ preserves inner product, as we shall show, this simple procedure will be effective.

First some comments on integrals of L^2 functions. If $f, g \in L^2[-\infty,\infty]$ then the integral $(f,g) = \int_{-\infty}^{\infty} f(t)\overline{g}(t)dt$ necessarily exists, whereas the integral (3.9) may not, because the exponential $e^{-i\lambda t}$ is not an element of L^2. However, the integral of $f \in L^2$ over any finite interval, say $[-N,N]$ does exist. Indeed for N a positive integer, let $\chi_{[-N,N]}$ be the indicator function for that interval:

$$\chi_{[-N,N]}(t) = \begin{cases} 1 & \text{if } -N \le t \le N \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Then $\chi_{[-N,N]} \in L^2[-\infty,\infty]$ so $\int_{-N}^{N} f(t)dt$ exists because

$$\int_{-N}^{N} |f(t)|dt = |(|f|, \chi_{[-N,N]})| \le ||f||_{L^2}||\chi_{[-N,N]}||_{L^2} = ||f||_{L^2}\sqrt{2N} < \infty$$

Now the space of step functions is dense in $L^2[-\infty,\infty]$, so we can find a convergent sequence of step functions $\{s_n\}$ such that $\lim_{n\to\infty}||f-s_n||_{L^2}=0$. Note that the sequence of functions $\{f_N=f\chi_{[-N,N]}\}$ converges to f pointwise as $N\to\infty$ and each $f_N\in L^2\cap L^1$.

Lemma 44 *$\{f_N\}$ is a Cauchy sequence in the norm of $L^2[-\infty,\infty]$ and $\lim_{n\to\infty}||f-f_n||_{L^2}=0$.*

PROOF: Given $\epsilon>0$ there is step function s_M such that $||f-s_M||<\frac{\epsilon}{2}$. Choose N so large that the support of s_M is contained in $[-N,N]$, i.e., $s_M(t)\chi_{[-N,N]}(t)=s_M(t)$ for all t. Then $||s_M-f_N||^2=\int_{-N}^{N}|s_M(t)-f(t)|^2dt\le\int_{-\infty}^{\infty}|s_M(t)-f(t)|^2dt=||s_M-f||^2$, so

$$||f-f_N||-||(f-s_M)+(s_M-f_N)||\le||f-s_M||+||s_M-f_N||\le 2||f-s_M||<\epsilon.$$

Q.E.D.

Here we will study the linear mapping $\mathcal{F}:L^2[-\infty,\infty]\to\hat{L}^2[-\infty,\infty]$ from the signal space to the frequency space. We will show that the mapping is *unitary*, i.e., it preserves the inner product and is 1-1 and onto. Moreover, the map $\mathcal{F}^*:\hat{L}^2[-\infty,\infty]\to L^2[-\infty,\infty]$ is also a unitary mapping and is the inverse of $\mathcal{F}$:

$$\mathcal{F}^*\mathcal{F}=I_{L^2},\qquad \mathcal{F}\mathcal{F}^*=I_{\hat{L}^2}$$

where $I_{L^2},I_{\hat{L}^2}$ are the identity operators on L^2 and $\hat{L}^2$, respectively. We know that the space of step functions is dense in L^2. Hence to show that $\mathcal{F}$ preseves inner product, it is enough to verify this fact for step functions and then go to the limit. Once we have done this, we can define $\mathcal{F}f$ for any $f\in L^2[-\infty,\infty]$. Indeed, if $\{s_n\}$ is a Cauchy sequence of step functions such that $\lim_{n\to\infty}||f-s_n||_{L^2}=0$, then $\{\mathcal{F}s_n\}$ is also a Cauchy sequence (indeed, $||s_n-s_m||=||\mathcal{F}s_n-\mathcal{F}s_m||$) so we can define $\mathcal{F}f$ by $\mathcal{F}f=\lim_{n\to\infty}\mathcal{F}s_n$. The standard methods of Section 1.3 show that $\mathcal{F}f$ is uniquely defined by this construction. Now the truncated functions f_N have Fourier transforms given by the convergent integrals

$$\mathcal{F}[f_N](\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}f(t)e^{-i\lambda t}dt$$

and $\lim_{N\to\infty}||f-f_N||_{L^2}=0$. Since $\mathcal{F}$ preserves inner product we have $||\mathcal{F}f-\mathcal{F}f_N||_{L^2}=||\mathcal{F}(f-f_N)||_{L^2}=||f-f_N||_{L^2}$, so $\lim_{N\to\infty}||\mathcal{F}f-\mathcal{F}f_N||_{L^2}=0$. We write

$$\mathcal{F}[f](\lambda)=\text{l.i.m.}_{N\to\infty}\mathcal{F}[f_N](\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}f(t)e^{-i\lambda t}dt$$

where 'l.i.m.' indicates that the convergence is in the mean (Hilbert space) sense, rather than pointwise.

We have already shown that the Fourier transform of the rectangular box function with support on $[c, d]$:

$$R_{c,d}(t) = \begin{cases} 1 & \text{if } c < t < d \\ \frac{1}{2} & \text{if } t = c, d \\ 0 & \text{otherwise.} \end{cases}$$

is

$$\mathcal{F}[R_{c,d}](\lambda) = \frac{4\pi^2}{\sqrt{2\pi}(d-c)} e^{i\pi\lambda(c+d)/(d-c)} \text{sinc}(\frac{2\pi\lambda}{d-c}).$$

and that $\mathcal{F}^*(\mathcal{F})R_{c,d}(t) = R_{c,d}(t)$. (Since here we are concerned only with convergence in the mean the value of a step function at a particular point is immaterial. Hence for this discussion we can ignore such niceties as the values of step functions at the points of their jump discontinuities.)

Lemma 45

$$(R_{a,b}, R_{c,d})_{L^2} = (\mathcal{F}R_{a,b}, \mathcal{F}R_{c,d})_{\hat{L}^2}$$

for all real numbers $a \leq b$ and $c \leq d$.

PROOF:

$$(\mathcal{F}R_{a,b}, \mathcal{F}R_{c,d})_{\hat{L}^2} = \int_{-\infty}^{\infty} \mathcal{F}[R_{a,b}](\lambda)\overline{\mathcal{F}}[R_{c,d}](\lambda)d\lambda$$

$$= \lim_{N\to\infty} \int_{-N}^{N} \left(\mathcal{F}[R_{a,b}](\lambda) \int_{c}^{d} \frac{e^{i\lambda t}}{\sqrt{2\pi}} dt \right) d\lambda$$

$$= \lim_{N\to\infty} \int_{c}^{d} \left(\int_{-N}^{N} \mathcal{F}[R_{a,b}](\lambda) \frac{e^{i\lambda t}}{\sqrt{2\pi}} d\lambda \right) dt.$$

Now the inside integral is converging to $R_{a,b}$ as $N \to \infty$ in both the pointwise and L^2 sense, as we have shown. Thus

$$(\mathcal{F}R_{a,b}, \mathcal{F}R_{c,d})_{\hat{L}^2} = \int_{c}^{d} R_{a,b} dt = (R_{a,b}, R_{c,d})_{L^2}.$$

Q.E.D.

Since any step functions u, v are finite linear combination of indicator functions R_{a_j,b_j} with complex coeficients, $u = \sum_j \alpha_j R_{a_j,b_j}$, $v = \sum_k \beta_k R_{c_k,d_k}$ we have

$$(\mathcal{F}u, \mathcal{F}v)_{\hat{L}^2} = \sum_{j,k} \alpha_j \overline{\beta}_k (\mathcal{F}R_{a_j,b_j}, \mathcal{F}R_{c_k,d_k})_{\hat{L}^2}$$

$$= \sum_{j,k} \alpha_j \overline{\beta}_k (R_{a_j,b_j}, R_{c_k,d_k})_{L^2} = (u,v)_{L^2}.$$

Thus $\mathcal{F}$ preserves inner product on step functions, and by taking Cauchy sequences of step functions, we have the

Theorem 56 *(Plancherel Formula) Let $f, g \in L^2[-\infty,\infty]$. Then*

$$(f,g)_{L^2} = (\mathcal{F}f, \mathcal{F}g)_{\hat{L}^2}, \qquad ||f||^2_{L^2} = ||\mathcal{F}f||^2_{\hat{L}^2}$$

In the engineering notation this reads

$$2\pi \int_{-\infty}^{\infty} f(t)\overline{g}(t)dt = \int_{-\infty}^{\infty} \hat{f}(\lambda)\overline{\hat{g}}(\lambda)d\lambda.$$

Theorem 57 *The map $\mathcal{F}^* : \hat{L}^2[-\infty,\infty] \to L^2[-\infty,\infty]$ has the following properties:*

1. *It preserves inner product, i.e.,*

   $$(\mathcal{F}^*\hat{f}, \mathcal{F}^*\hat{g})_{L^2} = (\hat{f},\hat{g})_{\hat{L}^2}$$

   for all $\hat{f}, \hat{g} \in \hat{L}^2[-\infty,\infty]$.

2. *$\mathcal{F}^*$ is the adjoint operator to $\mathcal{F} : L^2[-\infty,\infty] \to \hat{L}^2[-\infty,\infty]$, i.e.,*

   $$(\mathcal{F}f, \hat{g})_{\hat{L}^2} = (f, \mathcal{F}^*\hat{g})_{L^2},$$

   for all $f \in L^2[-\infty,\infty]$, $\hat{g} \in \hat{L}^2[-\infty,\infty]$.

PROOF:

1. This follows immediately from the facts that $\mathcal{F}$ preserves inner product and $\overline{\mathcal{F}}[\overline{f}](\lambda) = \mathcal{F}^*[f](\lambda)$.

2. $$(\mathcal{F}R_{a,b}, R_{c,d})_{\hat{L}^2} = (R_{a,b}, \mathcal{F}^*R_{c,d})_{L^2}$$

   as can be seen by an interchange in the order of integration. Then using the linearity of $\mathcal{F}$ and $\mathcal{F}^*$ we see that

   $$(\mathcal{F}u, v)_{\hat{L}^2} = (u, \mathcal{F}^*v)_{L^2},$$

   for all step functions u, v. Since the space of step functions is dense in $\hat{L}^2[-\infty,\infty]$ and in $L^2[-\infty,\infty]$

Q.E.D.

Theorem 58 *1. The Fourier transform $\mathcal{F} : L^2[-\infty, \infty] \to \hat{L}^2[-\infty, \infty]$ is a unitary transformation, i.e., it preserves the inner product and is 1-1 and onto.*

2. The adjoint map $\mathcal{F}^* : \hat{L}^2[-\infty, \infty] \to L^2[-\infty, \infty]$ is also a unitary mapping.

3. $\mathcal{F}^*$ is the inverse operator to $\mathcal{F}$:

$$\mathcal{F}^*\mathcal{F} = I_{L^2}, \qquad \mathcal{F}\mathcal{F}^* = I_{\hat{L}^2}$$

where $I_{L^2}, I_{\hat{L}^2}$ are the identity operators on L^2 and $\hat{L}^2$, respectively.

PROOF:

1. The only thing left to prove is that for every $\hat{g} \in \hat{L}^2[-\infty, \infty]$ there is a $f \in L^2[-\infty, \infty]$ such that $\mathcal{F}f = \hat{g}$, i.e., $\mathcal{R} \equiv \{\mathcal{F}f : f \in L^2[-\infty, \infty]\} = \hat{L}^2[-\infty, \infty]$. Suppose this isn't true. Then there exists a nonzero $\hat{h} \in \hat{L}^2[-\infty, \infty]$ such that $\hat{h} \perp \mathcal{R}$, i.e., $(\mathcal{F}f, \hat{h})_{\hat{L}^2} = 0$ for all $f \in L^2[-\infty, \infty]$. But this means that $(f, \mathcal{F}^*\hat{h})_{L^2} = 0$ for all $f \in L^2[-\infty, \infty]$, so $\mathcal{F}^*\hat{h} = \Theta$. But then $||\mathcal{F}^*\hat{h}||_{L^2} = ||\hat{h}||_{\hat{L}^2} = 0$ so $\hat{h} = \Theta$, a contradiction.

2. Same proof as for 1.

3. We have shown that $\mathcal{F}\mathcal{F}^*R_{a,b} = \mathcal{F}^*\mathcal{F}R_{a,b} = R_{a,b}$ for all indicator functions $R_{a,b}$. By linearity we have $\mathcal{F}\mathcal{F}^*s = \mathcal{F}^*\mathcal{F}s = s$for all step functions s. This implies that

$$(\mathcal{F}^*\mathcal{F}f, g)_{L^2} = (f, g)_{L^2}$$

for all $f, g \in L^2[-\infty, \infty]$. Thus

$$([\mathcal{F}^*\mathcal{F} - I_{L^2}]f, g)_{L^2} = 0$$

for all $f, g \in L^2[-\infty, \infty]$. Thus $\mathcal{F}^*\mathcal{F} = I_{L^2}$. An analogous argument gives $\mathcal{F}\mathcal{F}^* = I_{\hat{L}^2}$.

Q.E.D.

3.3 The Riemann-Lebesgue Lemma and pointwise convergence

Lemma 46 *(Riemann-Lebesgue) Suppose f is absolutely Riemann integrable in $(-\infty,\infty)$ (so that $f \in L^1[-\infty,\infty]$), and is bounded in any finite subinterval $[a,b]$, and let α,β be real. Then*

$$\lim_{\alpha\to+\infty}\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)dt = 0.$$

PROOF: Without loss of generality, we can assume that f is real, because we can break up the complex integral into its real and imaginary parts.

1. The statement is true if $f = R_{a,b}$ is an indicator function, for
$$\int_{-\infty}^{\infty} R_{a,b}(t)\sin(\alpha t+\beta)dt = \int_a^b \sin(\alpha t+\beta)dt = \frac{-1}{\alpha}\cos(\alpha t+\beta)|_a^b \to 0$$
as $\alpha \to +\infty$.

2. The statement is true if f is a step function, since a step function is a finite linear combination of indicator functions.

3. The statement is true if f is bounded and Riemann integrable on the finite interval $[a,b]$ and vanishes outside the interval. Indeed given any $\epsilon > 0$ there exist two step functions $\overline{s}$ (Darboux upper sum) and $\underline{s}$ (Darboux lower sum) with support in $[a,b]$ such that $\overline{s}(t) \geq f(t) \geq \underline{s}(t)$ for all $t \in [a,b]$ and $\int_a^b |\overline{s}-\underline{s}| < \frac{\epsilon}{2}$. Then
$$\int_a^b f(t)\sin(\alpha t+\beta)dt = \int_a^b [f(t)-\underline{s}(t)]\sin(\alpha t+\beta)dt + \int_a^b \underline{s}(t)\sin(\alpha t+\beta)dt.$$
Now
$$|\int_a^b [f(t)-\underline{s}(t)]\sin(\alpha t+\beta)dt| \leq \int_a^b |f(t)-\underline{s}(t)|dt \leq \int_a^b |\overline{s}-\underline{s}| < \frac{\epsilon}{2}$$
and (since $\underline{s}$ is a step function, by choosing α sufficiently large we can ensure
$$|\int_a^b \underline{s}(t)\sin(\alpha t+\beta)dt| < \frac{\epsilon}{2}.$$
Hence
$$|\int_a^b f(t)\sin(\alpha t+\beta)dt| < \epsilon$$
for α sufficiently large.

4. The statement of the lemma is true in general. Indeed

$$|\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)dt| \leq |\int_{-\infty}^{a} f(t)\sin(\alpha t+\beta)dt|$$

$$+|\int_{a}^{b} f(t)\sin(\alpha t+\beta)dt| + |\int_{b}^{\infty} f(t)\sin(\alpha t+\beta)dt|.$$

Given $\epsilon > 0$ we can choose a and b such the first and third integrals are each $< \frac{\epsilon}{3}$, and we can choose α so large the the second integral is $< \frac{\epsilon}{3}$. Hence the limit exists and is 0.

Q.E.D.

Theorem 59 *Let f be a complex valued function such that*

- *$f(t)$ is absolutely Riemann integrable on $(-\infty,\infty)$ (hence $f \in L^1[-\infty,\infty]$).*
- *$f(t)$ is piecewise continuous on $(-\infty,\infty)$, with only a finite number of discontinuities in any bounded interval.*
- *$f'(t)$ is piecewise continuous on $(-\infty,\infty)$, with only a finite number of discontinuities in any bounded interval.*
- *$f(t) = \frac{f(t+0)+f(t-0)}{2}$ at each point t.*

Let

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt$$

be the Fourier transform of f. Then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda t}d\lambda$$

for every $t \in (-\infty,\infty)$.

PROOF: For real $L > 0$ set

$$f_L(t) = \int_{-L}^{L} \hat{f}(\lambda)e^{i\lambda t}d\lambda = \frac{1}{2\pi}\int_{-L}^{L}\left[\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx\right]e^{i\lambda t}d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\left[\int_{-L}^{L} e^{i\lambda(t-x)}d\lambda\right]dx = \int_{-\infty}^{\infty} f(x)\Delta_L(t-x)dx,$$

where

$$\Delta_L(x) = \frac{1}{2\pi}\int_{-L}^{L} e^{i\lambda x} d\lambda = \begin{cases} \frac{L}{\pi} & \text{if } x = 0 \\ \frac{\sin Lx}{\pi x} & \text{otherwise.} \end{cases}$$

Using the integral

$$I = \int_0^\infty \frac{\sin x}{x} dx = \pi \int_0^\infty \text{sinc } x \ dx = \frac{\pi}{2}, \tag{3.11}$$

which is proved in complex variable theory (or in the wavelets notes) we have,

$$f_L(t) - f(t) = \int_{-\infty}^{\infty} \Delta_L(t-x)[f(x) - f(t)]dx$$

$$\int_0^\infty \Delta_L(x)[f(t+x) + f(t-x) - 2f(t)]dx$$

$$= \int_0^\infty \{\frac{f(t+x) + f(t-x) - 2f(t)}{\pi x}\} \sin Lx \ dx$$

The function in the curly braces satisfies the assumptions of the Riemann-Lebesgue Lemma. Hence $\lim_{L\to+\infty}[f_L(t) - f(t)] = 0$. Q.E.D

Note: Condition 4 is just for convenience; redefining f at the discrete points where there is a jump discontinutiy doesn't change the value of any of the integrals. The inverse Fourier transform converges to the midpoint of a jump discontinuity, just as does the Fourier series.