GROUP THEORY AND DIFFERENTIAL EQUATIONS

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1. Historical Background

The theory of groups in the first half of the nineteenth century played a central role in the development of mathematics. In the theory of equations the analysis of substitution groups on the roots of algebraic equations by J. Lagrange, P. Ruffini, N. Abel, and E. Galois culminated in the proof of the insolvability of the general quintic equation by radicals.

In geometry, the projective group, as interpreted by J. Poncelet and A. Möbius, led to the creation of projective geometry as a structure independent of Euclidean geometry. Also the non-Euclidean geometries of N. Lobachewski, J. Bolya, K. Gauss, B. Riemann, S. Lie, and H. Helmhotz emphasized their own groups of motions rather than the Euclidean motion group. Even in Euclidean geometry W. Hamilton, W. Clifford, H. Grassmann, and A. Cayley were investigating new invariants of the group of rigid motions.

Thus the study of transformation groups and their invariants was well established in 1872 when F. Klein announced his program at Erlangen to cast all geometry in this intrinsic form.

In 1869 S. Lie wrote some notes on canonical forms for first order, non-linear, differential equations. He continued working, sometimes in conjunction with F. Klein, until in 1874 Lie obtained the basis for their differential invariants. The classical formulation of this theory is found in <u>Theorie der Transformationgruppen</u>, S. Lie and F. Engel, in three volumes, 1888-1893. The results immediately applicable to differential equations are in

<u>Differentialgleichungen</u> <u>mit</u> <u>Bekannten</u> <u>Infinitesimalen</u> <u>Transformationen</u>, S. Lie and F. Scheffers, 1891.

A simplified English version of this latter is

<u>An</u> <u>Introduction</u> <u>to</u> <u>the</u> <u>Lie</u> <u>Theory</u> <u>of</u> <u>One-parameter</u> <u>Groups</u>, A. Cohen, 1911.

Also an important contribution to the classical theory is found in the text by L. Bianchi, 1903 (reprinted in 1918).

The study of abstract (as distinct from transformation) Lie groups was pursued by W. Killing, E. Cartan, C. Chevalley, and L. Pontrjagin. A survey of this modern theory is found in the texts on <u>Lie</u> <u>groups</u> and <u>Topological</u> <u>Groups</u>, respectively, of the last two authors. In recent years important work on Lie groups has been done by A. Gleason, H. Yamabe and many others. The text on <u>Transformation</u> <u>Groups</u> by D. Montgomery and L. Zippin returns to the analysis of transformation groups, but uses the modern techniques of global topological groups rather than the local groups of S. Lie.

Later in this course we shall deal with the monodromy group and with the Galois or rationality group of a linear differential equation. The former concept was introduced by B. Riemann in 1856 in a paper on the hypergeometric equation and then developed by L. Fuchs, L. Schlesinger, D. Hilbert, G.D. Birkhoff, and quite recently by H. Röhrl. The rationality group was invented independently by E. Picard in 1883 and E. Vessiot in 1889. Recently this theory has been incorporated into differential algebra by J. Ritt and E. Kolchin.

2. <u>Examples</u> <u>of</u> <u>Different</u> <u>Types</u> <u>of</u> <u>Differential</u> <u>Equations</u> <u>and</u> <u>Transformation</u> <u>Groups</u>.

An ordinary differential equation, say in the real plane R^2, is geometrically a family or network of curves, the solution curves of the equation. We say two such differential equations are the same, or of the same type, in case a one-to-one differentiable map, of the domain of definition of the first equation onto the domain of definition of the second equation, carries the first curve network onto the second. That is, we change both dependent and independent variable and seek a simplified canonical form for the differential equation. The canonical coordinates, in which the differential equation is to assume the simplified form, is to be found from a study of the internal symmetries of the solution curve family. Technically, we find transformation groups, say acting on R^2, for which the curve family is an invariant (that is, the family as a unit is an invariant -- the individual solution curves are permuted about under the transformations). Using these transformation groups we try to bring the invariant differential equation into a simplified form from which it can, for example, be integrated by quadrature alone.

The classical theory of S. Lie is purely local and refers to a region in the plane R^2. For the first part of these lectures we follow the methods of Lie.

Examples of types of differential equations (no proofs).

1. $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^4$ is not equivalent to $\frac{d^2y}{dx^2} = 0$

(that is, no local diffeomorphism $(x,y) \to (u,v)$ carries this equation to $\frac{d^2u}{dv^2} = 0$). However, $y'' = A(x,y)\, y'^3$ is equivalent to $y'' = 0$ if $A_{xx} \equiv 0$.

2. $y'' = xy + \tan y'$ is not equivalent to $y'' = P(x)\, y' + Q(x)\, y + R(x)$.
3. $y' = f(x,y)$ is equivalent (locally) to $y' = 0$.

Examples of transformation groups on R^2.

1. $T_t : \begin{array}{l} y \to y_1 = y + t \\ x \to x_1 = x \end{array}$ (translations), $T_s T_t = T_{s+t}$ is the group property. An invariant of the geometry under this group is a curve family $y' = f(x)$, which is integrable by quadratures $y = y_0 + \int_{x_0}^{x} f(\tau)\, d\tau$

Here the slope $f(x)$ is independent of y and hence $y' = f(x)$ is invariant under this translation group.

2. $T_t : \begin{array}{l} x \to x_1 = x \cos t - y \sin t \\ y \to y_1 = x \sin t + y \cos t \end{array}$ (rotations), $T_s T_t = T_{s+t}$.

Invariants of the geometry are curve families $\dfrac{y - xy'}{x + yy'} = f(x^2 + y^2)$.

Note: $\tan \psi = \dfrac{y - xy'}{x + yy'}$ where ψ is the angle between radius vector and solution curve — this is invariant under a rotation about the origin.

3. $T_t : \begin{array}{l} x \to x_1 = x e^t \\ y \to y_1 = y e^t \end{array}$ (dilations), $T_s T_t = T_{s+t}$.

Invariants of geometry are $y' = f(y/x)$, homogeneous differential equations.

4. $T_t : \begin{array}{l} x \to x_1 = x + t \\ y \to y_1 = y \end{array}$ and $R_s : \begin{array}{l} x \to x_1 = x \\ y \to y_1 = y + s \end{array}$

(two parameter commutative group of translations with general group element $T_t R_s$). Invariant of geometry is $y'' = f(y')$. Here $f(x,y,y') = f(y')$ is invariant under the rigid translations of the group. Let $v = y'$ so $v' = f(v)$, integrate by quadratures to find $v(x)$ and $y = \int v(x)\,dx$.

5. $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $ad - bc \neq 0$ (general linear group).

This is a non-commutative 4 parameter group, $GL(2,R)$ acting on R^2 .

6. $\begin{array}{l} x_1 = ax + by + \alpha \\ y_1 = cx + dy + \beta \end{array}$ affine group on R^2 . This is a 6 parameter non-commutative group.

7. $x_1 = \dfrac{a_1 x + a_2 y + a_3}{a_7 x + a_8 y + a_9}$, $y_1 = \dfrac{a_4 x + a_5 y + a_6}{a_7 x + a_8 y + a_9}$, $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} \neq 0$

(projective group of 8 essential parameters). Take (x, y) near origin of R^2 and use only projective transformations near the identity -- this is local transformation group. Invariant of the geometry is $y'' = 0$, solutions are all lines.

8. $z \to z_1 = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ with complex numbers $\alpha\delta - \beta\gamma \neq 0$

This is the 6 (real, essential) parameter conformal local transformation group on a plane region. Invariant of the geometry is $\dfrac{d}{dx}\left[\dfrac{y''}{(1+y'^2)^{3/2}}\right] = 0$, curve family of all circles and lines.

3. EXAMPLES OF ABSTRACT GROUPS

Definition. A group G is a non-empty set of objects together with a binary law of composition satisfying:

1. There exists a unique element 0 such that
$a + 0 = 0 + a = a$ (or $\exists\, i \ni : ai = ia = a$)
2. For each a there exists a unique element $(-a)$ such that
$a + (-a) = (-a) + a = 0$ (or $\exists\, a^{-1} \ni : aa^{-1} = a^{-1}a = i$)
3. $(a + b) + c = a + (b + c)$ (or $(ab)c = a(bc)$).

The group is abelian or commutative if $a + b = b + a$ (or $ab = ba$ in the multiplicative notation).

Examples:

1. The real numbers with addition, R^1. Same as positive reals R^1_+ using multiplication.

Definition. A function $h: G \to H$ (into H) such that $h(g_1 g_2) = h(g_1) h(g_2)$

is a homomorphism of G into H. If h is one-to-one onto G, then h^{-1} is also a homomorphism of G onto H.

<u>Exercise.</u> $h(O_G) = O_H$ and $h(-a) = -h(a)$. Also $R^1 \to R^1_+$ by $h(x) = e^x$ is an isomorphism.

2. The circle S^1, complex numbers z with $|z| = 1$ under multiplication.

3. R^n with vector addition.

4. Set of all linear transformations of R^n onto R^n, pick basis in R^n, then this is isomorphic with $GL(n, R)$; $a_{ij} \cdot b_{jk} = c_{ik}$, $\det(a_{ij}) \neq 0$.

5. All affine transformations of line R^1, $x \to ax + b$, $a \neq 0$. Isomorphic with subgroup of $GL(2, R)$ of form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.

<u>Definition.</u> A subset $H \subset G$ which forms a group under the composition law of G is a subgroup of G.

6. All affine transformations of R^n, $x^i \to x_1^i = a^i_j x^j + b^i$, $\det(a^i_j) \neq 0$, isomorphic with subgroup of $GL(n+1, R)$ of form $\begin{pmatrix} a^i_j & b^i \\ 0 & 1 \end{pmatrix}$.

7. All linear transformations of R^n with Euclidean inner product preserving length. Pick orthonormal basis in R^n, then isomorphic with $O(n)$; $A = (a_{ij})$ with $a_{ij} \cdot a_{kj} = \delta_{ik}$ or $A^T = A^{-1}$. Subgroup with $\det = +1$ is $O_+(n)$ or $SO(n)$.

8. All linear transformations of R^n with Minkowski inner product. Pick orthonormal basis of R^n, then isomorphic with Lorentz group $L(n)$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{with} \quad \begin{aligned} -a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2 &= -1 \\ -a_{21}^2 + a_{22}^2 + \cdots + a_{2n}^2 &= 1 \\ &\vdots \\ -a_{n1}^2 + a_{n2}^2 + \cdots + a_{nn}^2 &= 1 \end{aligned}$$

and each pair of rows orthogonal,

$i \neq j$, $-a_{i1} a_{j1} + a_{i2} a_{j2} + \cdots + a_{in} a_{jn} = 0$.

Subgroup preserving orientation ($\det = +1$) and future time sense $a_{i1} > 1$ is proper Lorentz group $L_p(n)$.

9. All linear transformation of vector space R^{n+1} where two transformations are considered the same if they have the same action on a set of rays through the origin (on projective space P^n). Pick basis, then this is isomorphic with $GL(n+1,R)/C(n+1) = PG(n+1,R)$ where $C(n+1)$ are scalar matrices.

Let N be a subgroup of G . Then $g_1 \sim g_2$ in case the (left) cosets $g_1 N$ and $g_2 N$ "coincide" is an equivalence relation which yields the coset decomposition of G .

If gNg^{-1} is the coset N , for each $g \in G$, then N is normal (or invariant) in G and we define the quotient or factor group as follows:

The elements of G/N are the left cosets of N and the composition is well-defined by $(g_1 N)(g_2 N) = g_1 g_2 N$. Also $G \to G/N$ by $g \to gN$ is a homomorphism of G onto G/N .

If $h: G \to H$ is a homomorphism of G onto H , then $h^{-1}(0) = N$ is a normal subgroup of G and G/N is isomorphic with H in a natural way. These results are all contained in Pontrjagin, Topological Groups, Chapter I.

10. All quaternions with unit norm $a + bi + cj + dk$ where $i^2 = j^2 = k^2 = -1$ (and cyclic permutations). Also the real coefficients satisfy $a^2 + b^2 + c^2 + d^2 = 1$. This group is the spinor group Spin 3.

Note. Each of these above 10 examples are Lie groups. They bear a natural geometry, which is that of the curve, surface, or manifold and (locally near the origin) the points and their group multiplications are given easily in terms of a finite number of local coordinates.

Note that S^1 and R^1 are locally isomorphic (technical definition later) and this is also the case for $O_+(3)$ and Spin 3.

11. All homeomorphisms (one-to-one and bicontinuous maps) of plane region $\Theta \subset R^2$ onto Θ form a group — not a Lie group.

12. Consider the torus $T^2(\theta_1, \theta_2)$ with θ_1 in S^1 and θ_2 in S^1 and componentwise addition, that is $T^2 = S^1 \times S^1$. This is a Lie group locally isomorphic with the plane R^2. Consider the subgroup N of T^2 corresponding to the line $y = \sqrt{2}\, x$ in R^2. Then N is a one-to-one continuous image of R^1 in T^2 and N is dense in T^2. Let H be the smallest group in T^2 containing N and a point $P \in T^2 - N$. Then H is not a Lie group for it is connected but not locally connected.

4. One-parameter transformation groups on R^2.

Definition. Let Θ be an open set of R^2. For each $t \in R^1$ let T_t be a homeomorphism of Θ onto Θ

$$T_t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \varphi(t,x,y) \\ \psi(t,x,y) \end{pmatrix}.$$

Assume $\varphi(t,x,y)$ and $\psi(t,x,y) \in C^\infty$ in $R^1 \times \Theta$ and $T_s T_t = T_{s+t}$ that is

$$\varphi(s, \varphi(t,x,y), \psi(t,x,y)) = \varphi(s+t, x, y)$$
$$\psi(s, \varphi(t,x,y), \psi(t,x,y)) = \psi(s+t, x, y)$$

for all $s, t \in R^1$ and $(x,y) \in \Theta$.

Then the functions $\varphi(t,x,y)$, $\psi(t,x,y)$ define a one-parameter transformation group of R^1 acting on Θ.

Note. A one-parameter transformation group is described by a homomorphism of R^1 into the group of homeomorphisms of Θ. Thus $T_0 = I$ (identity map of Θ) and $T_{-t} = (T_t)^{-1}$.

Examples.

1. $x \to x$ on $\Theta = R^2$, translation group.
$y \to y + t$

2. $x \to x\cos t - y\sin t$ on R^2, rotation group.
   $y \to x\sin t + y\cos t$

3. $x \to xe^t$ on R^2, dialation group.
   $y \to ye^t$

<u>Definition.</u> A differential system (or vector field) $\mathcal{U}: \frac{dx}{dt} = f(x,y),\ \frac{dy}{dt} = g(x,y)$ in C^∞ in $\Theta \subset R^2$ is called an infinitesimal one-parameter transformation group on Θ.

<u>Remark.</u> Let $\mathcal{U}: \dot{x} = f(x,y), \dot{y} = g(x,y)$ be an infinitesimal transformation group on Θ and assume that each solution curve is defined in Θ for all $-\infty < t < \infty$ (this is always the case if Θ is a compact manifold). Let $\varphi(t,x_0,y_0), \psi(t,x_0,y_0)$ be the solution of $\mathcal{U}$ through (x_0,y_0) at $t = 0$. Then $\varphi(t,x,y), \psi(t,x,y)$ define a one-parameter transformation group $\{T_t\}$ on Θ, which we say is generated by $\mathcal{U}$. For certainly $\varphi(t,x,y), \psi(t,x,y) \in C^\infty$ in $R^1 \times \Theta$. Also

$$\frac{\partial}{\partial s}\varphi(s+t,x_0,y_0) = f(\varphi(s+t,x_0,y_0), \psi(s+t,x_0,y_0))$$

$$\frac{\partial}{\partial s}\psi(s+t,x_0,y_0) = g(\varphi(s+t,x_0,y_0), \psi(s+t,x_0,y_0)).$$

Thus, for each fixed t,

$$\varphi(s+t,x_0,y_0) = \hat{\varphi}(s)$$
$$\psi(s+t,x_0,y_0) = \hat{\psi}(s)$$

is the unique solution of $\mathcal{U}$ through

$$\varphi(t,x_0,y_0) = x_1$$
$$\psi(t,x_0,y_0) = y_1$$

at $s = 0$.

Thus we have

$$\varphi(s+t,x_0,y_0) = \varphi(s,x_1,y_1)$$
$$\psi(s+t,x_0,y_0) = \psi(s,x_1,y_1)$$

which is the required group property.

Examples.

1. $\dot{x}=0,\ \dot{y}=1$ generates the translation group.

2. $\dot{x}=-y,\ \dot{y}=x$ generates the rotation group.

3. $\dot{x}=x,\ \dot{y}=y$ generates the dialation group.

4. $\dot{x}=0,\ \dot{y}=y^2$ has solution through $x_0, y_0>0$ of $x=x_0,\ y=\frac{1}{\frac{1}{y_0}-t}$ which is defined in R^2 only for $t<\frac{1}{y_0}$. Thus this infinitesimal transformation group does not generate an entire transformation group but only a local transformation group.

Definition. Let Θ be open in R^2 and let $\varphi(t,x,y),\ \psi(t,x,y)$ be in C^∞ in a neighborhood of $0\times\Theta$ in $R^1\times\Theta$ and satisfy

$$\varphi(s,\varphi(t,x,y),\psi(t,x,y)) = \varphi(s+t,x,y)$$
$$\psi(s,\varphi(t,x,y),\psi(t,x,y)) = \psi(s+t,x,y)$$

wherever defined. Assume that for each compact set $K\subset\Theta$ there exists a $t_K>0$ such that for $|t|<t_K$ and $(x,y)\in K$ the map

$$T_t : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \varphi(t,x,y) \\ \psi(t,x,y) \end{pmatrix}$$

is a homeomorphism of K onto some $K_t\subset\Theta$. Then the functions $\varphi(t,x,y),\ \psi(t,x,y)$ define a local 1-parameter transformation group on Θ.

Note. We identify two such local 1-parameter transformation groups $\{\varphi,\psi\}$ and $\{\varphi^*,\psi^*\}$ in case they coincide on some neighborhood of $0\times\Theta$ in $R^1\times\Theta$. It is easy to verify that $\varphi(0,x,y)=x,\ \psi(0,x,y)=y$ and that $T_{-t}=(T_t)^{-1}$ wherever defined.

Theorem 1. Each infinitesimal 1-parameter transformation group on Θ generates a local 1-parameter transformation group on Θ. Also each local 1-parameter transformation group on Θ is generated by one and only one infinitesimal one-parameter transformation group.

Proof.

If $\mathcal{U}: \dot{x} = f(x,y),\ \dot{y} = g(x,y)$ is an infinitesimal 1-parameter transformation group then $\varphi(t,x,y), \psi(t,x,y)$, where $\varphi(t,x_0,y_0), \psi(t,x_0,y_0)$ is the solution of $\mathcal{U}$ through (x_0,y_0) at $t=0$ is a local 1-parameter transformation group.

Now let $\{T_t\}, \varphi(t,x,y)$ and $\psi(t,x,y)$ be a local 1-parameter transformation group on θ. Define the infinitesimal generator $\mathcal{U}$ of $\{T_t\}$ by

$$\dot{x} = \left[\frac{\partial\varphi}{\partial t}(t,x,y)\right]_{t=0} = f(x,y), \quad \dot{y} = \left[\frac{\partial\psi}{\partial t}\right]_{t=0} = g(x,y).$$

Thus $\varphi_1(0,x,y) = f(x,y),\ \psi_1(0,x,y) = g(x,y)$ and we must show (where defined)

$$\varphi_1(t,x,y) = f(\varphi(t,x,y), \psi(t,x,y))$$
$$\psi_1(t,x,y) = g(\varphi(t,x,y), \psi(t,x,y)).$$

Now
$$\varphi(s+t,x,y) = \varphi(s, \varphi(t,x,y), \psi(t,x,y))$$
$$\psi(s+t,x,y) = \psi(s, \varphi(t,x,y), \psi(t,x,y)).$$

Thus
$$\varphi_1(s+t,x,y) = \varphi_1(s, \varphi(t,x,y), \psi(t,x,y))$$
$$\varphi_1(t,x,y) = \varphi_1(0, \varphi(t,x,y), \psi(t,x,y)).$$

Thus
$$\varphi_1(t,x,y) = f(\varphi(t,x,y), \psi(t,x,y))$$

and similarly
$$\psi_1(t,x,y) = g(\varphi(t,x,y), \psi(t,x,y))$$
as required.

The uniqueness of the infinitesimal generator of a 1-parameter local transformation group is obvious since distinct vector fields have distinct trajectories. Q. E. D.

Note. The critical points of $\mathcal{U}$ are points of θ where $f(x_0,y_0)=0,\ g(x_0,y_0)=0$. Other points are regular. A trajectory of $\mathcal{U}$ consists of a single point if and only if that point is a critical point of $\mathcal{U}$.

<u>Definition.</u> Local 1-parameter transformation groups $\{T_t\}$ on $\mathcal{O}$ and $\{T'_t\}$ on $\mathcal{O}'$ are isomorphic in case there is a diffeomorphism of $\mathcal{O}$ onto $\mathcal{O}'$ and a constant factor change in time scale in R^1, which carries the transformations of $\{T_t\}$ onto those of $\{T'_t\}$ (where defined).

<u>Remark.</u> The only continuous isomorphisms of R^1 onto R^1 (or even local isomorphisms) are $t \to ct$ for a constant $c \neq 0$. One sometimes permits only orientation preserving isomorphisms, $c > 0$.

<u>Theorem 2.</u> Local 1-parameter transformation groups $\{T_t\}$ in $\mathcal{O}$ and $\{T'_t\}$ in $\mathcal{O}'$ are isomorphic if and only if there is a diffeomorphism of $\mathcal{O}$ onto $\mathcal{O}'$ which carries the infinitesimal generator $\mathcal{U}$ of $\{T_t\}$ onto $c\mathcal{U}'$, where $c \neq 0$ and $\mathcal{U}'$ is the infinitesimal generator of $\{T'_t\}$ in $\mathcal{O}'$.

<u>Definition.</u> The differential operator (on $C^\infty(\mathcal{O})$) $\mathcal{U} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$ is also called the infinitesimal generator or "symbol for the generator" for the local group generated by

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y).$$

The local transformation group is often written

$$\varphi(t,x,y) = e^{t\mathcal{U}}x, \quad \psi(t,x,y) = e^{t\mathcal{U}}y.$$

Let $\mathcal{U}: \dot{x} = f(x,y), \dot{y} = g(x,y)$ be real analytic C^ω in $\mathcal{O}$. Let $h(x,y) \in C^\omega$ in $\mathcal{O}$ and define $\hat{h}(t,x,y) = h(\varphi(t,x,y), \psi(t,x,y))$ using the local group generated by $\mathcal{U}$.

Then $\frac{\partial \hat{h}}{\partial t} = h_1 \frac{\partial \varphi}{\partial t} + h_2 \frac{\partial \psi}{\partial t}$ and $\left[\frac{\partial \hat{h}}{\partial t}\right]_{t=0} = h_1(x,y)f(x,y) + h_2(x,y)g(x,y) = \mathcal{U}h$.

In fact, this equation could be used as a definition for $\mathcal{U}$.

Assume

$$\left[\frac{\partial^{n-1}\hat{h}}{\partial t^{n-1}}\right]_{t=0} = \mathcal{U}^{n-1}h$$

Then

$$\mathcal{U}[\mathcal{U}^{n-1}h] = \frac{\partial}{\partial s}\left[\frac{\partial^{n-1}h}{\partial t^{n-1}}(\varphi(t,\varphi(s,x,y),\psi(s,x,y)), \psi(t,\varphi(s,x,y),\psi(s,x,y))\right]_{\substack{t=0\\s=0}}.$$

Thus

$$u^n h = \left[\frac{\partial}{\partial s} \frac{\partial^{n-1}}{\partial t^{n-1}} h(\varphi(t+s,x,y), \psi(t+s,x,y)) \right]_{\substack{t=0 \\ s=0}} ,$$

or

$$u^n h = \left[\frac{\partial^n \hat{h}}{\partial t^n} \right]_{t=0} .$$

Therefore, by Taylor's series for small $|t|$

$$\hat{h}(t,x,y) = h(x,y) + t\,uh + \frac{t^2}{2!} u^2 h + \frac{t^3}{3!} u^3 h + \cdots$$

or

$$\hat{h}(t,x,y) = e^{tu} h(x,y) .$$

If we let $h(x,y) = x$ so $\hat{h}(t,x,y) = \varphi(t,x,y)$

then $\varphi(t,x,y) = e^{tu} x$ and also $\psi(t,x,y) = e^{tu} y$.

Note that the concepts of a one-parameter local transformation group, its infinitesimal generator, and the symbol or operator $u = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y}$ are all defined without reference to the coordinates in $\mathcal{O}$ and thus they are concepts of differential geometry.

<u>Example.</u> $T_t: \begin{array}{l} x \to x \\ y \to y + t \end{array}$ has $u = \frac{\partial}{\partial y}$

$T_t: \begin{array}{l} x \to x\cos t - y \sin t \\ y \to x \sin t + y \cos t \end{array}$ has $u = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$

$T_t: \begin{array}{l} x \to x e^t \\ y \to y e^t \end{array}$ has $u = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$.

<u>Example.</u> $u = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$; $f = x$, $g = y$.

Then $ux = x$, $u^2 x = x$, $\ldots$, $u^n x = x$

and $uy = y$, $u^2 y = y$, $\ldots$, $u^n y = y$.

So $$e^{tu} x = x + tx + \frac{t^2}{2!} x + \cdots = x e^t$$
$$e^{tu} y = y + ty + \frac{t^2}{2!} y + \cdots = y e^t .$$

Let $u = f(x,y) \frac{\partial}{\partial x} + g(x,y) \frac{\partial}{\partial y}$ be the infinitesimal generator of a 1-parameter local transformation group on $\mathcal{O}$. Note that

$$u = (ux) \frac{\partial}{\partial x} + (uy) \frac{\partial}{\partial y}$$

Let $\binom{x}{y} \to \binom{u(x,y)}{v(x,y)}$ be a diffeomorphism of $\mathcal{O}$ onto $\mathcal{O}' \subset R^2$ or a change of coordinates in $\mathcal{O}$. Then the infinitesimal generator

$$\mathcal{U}: \quad \dot{x} = f(x,y), \quad \dot{y} = g(x,y)$$

becomes, in the new coordinates,

$$\mathcal{U}: \quad \dot{u} = \frac{\partial u}{\partial x} f + \frac{\partial u}{\partial y} g, \quad \dot{v} = \frac{\partial v}{\partial x} f + \frac{\partial v}{\partial y} g$$

or

$$\mathcal{U} = \left(\frac{\partial u}{\partial x} f + \frac{\partial u}{\partial y} g \right) \frac{\partial}{\partial u} + \left(\frac{\partial v}{\partial x} f + \frac{\partial v}{\partial y} g \right) \frac{\partial}{\partial v} .$$

This can be written

$$\mathcal{U} = (\mathcal{U} u) \frac{\partial}{\partial u} + (\mathcal{U} v) \frac{\partial}{\partial v} .$$

<u>Theorem 3.</u> Let $\mathcal{U} = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y}$ be an infinitesimal transformation group in $\mathcal{O}$. Let $P \in \mathcal{O}$ be a non-critical point of $\mathcal{U}$. Then there exist local coordinates near P, (that is $u(x,y), v(x,y) \in C^\infty$ with $\left| \frac{\partial(u,v)}{\partial(x,y)} \right| \neq 0$) in terms of which $\mathcal{U} = \frac{\partial}{\partial v}$

<u>Proof.</u>

Consider the local transformation group $\varphi(t,x,y), \psi(t,x,y)$ generated by $\mathcal{U}$. Let L be a line segment through P orthogonal to the trajectory of the local transformation group through P and let u be the distance along L measured from P. For each point $Q: (x,y)$ near P define $u(x,y)$ as the intersection coordinate of the trajectory through Q with L and let $v(x,y)$ be the value of t which carries this intersection point along the trajectory to Q. Then (u,v) are local (C^∞ with C^∞ inverse) coordinates near P and the trajectories of the transformation group become

$T_t: u \to u, \quad v = v + t$ so $\mathcal{U} = \frac{\partial}{\partial v}$.

Q. E. D.

5. Invariants of one-parameter transformation groups on R^2.

Definition. Let $\mathcal{U} = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ be the infinitesimal generator of a 1-parameter local transformation group in $\Theta \subset R^2$. A function $h(x,y) \in C^\infty(\Theta)$ is invariant under the transformation group in case $h(x,y)$ is constant along each trajectory of $\mathcal{U}$ in Θ.

Theorem 4. $h(x,y) \in C^\infty(\Theta)$ is invariant under the one-parameter local transformation group generated by $\mathcal{U} = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$, if and only if $\mathcal{U}h \equiv 0$ in Θ.

Proof.

If $h(x,y)$ is an invariant of $\mathcal{U}$, then $\hat{h}(t,x,y) = h(\varphi(t,x,y), \psi(t,x,y))$ is independent of t. That is, $\frac{\partial \hat{h}}{\partial t}(t,x,y) = 0$. But $\left[\frac{\partial \hat{h}}{\partial t}\right]_{t=0} = \mathcal{U}h$. Thus $\mathcal{U}h$ for each $(x,y) \in \Theta$.

Conversely, suppose $\mathcal{U}h \equiv 0$ in Θ.

Then
$$\hat{h}(t+s, x_0, y_0) = h(\varphi(t+s, x_0, y_0), \psi(t+s, x_0, y_0))$$
$$\hat{h}(t+s, x_0, y_0) = h(\varphi(s, x_1, y_1), \psi(s, x_1, y_1))$$
where $x_1 = \varphi(t, x_0, y_0)$, $y_1 = \psi(t, x_0, y_0)$.

Then $$\left[\frac{\partial \hat{h}}{\partial s}(t+s, x_0, y_0)\right]_{s=0} = \left[\mathcal{U}h\right]_{\substack{x=x_1\\ y=y_1}} = 0.$$

Thus $\frac{\partial \hat{h}}{\partial t}(t, x_0, y_0) = 0$ for each t, where defined.

Q. E. D.

Definition. A differential equation $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$, where $M(x,y)$ and $N(x,y) \in C^\infty$ in Θ and do not vanish simultaneously, is a differentiable line element field in Θ.

Note. We often write a first order differential equation as $\frac{dw}{dx} = W(x,y)$ but we mean a line element field where vertical line elements are allowed. The solutions of $\frac{dy}{dx} = w(x,y)$ are C^∞ curves — not parametrized or sensed.

<u>Definition.</u> A differential equation $\mathcal{D}$: $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$ or $M(x,y)\,dx + N(x,y)\,dy = 0$ in $\mathcal{O} \subset R^2$ is invariant under the local 1-parameter group $\{T_t\}$ generated by $\mathcal{U} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$ in case each diffeomorphism of an open set $\mathcal{O}_0$ onto $\mathcal{O}_1 \subset \mathcal{O}$, effected by a transformation of $\{T_t\}$, carries the line element field of $\mathcal{D}$ in $\mathcal{O}_0$ onto that of $\mathcal{D}$ in $\mathcal{O}_1$.

<u>Note.</u> If $\mathcal{D}$ is invariant under $\{T_t\}$ then the solution curve family of $\mathcal{D}$ is mapped onto itself by each transformation of $\{T_t\}$, wherever defined.

<u>Definition.</u> The manifold $\mathcal{L}(\mathcal{O})$ of line elements of $\mathcal{O}$ is a differentiable 3-manifold which is diffeomorphic with $\mathcal{O} \times S^1$, in a natural way. The projection $\pi : \mathcal{L}(\mathcal{O}) \to \mathcal{O}$ (onto) is differentiable and for each point $\pi^{-1}(P)$ is the set of line elements based at P and this is diffeomorphic with S^1.

A differential equation $\mathcal{D}: M(x,y)dx + N(x,y)dy = 0$ $(M^2+N^2>0$ in $\mathcal{O}$) is a differentiable surface in $\mathcal{L}(\mathcal{O})$ above $\mathcal{O}$. That is a differentiable function $\ell : \mathcal{O} \to \mathcal{L}(\mathcal{O})$ such that $\pi \ell =$ identity.

<u>Remark.</u> In $\mathcal{L}(\mathcal{O})$ we can use local coordinates (x,y,p) (where $p = \frac{dy}{dx}$ is the slope of a line element) and also another local coordinate system in obtained by interchanging x and y in the coordinates in $\mathcal{O}$. A diffeomorphism $u=u(x,y)$, $v=v(x,y)$ between open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}$ induces a diffeomorphism between $\pi^{-1}\mathcal{O}_1$ and $\pi^{-1}\mathcal{O}_2$ in $\mathcal{L}(\mathcal{O})$. In local coordinates the induced diffeomorphism is $(x,y,p) \to (x,y,q)$ where

$$q = \frac{\frac{dv(x,y(x))}{dx}}{\frac{du(x,y(x))}{dx}} = \frac{\frac{dv}{dx} + \frac{dv}{dy}p}{\frac{du}{dx} + \frac{du}{dy}p}$$

A local transformation group $\varphi(t,x,y)$, $\psi(t,x,y)$ in $\mathcal{O}$ induces a local transformation group in $\mathcal{L}(\Theta)$ by $(x,y,p) \to \varphi(t,x,y), \psi(t,x,y), \chi(t,x,y,p)$ where, in local coordinates,

$$\chi(t,x,y,p) = \frac{\frac{\partial\psi(t,x,y)}{\partial x} + \frac{\partial\psi}{\partial y}p}{\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}p}.$$

Theorem 5. Let $\mathcal{U} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$ be the infinitesimal generator of a local 1-parameter group on Θ. The infinitesimal generator of the induced local 1-parameter transformation group in $\mathcal{L}(\Theta)$ is

$$\mathcal{U}' = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y} + h(x,y,p)\frac{\partial}{\partial p}$$

where (in local coordinates)

$$h(x,y,p) = \frac{\partial g}{\partial x} + \left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right)p - \left(\frac{\partial f}{\partial y}\right)p^2.$$

Proof.

The induced transformation group is $\varphi(t,x,y)$, $\psi(t,x,y)$ and $\chi(t,x,y,p) = \frac{\psi_x + \psi_y p}{\varphi_x + \varphi_y p}$. Then $\left(\frac{\partial\varphi}{\partial t}\right)_{t=0} = f(x,y)$, $\left(\frac{\partial\psi}{\partial t}\right)_{t=0} = g(x,y)$

and

$$\left[\frac{\partial\chi}{\partial t}(t,x,y,p)\right]_{t=0} = \left[\frac{(\varphi_x + \varphi_y p)(\psi_{xt} + \psi_{yt} p) - (\psi_x + \psi_y p)(\varphi_{xt} + \varphi_{yt} p)}{(\varphi_x + \varphi_y p)^2}\right]_{t=0}$$

Thus $h(x,y,p) = \dfrac{1\cdot(g_x + g_y p) - (p)(f_x + f_y p)}{1^2}$

as required.

Q. E. D.

Note. $\mathcal{U}'$ is called the once-extended infinitesimal transformation group.

Theorem 6. A differential equation θ: $M(x,y)dx + N(x,y)dy = 0$ ($M^2+N^2>0$ in Θ) is invariant under the local transformation group generated by

$$\mathcal{U} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$$

if and only if the surface $\mathcal{D}$ in $\mathcal{L}(\mathcal{O})$ is invariant under the local group generated by $\mathcal{U}'$. This occurs if and only if the vector field of $\mathcal{U}'$ is everywhere tangent to the surface $\mathcal{D}$. If, in local coordinates (x, y, p) in $\mathcal{L}(\mathcal{O})$, $\mathcal{D}$ is the surface $p = w(x,y)$ (or $w(x,y) = -\frac{M}{N}$), then $\mathcal{D}$ is invariant under $\mathcal{U}$ if and only if $\mathcal{U}'[p - w(x,y)] = 0$ at points of $\mathcal{D}$.

Proof.

It is clear that $\mathcal{D}$ is invariant under the transformations generated by $\mathcal{U}$ if and only if $\mathcal{U}'$ is everywhere tangent to the surface $\mathcal{D}$ in $\mathcal{L}(\mathcal{O})$. The tangent vector f, g, h is in the surface $p = w(x,y)$ just in case $f \cdot (-w_x) + g \cdot (-w_y) + h \cdot (1) = 0$ or $\mathcal{U}'[p - w(x,y)] = 0$ on $\mathcal{D}$.

Q. E. D.

Example 1. $\mathcal{U} = \frac{\partial}{\partial y}$, $\mathcal{U}' = \frac{\partial}{\partial y}$ leaves $y' = f(x)$ invariant.

2. $\mathcal{U} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, $\mathcal{U}' = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (1+p^2)\frac{\partial}{\partial p}$

leaves
$$\frac{y - xy'}{x + yy'} = F(x^2 + y^2) \quad \text{or} \quad y' = \frac{y - xF}{x + yF}$$
invariant.

3. $\mathcal{U} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $\mathcal{U}' = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ leaves $y' = F(y/x)$ invariant.

Theorem 7. Let the differential equation $\mathcal{D}$: $M(x,y)\,dx + N(x,y)\,dy = 0$ ($M^2 + N^2 > 0$ in $\mathcal{O}$) be invariant under the transformation group generated by $\mathcal{U} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$. Assume the trajectories of $\mathcal{U}$ in $\mathcal{O}$ are nowhere tangent to the line elements of $\mathcal{D}$, that is, $fM + gN \neq 0$. Then
$$\mu = \frac{1}{fM + gN}$$
is an integrating factor for $\mathcal{D}$, that is, $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$.

Proof.

We verify $(\mu M)_y = (\mu M)_x$ near a point $P \in \mathcal{O}$ and assume $N(x,y) \neq 0$ near P otherwise $M(x,y) \neq 0$ and interchange the roles of x and y and use other local coordinates in $\mathcal{L}(\mathcal{O})$.

Now $\mathcal{U}'[-p + \frac{M}{N}] = 0$ on $\mathcal{D}$ in the subset of $\mathcal{L}(\mathcal{O})$ lying above a neighborhood of P. Thus

$$f \frac{\partial}{\partial x}\left(\frac{M}{N}\right) + g \frac{\partial}{\partial y}\left(\frac{M}{N}\right) + g_x + (g_y - f_x)p - f_y p^2 = 0$$

where $p = -\frac{M}{N}$.

Thus

$$f \frac{\partial}{\partial x}\left(\frac{M}{N}\right) + g \frac{\partial}{\partial y}\left(\frac{M}{N}\right) + g_x + (g_y - f_x)\left(-\frac{M}{N}\right) - f_y\left(\frac{M^2}{N^2}\right) = 0$$

identically in (x,y) near P in $\mathcal{O}$. But this is just the assertion $(\mu M)_x = (\mu M)_y$. Q.E.D.

Remark. The condition that $\int M\,dx + N\,dy$ is (locally) independent of the path does not depend on the local coordinates in which the differential $M\,dx + N\,dy$ is expressed. Thus near $P \in \mathcal{O}$ select local coordinates (still called (x,y)) so that $\mathcal{U} = \frac{\partial}{\partial y}$. Then $N(x,y) \neq 0$ near P and $-\frac{M(x,y)}{N(x,y)} = w(x)$ or $M(x,y) = -w(x)\,N(x,y)$. Thus we must show that $\mu = \frac{1}{N(x,y)}$ is an integrating factor for $-w(x)\,N(x,y)\,dx + N(x,y)\,dy = 0$. But certainly $-w(x)\,dx + dy$ is locally exact.

Example.

1. Linear differential equation $\frac{dy}{dx} + P(x)\,y = Q(x)$

Group $\mathcal{U} = e^{-\int P(x)\,dx} \frac{\partial}{\partial y}$

$$\mathcal{U}' = e^{-\int P\,dx} \frac{\partial}{\partial y} - P(x)\,e^{-\int P\,dx} \frac{\partial}{\partial p}$$

Integrating factor

$$\mu = e^{\int P\,dx}$$

2. Bernoulli equation $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^s$, $s \neq 0, 1$

Group $\mathcal{U} = \left[\exp \int (s-1)\, P(x)\, dx\right] y^s \frac{\partial}{\partial y}$

$$\mathcal{U}' = \left[\exp \int (s-1)\, P(x)\, dx\right] y^s \frac{\partial}{\partial y} + \left[(s-1)\, P(x)\, y^s e^{\int (s-1) P dx} + s y^{s-1} p e^{\int (s-1) P dx}\right] \frac{\partial}{\partial p}$$

$$\mu = y^{-s} \exp \int (1-s)\, P(x)\, dx$$

3. Homogeneous equation $\quad y' = F(y/x)$

Group $\mathcal{U} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$

$$\mathcal{U}' = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

$$\mu = (-xF + y)^{-1}$$

4. Variables separable $\quad y' = \varphi(x)\psi(y)$

Group $\mathcal{U} = \psi(y) \frac{\partial}{\partial y}, \quad \mathcal{U}' = \psi(y) \frac{\partial}{\partial y} + \frac{d\psi}{dy} p \frac{\partial}{\partial p}$

Integrating factor $\quad \mu = \dfrac{1}{\psi(y)}$

Theorem 8. Let $\mathcal{D}: M(x,y)\,dx + N\,dy = 0,\ M^2 + N^2 > 0$ be invariant under the transformation group generated by $\mathcal{U} = f(x,y) \frac{\partial}{\partial x} + g(x,y) \frac{\partial}{\partial y}$ in $\mathcal{O}$. Let $P \in \mathcal{O}$ be a non-critical point of $\mathcal{U}$. Then in a neighborhood of P there exist local coordinates (canonical coordinates of $\mathcal{U}$) in which we obtain

$$\mathcal{D}:\ w(u)\, dv - du = 0 \quad \text{or} \quad \frac{dv}{du} = \frac{1}{w(u)} \qquad \text{if} \quad w(u) \neq 0 .$$

Proof.

Take local coordinates $u(x,y), v(x,y)$ near P so that $\mathcal{U} = \frac{\partial}{\partial v}$. Then the slopes of the line elements of $\mathcal{D}$ do not depend on v. If the line element of $\mathcal{D}$ at P is parallel to the v-axis, write $du = 0$

Q.E.D.

Here $\mathcal{D}$ can be "solved by quadratures" in a coordinate system determined by the infinitesimal group $\mathcal{U}$.

6. Canonical forms for certain second order differential equations.

A first order differential equation

$$\mathcal{D}: \quad M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (M^2+N^2>0 \ \text{ in } \mathcal{O})$$

is locally equivalent (under change of local coordinates in $\mathcal{O}$) with $\frac{dy}{dx} = 0$.

Not all second order differential equations $y'' = w(x,y,y')$ are so equivalent to $y'' = 0$.

Let P be a point in $\mathcal{O} \subset R^2$ and let $C: x(t), y(t)$ and $\hat{C}: \hat{x}(t), \hat{y}(t)$ be differentiable curves through P at $t=0$. We say C and $\hat{C}$ define the same line element at P in case they have linearly dependent, but non-zero, tangent vectors at P. If C and $\hat{C}$ have the same line element at P, choose new coordinates $(\bar{x},\bar{y})$ so that $C: \bar{y} = \varphi(\bar{x})$ and $\hat{C}: \bar{y} = \hat{\varphi}(\bar{x})$ pass through $P: \bar{x}=0, \bar{y}=0$. We say C and $\hat{C}$ define the same curvature line element in case $\varphi''(0)$ and $\hat{\varphi}''(0)$ are equal. Thus, in appropriate coordinates near P, a curvature line element has four coordinates $x, y, p = y', r = y''$.

The manifold $K(\mathcal{O})$ of all curvature line elements of $\mathcal{O}$ is a differentiable 4-manifold, diffeomorphic with $\mathcal{O} \times S^1 \times R^1$ in a natural way. There is a differentiable projection $\sigma: K(\mathcal{O}) \to \mathcal{L}(\mathcal{O})$ such that $\sigma^{-1}(x_0, y_0, p_0)$ is diffeomorphic with R^1 for each point (x_0, y_0, p_0) in $\mathcal{L}(\mathcal{O})$.

Definition. A second order differential equation, written $y'' = w(x,y,y')$ is a differentiable map $w: S \to K(\mathcal{O})$, where S is open in $\mathcal{L}(\mathcal{O})$, such that σw identity on S .

Thus $y'' = w(x,y,y')$ is a 3-surface in $K(\mathcal{O})$ lying above an open set $S \subset \mathcal{L}(\mathcal{O})$.

A diffeomorphism of open sets $\mathcal{O}_1 \to \mathcal{O}_2$ in R^2 induces a diffeomorphism of $\mathcal{L}(\mathcal{O}_1)$ onto $\mathcal{L}(\mathcal{O}_2)$ and also $K(\mathcal{O}_1)$ onto $K(\mathcal{O}_2)$. In local coordinates, $(x,y) \to (u(x,y), v(x,y))$ induces $p \to \dfrac{v_x + v_y p}{u_x + u_y p}$ and

$$r \to [(u_x + u_y p)(v_{xx} + 2v_{xy}p + v_{yy}p^2 + v_y r) - (v_x + v_y p)(u_{xx} + 2u_{xy}p + u_{yy}p^2 + u_y r)](u_x + u_y p)^{-3}$$

Thus an 1-parameter local transformation group, generated by $\mathcal{U} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$ on $\mathcal{O}$, induces a 1-parameter local transformation group on $K(\mathcal{O})$ with infinitesimal generator $\mathcal{U}'' = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h(x,y,p)\frac{\partial}{\partial p} + k(x,y,p,r)\frac{\partial}{\partial r}$

where $h(x,y,p) = g_x + (g_y - f_x)p - f_y p^2$ and

$k(x,y,p,r) = h_x + h_y p + (h_p - f_x - f_y p)r$

or

$k(x,y,p,r) = g_{xx} + p(2g_{xy} - f_{xx}) + p^2(g_{yy} - 2f_{xy}) - f_{yy}p^3 + r(g_y - 2f_x - 3f_y p)$

__Definition.__ Let $y'' = w(x,y,y')$ be a second order differential equation over $S \subset \mathcal{L}(\mathcal{O})$ and let $\mathcal{U} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$ generate a local 1-parameter transformation group $\{T_t\}$ in $\pi S \subset \mathcal{O}$. Then $y'' = w(x,y,y')$ is invariant under $\mathcal{U}$ (or under the transformations generated by $\mathcal{U}$) in case the curvature elements of the differential equation are carried onto curvature elements of the same equation by the induced transformations of $\{T_t\}$, wherever defined.

__Theorem 9.__ Let $y'' = w(x,y,y')$ be a second order differential equation in $K(\mathcal{O})$ over an open set $S \subset \mathcal{L}(\mathcal{O})$. Let $\mathcal{U} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$ generate a local 1-parameter transformation group $\{T_t\}$ in $\pi S \subset \mathcal{O}$. Then $y'' = w(x,y,y')$ is invariant under $\{T_t\}$ in case $\mathcal{U}'' = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial p} + k\frac{\partial}{\partial r}$ is everywhere tangent to the hyper-surface of $y'' = w(x,y,y')$ in $K(\mathcal{O})$. In local coordinates (x,y,p,r) this occurs if and only if $\mathcal{U}''(r - \omega(x,y,p)) = 0$ wherever $r = \omega(x,y,p)$.

__Example.__ $y'' = P(x)y' + Q(x)y + R(x)$ in C^∞ over $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < y' < \infty$. Let $\varphi(x)$ be a solution of the homogeneous equation and define the local transformation group by $\mathcal{U} = \varphi(x)\frac{\partial}{\partial x}$.

Then $\mathcal{U}'' = \varphi(x)\frac{\partial}{\partial y} + \varphi'(x)\frac{\partial}{\partial p} + \varphi''(x)\frac{\partial}{\partial r}$. Then $\mathcal{U}''[r - P(x)r - Q(x)y - R(x)] =$
$= -\varphi(x)Q(x) - \varphi'(x)P(x) + \varphi''(x) = 0$.

Thus, near each point $(x,y,p) \in \mathcal{L}(R^2)$ the differential equation is invariant under the local transformation group generated by $\mathcal{U}$.

Example. Consider $y'' = xy + \tan y'$ in $(x,y) \in R^2$ and $|p| < \pi/2$. Suppose in some subregion $\mathcal{O} \subset R^2$, the infinitesimal transformation group $\mathcal{U} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$ (not the identity map) leaves $y'' = xy + \tan y'$ invariant. Then

$$-yf - xg + [g_x + (g_y - f_x)p - f_y p^2]\sec^2 p + g_{xx} + p(2g_{xy} - f_{xx})$$
$$+ p^2(g_{yy} - 2f_{xy}) - f_{yy}p^3 + (xy + \tan p)(g_y - 2f_x - 3f_y p) \equiv 0$$

in (x,y,p) in S.

This requires: $-yf - xg + g_{xx} + xy\,g_y - 2xy\,f_x \equiv 0$,

$g_x \equiv 0, \quad g_y - f_x \equiv 0, \quad f_y \equiv 0,$

$2g_{xy} - f_{xx} - 3xy\,f_y \equiv 0,$

$g_{yy} - 2f_{xy} \equiv 0, \quad f_{yy} \equiv 0, \quad g_y - 2f_x \equiv 0.$

Then $f = f(x)$, $g = g(y)$. But $2f' = f'$ so $f' = 0$ and $f =$ constant. Then $g =$ constant. Then $-yf - xg \equiv 0$ which is impossible. Thus $y'' = xy + \tan y'$ is nowhere invariant under a local 1-parameter transformation group. Therefore one cannot introduce new local coordinates in an open set in R^2, $u(x,y)$, $v(x,y)$ so that (near the slope $\frac{dy}{dx} = p_0$), this differential equation has a solution curve family which is diffeomorphic with the solutions of $v'' = P(u)v' + Q(u)v + R(u)$.

Examples. Consider $y'' = \omega(x,y,y')$ invariant under

a.) $\mathcal{U}_1 = \frac{\partial}{\partial x}$ and $\mathcal{U}_2 = \frac{\partial}{\partial x}$.

Then $y'' = \omega(y')$. Set $y' = v$ for quadrature.

b.) $\mathcal{U}_1 = \frac{\partial}{\partial y}$, $\mathcal{U}_2 = x\frac{\partial}{\partial y}$.

Then $y'' = \omega(x, y')$. But $\mathcal{U}_2'' = x\frac{\partial}{\partial y} + \frac{\partial}{\partial p}$. Then $\mathcal{U}_2''(r - \omega(x,p)) = -\frac{d\omega}{dp} = 0$ so $w = w(x)$. Then $y'' = \omega(x)$ and solve by quadratures.

c.) $\mathcal{U}_1 = \frac{\partial}{\partial y}$, $\mathcal{U}_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Then $w = w(x, y')$. But $\mathcal{U}_2'' = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - r\frac{\partial}{\partial r}$. Then $\mathcal{U}_2''(r - w(x,p)) = -x\frac{\partial w}{\partial x} - r = 0$ where $r = \omega(x,p)$. Thus $x\frac{\partial w}{\partial x} + w \equiv 0$ in (x,p). So $w = c(p)/x$. Thus $y'' = c(y')/x$ (or $y'' = 0$). Let $y' = v$ so $v' = c(v)/x$ and solve by quadrature.

d.) $\mathcal{U}_1 = \frac{\partial}{\partial y}$, $\mathcal{U}_2 = y\frac{\partial}{\partial y}$.

Then $y'' = w(x, y')$. But $\mathcal{U}_2'' = y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} + r\frac{\partial}{\partial r}$.

Then $\mathcal{U}_2''(r - w(x,p)) = -p\frac{\partial w}{\partial p} + r = 0$ where $r = w(x,p)$.

Then $-p\frac{\partial w}{\partial p} + w \equiv 0$ in (x,p). Thus $w = c(x)p$.

Then $y'' = c(x)y'$. Let $y' = v$ so $v' = c(x)v$ and solve by quadratures.

The pair of infinitesimal groups a.) and c.) have non-tangent trajectories whereas b.) and d.) have tangent trajectories. But a.) and c.) are different in that for a.) $[\mathcal{U}_1, \mathcal{U}_2] = \mathcal{U}_1\mathcal{U}_2 - \mathcal{U}_2\mathcal{U}_1 = 0$ whereas for c.) $[\mathcal{U}_1, \mathcal{U}_2] = \mathcal{U}_1\mathcal{U}_2 - \mathcal{U}_2\mathcal{U}_1 = \mathcal{U}_1$. Also b.) and d.) are different in that for b.) $[\mathcal{U}_1, \mathcal{U}_2] = \mathcal{U}_1\mathcal{U}_2 - \mathcal{U}_2\mathcal{U}_1 = 0$ whereas for d.) $[\mathcal{U}_1, \mathcal{U}_2] = \mathcal{U}_1\mathcal{U}_2 - \mathcal{U}_2\mathcal{U}_1 = \mathcal{U}_1$.

It will be shown later that cases a.) b.) c.) and d.) represent all the 2-parameter local transformation groups on R^2. Thus a second order differential equation which is invariant under a 2-parameter local transformation group in R^2 has a canonical form which can be "solved by quadratures."

It is interesting to note that the cases b.) and d.) yield the canonical forms $y'' = \omega(x)$ and $y = c(x)y'$ which are linear and which are known to be equivalent to $y'' = 0$.

In $\mathcal{L}(\Theta)$ there is a distinguished class of non-singular differentiable (non-parametrized) curves known as line-element unions. These are the vertical fibers (x_0, y_0, p_0) and also the lifted curves of non-singular curves in Θ , that is $x(t), y(t), p = \dot{y}/\dot{x}$. Thus all differentiable curves $x(t), y(t), p(t)$ with $\dot{x}^2 + \dot{y}^2 + \dot{p}^2 > 0$ and $\dot{y} - p\dot{x} = 0$.

<u>Definition</u>. A contact transformation of an open set $S \subset \mathcal{L}(\Theta)$ is a diffeomorphism of S onto itself which (together with its inverse) preserves the class of all unions of line elements.

Each diffeomorphism of Θ onto itself induces a contact transformation of $\mathcal{L}(\Theta)$ onto itself, $(x,y,p) \to \left(u(x,y), v(x,y), \frac{v_x + v_y p}{u_x + u_y p} \right)$.

Let $\Phi: (x,y,p) \longrightarrow (X(x,y,p), Y(x,y,p), P(x,y,p))$ be a contact transformation, in local coordinates. Then each union $x(t), y(t), p(t)$ with $\dot{x}^2 + \dot{y}^2 + \dot{p}^2 > 0$ and $\dot{y} - p\dot{x} = 0$ must transform to a union $X(t) = X(x(t), y(t), p(t))$ $Y(t)$, $P(t)$. Thus $\dot{Y} - P\dot{X} = 0$. This is guaranteed if $dy - p\,dx = \rho(x,y,p)\,[dY - P\,dX]$ for a positive function $\rho(x,y,p)$.

<u>Example</u>. Let S be $-\infty < x, y, y' < \infty$ and let Φ be
$X = p, \quad Y = y - xp, \quad P = -x$.

If $\mathcal{D}$ is a differential system $M(x,y)dx + N(x,y)\,dy = 0$ $(M^2 + N^2 > 0$ in Θ), a surface in an open set $S \subset \mathcal{L}(\Theta)$, then the solution curves of $\mathcal{D}$ in Θ lift to element unions in S . In fact, the process of finding the solution curves of $\mathcal{D}$ in Θ , consists in decomposing the corresponding surface in S into the disjoint union of line element unions which then project onto the solutions of $\mathcal{D}$ in Θ . A contact transformation of S onto S maps the surface of $\mathcal{D}$ again onto a surface $\mathcal{D}^*$ in S which is the union of disjoint line element unions. On ϑ^* this decomposition might be easy and then the inverse of the given contact transformation finds the solutions of ϑ, as required.

Theorem 10. Let $y''=w(x,y,y')$ be defined over an open set $S\subset \mathcal{L}(\vartheta)$. Then for each point $P\in S$, there exists a neighborhood $N_P\subset S$ and a contact transformation of N_P onto itself which carries the solutions of $y''=\omega(x,y,y')$ onto those of $y''=0$ in N_P.

Proof.

Choose the local coordinates in S so that P is $(0,0,0)$. Then for each point $Q(x,y,p)$ near P the solution curve of the system

$$\frac{dy}{dx}=p, \quad \frac{dp}{dx}=w(x,y,p)$$

hits the plane $x=0$ at the point $y_0(x,y,p),\ p_0(x,y,p)$.

Consider the change of local coordinates in N_P, $(x,y,p)\to(u,v,q)$ where $y_0(x,y,p)=v-uq$, $p_0(x,y,p)=q$, $u=-\frac{\partial y_0}{\partial p}\Big/\frac{\partial p_0}{\partial p}$. It is easy to check that $x=0, y=0, p=0$ corresponds to $u=0, v=0, q=0$ and that $\left.\frac{\partial(u,v,q)}{\partial(x,y,p)}\right|_P\neq 0$. The first two equations guarantee that the solution curves of $y''=w$ in N_P fit onto the lines $\frac{dv}{du}=q$, $\frac{dq}{du}=0$, that is, the solutions of $\frac{d^2v}{du^2}=0$. The third equation specifies the change $x\to u$ along each solution of $y''=w$ so that the map is a contact transformation, cf. Lie and Scheffers Berührungstransformationen, p. 83.

Q. E. D.

Note: $y''=(y'')^4$ is not equivalent to $y'''=0$ under a contact transformation, cf. Berührungstransformationen, p. 86.

7. Topological Groups.

<u>Definition.</u> A topological group G is a group which is also a Hausdorff space and 1.) $x, y \to xy : G \times G \to G$

2.) $x \to x^{-1} : G \to G$

are continuous.

<u>Note.</u> An equivalent definition merely requires the map $x, y \to x^{-1}y : G \times G \to G$ be continuous (using the product topology on $G \times G$).

<u>Remark.</u> For each $g \in G$, the maps

$$x \to gx, \quad L_g : G \to G$$

$$x \to xg, \quad R_g : G \to G$$

and $\quad x \to x^{-1} \qquad G \to G$

are homeomorphisms of G onto G . Thus each point x of G has a neighborhood N_x homeomorphic with a neighborhood of the identity $N_e = x^{-1} N_x$.

<u>Example.</u> R^n with vector addition.

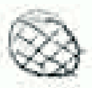
T^n with angular coordinate addition.

S', Spin 3

$GL(n,R)$, $O(n)$, $L(n)$

$GA(1,R) \approx \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a \neq 0$.

<u>Example.</u>  not a topological group.

 not a topological group.

<u>Definition.</u> Let G_1 and G_2 be topological groups. An abstract homomorphism $f : G_1 \to G_2$, which is continuous is called a homomorphism. If f is also a homeomorphism of G_1 onto G_2 then f is an isomorphism. An isomorphism of G_1 onto itself is an automorphism. For each $g \in G$

$$I_g : x \to g x g^{-1}$$

is an (inner) automorphism of G .

Definition. A subset H of a topological group is called a subgroup in case H is a subgroup of the abstract group G and, with the subspace topology, H is a topological group.

Definition. Let G_1 and G_2 be topological groups. Then the abstract direct product group $G_1 \times G_2$, with the product topology, is the product topological group.

Example. $T^n = S^1 \times S^1 \times \cdots \times S^1$ (n factors).

Theorem 11. Let G be a topological group and $H \subset G$ is an abstract subgroup of G . Then, in the subspace topology, H is a topological group. Also $\bar{H}$ is a closed subgroup of G . If H is normal or abelian, so is $\bar{H}$.

Proof.

It is clear that H is a Hausdorff space and thus a topological subgroup of G .

Let $x \in \bar{H}$ and $y \in \bar{H}$ and let U_z be a neighborhood of $z = xy$. Suppose $H \cap U_z$ is empty. Then for $\hat{x}$ near x and $\hat{y}$ near y , with $\hat{x} \in H$, $\hat{y} \in H$ we obtain $\hat{x} \cdot \hat{y} \in H \cap U_z$, which is impossible. Thus $xy \in \bar{H}$. Similarly $x^{-1} \in H$ and so $\bar{H}$ is a subgroup of G .

Assume H is abelian. Let $x \in \bar{H}$, $y \in \bar{H}$ and suppose $xy \neq yx$. Then separate xy and yx by neighborhoods U_{xy} and U_{yx} , respectively. Then, for $\hat{x}$ near x and $\hat{y}$ near y , with $\hat{x} \in H$, $\hat{y} \in H$, we obtain $\hat{x} \cdot \hat{y} \in U_{xy} \cap U_{yx}$, which is impossible. Thus $\bar{H}$ is abelian.

Now H is normal just in case the set H is invariant under each inner automorphism of G. But then the closure $\bar{H}$ is also invariant.

Q. E. D.

Theorem 12. Let H be a closed subgroup of the topological group G. Then the right (or left) coset space G/H, with the identification topology, is a Hausdorff space. Also the natural projection

$$p: G \to G/H$$

is an open, continuous map onto G/H.

Proof.

By the definition of the quotient topology, p is continuous onto G/H.

Let $\mathcal{O} \subset G$ be open. Then $p(\mathcal{O})$ is the set of all cosets in G/H which intersect $\mathcal{O}$ in G, that is, $p^{-1}[p(\mathcal{O})]$ is the saturation of $\mathcal{O}$. But $p^{-1}[p(\mathcal{O})] = H\mathcal{O} = \bigcup_{h \in H} h\mathcal{O}$. Now $h\mathcal{O}$ is open and hence $p^{-1}[p(\mathcal{O})]$ is open in G. Thus $p(\mathcal{O})$ is open in G/H so p is an open map.

Let Hx and Hy be distinct cosets of G/H, so $x \notin Hy$. Let $z = y^{-1}x$ and we need only show that there exist open neighborhoods U_z of z and U_e of the identity e whose saturations (by right cosets of H) are disjoint. Suppose the contrary. Then there exists a sequence (directed system in case there is no countable neighborhood base at e)

$$z_n \to z, \quad e_n \to e, \quad h_n \in H$$

with $z_n = h_n e$ or $h_n = z_n e_n^{-1}$.

Now $e_n^{-1} \to e$ and so $h_n \to z$. Thus $z \in H$ or $x \in Hy$ which is a contradiction. Therefore G/H is Hausdorff.

Q. E. D.

<u>Corollary.</u> If N is a closed normal subgroup of the topological group G, then the abstract quotient group G/N, with the quotient topology, is a topological group. Also the natural projection $p: G \to G/N$ is an open homomorphism onto G/N.

<u>Theorem 13.</u> Let $f: G \to H$ be a homomorphism of the topological group G onto the topological group H. Then the kernel

$$N = f^{-1}(e)$$

is a closed normal subgroup of G. Also there exists a continuous one-to-one homomorphism $\varphi: G/N \to H$, onto H, such that $f = \varphi p$.

<u>Proof.</u>

The function φ such that $f = \varphi p$ is an abstract group isomorphism. Let $O \subset H$ be open. Then $\varphi^{-1}(O) = p f^{-1}(O)$ is open in G/N. Thus φ is continuous.

Q. E. D.

<u>Corollary.</u> If f is open, φ is an isomorphism of G/N onto H.

<u>Theorem 14.</u> Let G be a topological group and let K be the component of the identity e of G. Then K is a closed normal subgroup of G.

<u>Proof.</u>

A component is necessarily closed since $\bar{K}$ is connected if K is connected. An automorphism of G maps $e \to e$ and the component of e again onto the component of e.

Q. E. D.

<u>Examples.</u> $GL_+(n,R) = SL(n,R)$, $O_+(n) = SO(n)$, $L_p(n)$.

<u>Theorem 15.</u> Let G be a connected topological group. Then each neighborhood $\mathcal{U}$ of the identity e generates all of G (by finite products of elements in $\mathcal{U}$).

<u>Proof.</u>

Let $E = \bigcup_{n=1}^{\infty} \mathcal{U}^{n} \subset G$. Since E is the union of open sets, E is open in G. Let $x \in \bar{E}$ and take an open neighborhood W of e so W^{-1} and $W \subset \mathcal{U}$. Then there exists $y \in Wx$, where $y \in E$ so $y = v_1 v_2 \cdots v_k$ with $v_i \in \mathcal{U}$. Then $v_1 v_2 \cdots v_k = wx$ and $x = w^{-1} v_1 v_2 \cdots v_k$. Thus E is closed in G and $G - E$ is open. Since G is connected, $G - E$ is empty.

Q. E. D.

<u>Theorem 16.</u> Let G be a connected topological group. A discrete subgroup N is closed. If N is also normal, then N lies in the center of G.

<u>Proof.</u>

Let N be discrete so that e is open in N. Thus there exists an open neighborhood $\mathcal{U}_1$ of e in G such that $N \cap \mathcal{U}_1 = e$. Let $x \in \bar{N}$. Then $\mathcal{U}_1 x$ intersects N, or $v_1 x = y_1$ with $v_1 \in \mathcal{U}_1$ and $y_1 \in N$. Suppose $v_1 \neq e$. Then use a smaller neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$, $v_1 \notin \mathcal{U}_2$ and write $v_2 x = y_2$ with $v \in \mathcal{U}_2$, $y_2 \in N$. Then $v_2 v_1^{-1} = y_2 y_1^{-1} \in N$. Thus, if $\mathcal{U}_1$ is sufficiently small, $y_2 y_1^{-1} = e$ so $v_1 = v_2$ which is a contradiction.

Now assume N is discrete and normal in G. Now $gxg^{-1} \in N$ for each $x \in N$ and $g \in G$. But $exe^{-1} = x$ and so $gxg^{-1} = e$ for each g sufficiently near e in G. Thus there exists a neighborhood $\mathcal{U}$ of e in G such that $gx = xg$ for each $g \in \mathcal{U}$ and $x \in N$. Thus N is in the center of G, since $\mathcal{U}$ generates the connected group G. Q. E. D.

<u>Definition.</u> Let G be a topological group and X a Hausdorff space. Assume that for each $g \in G$ there exists a homeomorphism of X onto X,

$$T_g : X \to X \quad : \quad x \to \varphi(x, g)$$

such that:

1.) $T_{g_1} T_{g_2} = T_{g_2 g_1}$ or $\varphi(x, g_2 g_1) = \varphi(\varphi(x, g_1), g_2)$.

2.) The function $(g, x) \to \varphi(x, g) : G \times X \to X$ is continuous (in both variables together). Then $\varphi : G \times X \to X$ is called a topological transformation group of G acting on X by the function φ. If, in addition,

3.) $T_g = I$ (identity) $\Leftrightarrow g = e$, then G acts effectively on X.

<u>Remark.</u> The axioms 1.) and 2.) of $\varphi : G \times X \to X$ insure that $T_e = I$ and $T_{g^{-1}} = (T_g)^{-1}$. Axiom 3.) insures that $T_{g_1} = T_{g_2} \Leftrightarrow g_1 = g_2$.

<u>Definition.</u> Topological transformation groups $\varphi_1 : G_1 \times X_1 \to X_1$ and $\varphi_2 : G_2 \times X_2 \to X_2$ are isomorphic in case there exists an isomorphism $f : G_1 \to G_2$ (onto) and a homeomorphism $\psi : X_1 \to X_2$ (onto) such that $\psi[\varphi_1(x_1, g_1)] = \varphi_2[\psi(x_1), f(g_1)]$ for all $(x_1, g_1) \in X_1 \times G_1$.

<u>Theorem 17.</u> Let $\varphi : G \times X \to X$ be a topological transformation group. Then the set N of $g \in G$ for which $T_g = I$ is a closed normal subgroup N of G. Moreover $G/N : X$ is then an effective transformation group under $T_{Ng} = T_g$.

<u>Proof.</u>

Clearly N is a subgroup of G. Also for a directed set $g_\alpha \to g$ with $\varphi(x, g_\alpha) = x$ for all $x \in X$, we must have $\varphi(x, g) = x$. Thus N is closed in G.

For each coset $Ng \in G/N$ define the action on X by $T_{Ng} = T_g$, that is $\bar{\varphi}(x, Ng) = \varphi(x, g)$. This is well-defined

since if $Nh = Ng$, so $hg^{-1} \in N$ we have
$\hat{\varphi}(x, Nh) = \varphi(x, h) = \varphi(x, ng) = \varphi[\varphi(x, g), n] = \varphi(x, g)$.

$$\text{Also } \varphi(x, hgh^{-1}) = \varphi(\varphi(x, h^{-1}), hg)$$
$$= \varphi(\varphi(\varphi(x, h^{-1}), g), h)$$
$$= \varphi(\varphi(x, h^{-1}), h)$$
$$= \varphi(x, hh^{-1}) = \varphi(x, e) = x,$$

if $g \in N$. Thus N is normal.

Finally suppose $\hat{\varphi}(x, Ng) = x$ for all $x \in \mathfrak{X}$. Then $\varphi(x,g) = x$ so $g \in N$. Thus $G/N : \mathfrak{X}$ is effective.

Q. E. D.

Remark. We shall usually consider effective transformation groups.

Definition. A topological transformation group

$$\varphi : G \times \mathfrak{X} \to \mathfrak{X}$$

is transitive in case for each pair $x \in \mathfrak{X}$, $y \in \mathfrak{X}$ there exists a $g \in G$ such that $\varphi(x, g) = y$.

Theorem 18. Let G be a topological group and N a closed subgroup of G . Then the group G acts transitively on the quotient space of left cosets G/N . If the only normal subgroup of G which is contained in N is the identity e (that is, N is abnormal), then $G : G/N$ is effective.

Proof.

For each $g_1 \in G$ and coset gN define $\varphi(gN, g_1) = g_1 g N$. Then $\varphi(gN, g_2 g_1) = g_2 g_1 g N = \varphi(g_1 g N, g_2)$ as required.

We must show that $gN \to g_1 g N$, is a homeomorphism of G/N onto itself. This transformation of G/N onto itself is well-defined since if $gN = hN$, then $g = hn$ and so $g_1 g N = g_1 h n N = g_1 h N$.

Also the transformation is one-to-one since: $g_1 g N = g_1 h N \Rightarrow g_1 g = g_1 h n$ or $g = hn$ so $gN = hN$.

Also the transformation is continuous in the pair g_1, gN since $(g_1, g) \to g_1 g : G \times G \to G$ is continuous and $g_1 g \to g_1 g N : G \to G/N$ is continuous. Thus $(g_1, g) \to g_1 g N$ from $G \times G \to G/N$ is continuous. Since the natural projection $p: G \to G/N$ is open, the map $(g_1, gN) \to g_1 g N$ is continuous. Moreover this map is a homeomorphism of G/N onto itself since the inverse of the map $T_{g_1} : gN \to g_1 g N$ is the continuous map $T_{g_1^{-1}}$.

It is easy to see that the transformation group $G : G/N$ is transitive. For given gN and hN, take $g_1 = h g^{-1}$ and then $g_1 g N = hN$.

Now let N be abnormal. The subgroup $\hat{N} \subset G$ which acts as the identity on G/N is closed and normal in G. If $\hat{g} \in \hat{N}$, then $\hat{g}(N) = N$ or $\hat{g} \in N$. Thus $\hat{N} \subset N$. Since N is abnormal, $\hat{N} = (e)$. Thus $G : G/N$ is effective.

Q. E. D.

<u>Remark.</u> The left coset space G/N is called a homogeneous space since $G : G/N$ is transitive.

<u>Example.</u> Consider the rotation group $O_+(n)$ and the closed subgroup $N = \begin{pmatrix} 1 & 0 \\ 0 & O_+(n-1) \end{pmatrix}$. Then the left coset space $O_+(n)/N$ is topologically the sphere S^{n-1} and $O_+(n) : S^{n-1}$ acts transitively and effectively.

<u>Definition.</u> Let $\varphi : G \times X \to X$ be a transformation group. If $\varphi(x_0, g)$ is an open map of $G \to X$, for a fixed $x_0 \in X$, then the transformation group is called locally transitive at x_0.

<u>Note.</u> If $\varphi : G \times X \to X$ is transitive, and if it is locally transitive at $x_0 \in X$, then it is locally transitive at each point $x \in X$.

Note. If N is a closed subgroup of the topological group G, then $G : G/N$ is transitive and locally transitive.

Theorem 19. Let $\varphi : G \times X \to X$ be a transitive, effective transformation group. For each point $x \in X$, the subgroup $N_x \subset G$ such that $\varphi(x, N_x) = x$ is called the isotropy (or stability) subgroup for x. Then N_x is a closed abnormal subgroup of G, and N_x is conjugate to N_y in G. Fix a point $z \in X$ and write $N_z = N$. Then $G : G/N$ is isomorphic with $\varphi : G \times X \to X$, provided G/N is compact or $\varphi : G \times X \to X$ is locally transitive.

Proof.

If $\varphi(x, g_\alpha) = x$ for a directed system $g_\alpha \to g$ in G, then, by continuity, $\varphi(x, g) = x$ so that N_x is a closed subgroup of G.

Take points $x, y \in X$ and write $\varphi(x, g) = y$ or $\varphi(y, g^{-1}) = x$, since $\varphi : G \times X \to X$ is transitive. Then $g N_x g^{-1} = N_y$. For

$$\begin{aligned} \varphi(y, g N_x g^{-1}) &= \varphi(\varphi(y, g^{-1}), g N_x) \\ &= \varphi[\varphi(\varphi(y, g^{-1}), N_x), g] \\ &= \varphi(\varphi(y, g^{-1}), g) \\ &= \varphi(y, e) = y. \end{aligned}$$

Thus $g N_x g^{-1} \subset N_y$. By symmetry $g^{-1} N_y g \subset N_x$ so $N_y \subset g N_x g^{-1}$ and $g N_x g^{-1} = N_y$.

Let $\hat{N}$ be a normal subgroup of G which lies in N_x. Then $g \hat{N} g^{-1} = \hat{N} \subset N_y$. Then $\hat{N}$ acts as the identity on all X so $\hat{N} = (e)$. Thus each N_x is abnormal.

To define the isomorphism of $G : G/N$ onto $\varphi : G \times X \to X$ use the identity automorphism of G onto G. Map G/N onto X as follows. For each left coset pN of G/N there corresponds a point $x \in X$ by

$\varphi(z, pN) = \varphi(z, p) = x$. Also if $\varphi(z, p_1) = x$, then $\varphi(z, p_1^{-1} p) = z$ so $p_1^{-1} p \in N$ and $p \in p_1 N$. Thus the map $\psi : G/N \to X$; $pN \to x$

is one-to-one onto X. By the continuity of φ, and since the natural projection $G \to G/N$ is open, we see that ψ is continuous.

If G/N is compact, ψ is a homeomorphism. If $\varphi : G \times X \to X$ is locally transitive, then ψ is an open map and hence a homeomorphism.

It is clear that the action $G : G/N$ is the same as that of $\varphi : G \times X \to X$. For write $\varphi(z, p) = x$ and $\psi : x \leftrightarrow pN$. We must show that $\psi : \varphi(x, g) \leftarrow gpN$. But $\varphi(z, gp) = \varphi(x, g)$ and so $\varphi(x, g) \leftrightarrow gpN$. Q. E. D.

<u>Example</u>. $G : G$ so G acts transitively and effectively on itself by left multiplication.

<u>Examples</u>. $GL(n, R)$ acts on R^n, not transitive.

$GA(n, R)$ is transitive on R^n.

$$\begin{pmatrix} GL(n,R) & R^n \\ 0 & 1 \end{pmatrix} \text{ on } \begin{pmatrix} I & R^n \\ 0 & 1 \end{pmatrix} = GA(n,R)/GL(n,R).$$

<u>Definition</u>. Topological groups G_1 and G_2 are locally isomorphic in case there exist neighborhoods $U_1 \subset G_1$ and $U_2 \subset G_2$ of the identities such that: there is a homeomorphism $f : U_1 \to U_2$ (onto) such that if x_1, y_1 and x_1^{-1}, $x_1 y_1 \in U_1$ then so are $f(x_1), f(y_1), f(x_1^{-1}), f(x_1 y_1) \in U_2$ and $f(x_1 y_1) = f(x_1) f(y_1)$, $f(x_1^{-1}) = f(x_1)^{-1}$.

<u>Theorem 20</u>. Let N be a discrete normal subgroup of the topological group G. Then $p : G \to G/N$ is a homomorphism onto the factor group G/N and also p is a local isomorphism of G with G/N.

Proof.

Let W be an open set in G such that $W \cap N = e$. Let $\mathcal{U}$ be open in G with $\mathcal{U}\mathcal{U}^{-1} \subset W$ and write $p(\mathcal{U}) = \mathcal{U}'$ open in G/N. Now p is a homeomorphism between $\mathcal{U}$ and $\mathcal{U}'$. For if $x, y \in \mathcal{U}$ and $p(x) = p(y)$, then $xy^{-1} \in N$ and hence $xy^{-1} = e$ or $x = y$.

Q. E. D.

8. Lie Groups.

Definition. A Lie group is a topological group G which is also a differentiable manifold, and

1.) $x, y \to xy \;:\; G \times G \to G$
2.) $x \to x^{-1} \;:\; G \to G$

are differentiable maps.

Remark. For each $g \in G$ we have

$$x \to gx, \quad L_g : G \to G$$
$$x \to xg, \quad R_g : G \to G$$
$$x \to x^{-1}, \quad G \to G$$

are diffeomorphisms of G onto G.

Definition. Let G_1 and G_2 be Lie groups. A (Lie) homomorphism $f : G_1$ into G_2 is a continuous homomorphism which is a differentiable map. If f is a diffeomorphism of G_1 onto G_2, then f is a (Lie) ismorphism. An isomorphism of G_1 onto itself is an automorphism. For each $g \in G_1$,

$$I_g : x \to gxg^{-1}$$

is an (inner) automorphism.

Remark. Let G be a connected topological group, with a countable base for the topology. Assume there exists a neighborhood $\mathcal{U} \subset V$ of e in

G and a homeomorphism f of V onto an open ball $B^n \subset R^n$. In the local coordinates in V assume the group operations are differentiable, that is, $\mathcal{U}\mathcal{U}\mathcal{U}^{-1} \subset V$ and

$$z^i = (xy)^i = f^i(x^1, x^2, \ldots, x^n, y^1, \ldots, y^n) \quad (i = 1, 2, \ldots, n),$$

and $(x^{-1})^i = h^i(x^1, \ldots, x^n)$ for $x, y \in \mathcal{U}$ are differentiable real functions. Then G is a Lie group.

To show this note that G is topologically homogeneous and thus G is a topological manifold. Let $\mathcal{B}$ be the family of local coordinate systems obtained by right group translations of $\mathcal{U}$. Let $g_1\mathcal{U}$ and $g_2\mathcal{U}$ overlap at P. Then the transformation of coordinates near P corresponds to the map $x \to g_2^{-1} g_1 x$ in $\mathcal{U}$, which is differentiable. Thus G is a differentiable manifold. Use the fact that a neighborhood of G generates G to prove that the group operations are differentiable.

<u>Example.</u> $GL_+(n)$. Near I use the n^2 coordinates of $\mathcal{M}_n$ ($n \times n$ real matrices) near O with $M \leftrightarrow e^M$.

<u>Example.</u> $O_+(n)$. Near I use the $\frac{n(n-1)}{2}$ coordinates of S near O with $-S = S^T$. Then $\exp S$ is one-to-one with a neighborhood of I in $O_+(n)$.

<u>Definition.</u> Let G_1 and G_2 be Lie groups. Then the topological direct product group $G_1 \times G_2$, with the product differentiable structure is a Lie group.

<u>Definition.</u> Let G be a Lie group and M^n a differentiable manifold. Assume that for each $g \in G$ there exists a diffeomorphism of M^n onto M^n

$$T_g : M^n \to M^n : x \to \varphi(x, g)$$

such that:

1.) $\varphi : G \times M^n \to M^n$ is a topological transformation group.

2.) The function $(g,x) \to \varphi(x,g) : G \times M^n \to M^n$ is differentiable (in both variables together). Then $\varphi : G \times M^n \to M^n$ is a Lie transformation group.

<u>Definition.</u> Lie transformation groups $\varphi_1 : G_1 \times M_1^n \to M_1^n$ and $\varphi_2 : G_2 \times M_2^n \to M_2^n$ are isomorphic in case there exists an isomorphism $f : G_1 \to G_2$ (onto) and a diffeomorphism $\psi : M_1^n \to M_2^n$ (onto) such that $\psi[\varphi_1(x_1, g_1)] = \varphi_2[\psi(x_1), f(g_1)]$ for all $(x_1, g_1) \in M_1^n \times G_1$.

<u>Definition.</u> Let G be a Lie group and M^n a differentiable manifold. Consider the differentiable manifold $G \times M^n$ and let

$$(x,g) \to \varphi(x,g)$$

be a differentiable map from an open neighborhood of $e \times M^n$ in $G \times M^n$ into M^n. Require

1.) $\varphi(\varphi(x,g_1),g_2) = \varphi(x, g_2 g_1)$ wherever defined and

2.) for each compact subset $K \subset M^n$ there exists a neighborhood N_K of e in G such that the map $T_g : x \to \varphi(x,g)$ is defined for each $x \in K$ and $g \in N_K$, and furthermore T_g is a homeomorphism of K onto some $K_g \subset M^n$. Then $\varphi : G \times M^n \to M^n$ is a local Lie transformation group.

<u>Note.</u> Identify two such local Lie transformation groups $\varphi_1 : G \times M^n \to M^n$ and $\varphi_2 : G \times M^n \to M^n$ in case they coincide on some neighborhood of $e \times M^n$ in $G \times M^n$. It is easy to verify that $\varphi(x,e) = x$ for all $x \in M^n$ and $T_{g^{-1}} = (T_g)^{-1}$ wherever defined.

<u>Definition.</u> Lie groups G_1 and G_2 are locally isomorphic in case there exist open neighborhoods U_1 and U_2 of the corresponding identities with a diffeomorphism f of U_1 onto U_2 such that f defines a local isomorphism between G_1 and G_2 as topological groups.

Definition. Let $\varphi_1 : G_1 \times M_1^n \to M_1^n$ and

$$\varphi_2 : G_2 \times M_2^n \to M_2^n$$

be local Lie transformation groups. Assume there exists a local isomorphism f of G_1 with G_2 and a diffeomorphism $\psi : M_1^n \to M_2^n$ (onto) such that $\psi[\varphi_1(x_1, g_1)] = \varphi_2[\psi(x_1), f(g_1)]$ for all (g_1, x_1) in an open neighborhood of $e_1 \times M_1^n$ in $G_1 \times M_1^n$. Then the two local Lie transformation groups are isomorphic.

Note. If $G = R^1$ and M^n is an open plane set, then these definitions coincide with those given for one-parameter transformation groups. However here we have the new problem of finding (up to local isomorphism) all n-parameter (or n-dimensional) Lie groups. This will be done through a study of the one-parameter subgroups of a Lie group G and through the infinitesimal group (or Lie algebra)" of G.

Definition. A Lie subgroup N of a Lie group G is a submanifold of G which is also a subgroup of the abstract group G.

Note. N may not be closed in G and the manifold topology on N may not be the inherited topology of G.

Theorem 21. Let G be a Lie group and let $v(e)$ be a tangent vector at the origin. Let v be the vector field on G defined by right group multiplications, that is $v(x) = dR_x v(e)$. Then v is a differentiable, right invariant vector field. The integral curve of v through e is a one-dimensional subgroup $\{g(t)\}$, in fact, a homomorphism of R^1 into G, and the other integral curves of v are right cosets of $\{g(t)\}$. Furthermore $\{g(t)\}$ is the unique (connected) 1-dimensional subgroup of G whose tangent vector at the origin is $v(e)$. Thus there is established a one-to-one correspondence between right-invariant vector fields of G and one-dimensional (connected) Lie subgroups of G .

Proof.

Let $\mathcal{U}$ be a neighborhood of e with local coordinates (x) so that

$$z^i = (xy)^i = f^i(x^1, \cdots, x^n, y^1, \cdots, y^n).$$

The vector $v(e)$ is, say $v_0^1 \frac{\partial}{\partial x^1} + \cdots + v_0^n \frac{\partial}{\partial x^n}$. Then at ω in $\mathcal{U}$ the vector $v(\omega)$ is represented by the curve $\varphi(t)\omega$ where $\dot{\varphi}^i(0) = v_0^i$. Thus we consider the curve

$$\psi^i(t) = f^i(\varphi^1(t), \cdots, \varphi^n(t), \omega^1, \cdots, \omega^n).$$

The components of $v(\omega)$ are $\frac{\partial f^i}{\partial x^j}(0,\omega) v_0^j + \frac{\partial f^i}{\partial y^j}(0,\omega)\cdot 0$.

Thus v is differentiable in $\mathcal{U}$.

Now near a point $g \in G$ use the local coordinates of $\mathcal{U}g$. The vector field v is clearly right-invariant. Since R_g is a diffeomorphism of G onto G carrying $\mathcal{U}$ to $\mathcal{U}g$, we see that v is differentiable everywhere.

Consider the integral curve $g(t)$ of v , through e at $t = 0$. Then the tangent vector to $g(t)$ is a vector of the field v . Thus, in the local coordinates near e ,

$$\frac{dg^i(t)}{dt} = \frac{\partial f^i}{\partial x^j}(0, g(t))\, g^j(t).$$

In Pontrjagin, theorem 46, it is shown that $g(t)\cdot g(u) = g(t+u)$ and hence $\{g(t)\}$ is a homomorphism (which is a local isomorphism) of R^1 onto a one-dimensional subgroup of G . It is also shown that each one-parameter subgroup of G , with the initial tangent vector v , satisfies the above differential equation and thus coincides with $\{g(t)\}$.

Q. E. D.

__Definition.__ For each tangent vector v at e in a Lie group G let $\{g(t)\}$ be the one-parameter subgroup initiating at v . Define $\exp v = g_v(1)$. Then $\exp\ T_e \to G$ is a diffeomorphism of a neighborhood of the origin in the vector space T_e onto a neighborhood of e in G. For each choice of basis in T_e we thus define the canonical coordinates (of the first kind) in a neighborhood of e in G. In canonical coordinates the one-parameter subgroups of G all have linear equations $g^i(t) = a^i t$, $|t|$ small.

Example. Let $G = GL_+(n,R)$. Then canonical coordinates are defined by the map $\exp\ M_n \to GL_+(n,R)$, in a neighborhood of the zero in the linear space of all $n \times n$ real matrices M_n. Thus $\begin{pmatrix} 0 & t \\ 2t & 0 \end{pmatrix}$ is the one-parameter subgroup in $GL_+(2,R)$, otherwise described by $e^{\begin{pmatrix} 0 & t \\ 2t & 0 \end{pmatrix}}$.

Theorem 22. A closed topological subgroup N of a Lie group G is a Lie subgroup of G. A Lie subgroup of G which is a closed subset of G inherits its topology and differentiable structure from G. In fact, if N is a closed Lie subgroup of G then there exist canonical coordinates $(x^1, \ldots, x^n)$ in a neighborhood U of e in G such that $N \cap U$ is exactly the locus $x^1 = 0, x^2 = 0, \ldots, x^m = 0$. The coordinates $(x^1, x^2, \ldots, x^m)$ can then be used to make the quotient space of (left) cosets G/N into a differentiable manifold. The natural projection $p: G \to G/N$ is a differentiable map.

Note. Dim G — dim N = dim G/N.

Theorem 23. Let N be a normal, closed Lie subgroup of a Lie group G. Then G/N is a Lie group. Also let $f: G \to H$ be a homomorphism of G onto a Lie group H. Then the kernel

$$N = f^{-1}(e) \subset G$$

is a normal, closed Lie subgroup of G and there exists a (diffeomorphism) isomorphism $\varphi: G/N \to H$ (onto) such that $f = \varphi p$.

<u>Theorem 24.</u> Let $\varphi: G\times M^n \to M^n$ be a Lie transformation group. Let N be the subgroup of G which acts as the identity transformation on M^n. Then N is a normal, closed Lie subgroup of G and $G/N \times M^n \to M^n$ is an effective Lie transformation group.

<u>Proof.</u>

We need only check that $\bar\varphi(x, Ng) = \varphi(x,g)$ is differentiable on $G/N \times M^n \to M^n$ which is clear.

Q. E. D.

<u>Theorem 25.</u> Let G be a Lie group and N a closed Lie subgroup of G. Then the topological transformation group $G\times G/N \to G/N$ is a Lie transformation group. The action is transitive and is effective if and only if the only closed Lie subgroup of G which lies in N is e.

<u>Proof.</u>

There are no non-trivial normal topological subgroups of G in N if and only if there are no non-trivial normal, closed Lie subgroups of G in N.

Q. E. D.

<u>Theorem 26.</u> Let $\varphi: G\times M^n \to M^n$ be a Lie transformation group which is transitive, effective. Assume, for a point $z \in M^n$, the map $G\to M^n: g \to \varphi(z,g)$ carries the tangent space at e onto the tangent space at z. Let N be the stability subgroup of z. Then $G\times G/N \to G/N$ is isomorphic with $\varphi: G\times M^n\to M^n$.

<u>Proof.</u>

Since $\varphi: G\times M^n \to M^n$ is transitive, effective, and also locally transitive there exists a homeomorphism ψ of the space of left cosets onto M^n which makes $G \times G/N \to G/N$ and $\varphi: G \times M^n \to M^n$ isomorphic as topological transformation groups. We need only show that is a diffeomorphism. But ψ is differentiable and carries the n-dimensional tangent space at (N) in G/N onto the n-dimensional tangent space at z in M^n. Thus ψ^{-1} differentiable near z. Using the transitivity of $\varphi: G \times M^n \to M^n$ we see that ψ is a diffeomorphism.

Q. E. D.

9. Lie Algebras.

A real linear algebra is a real linear vector space (possible infinite dimensional) together with a product between vectors such that

$$v(c_1 u + c_2 w) = c_1 vu + c_2 vw$$

$$(c_1 u + c_2 w)v = c_1 uv + c_2 wv \qquad \text{(bilinear).}$$

We do not require commutativity $uv = vu$, or associativity $(uv)w = u(vw)$ or the existence of a unit ε such that $\varepsilon v = v \varepsilon = v$.

Definition. A Lie algebra L is a real linear algebra such that

$$[u, v] = -[v, u] \qquad \text{(anti-commutative)}$$

and

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0 \qquad \text{(Jacoby identity).}$$

Note. $[u,u] = 0$ for all $u \in L$. The only commutative Lie algebra is the trivial multiplication $[u,v] = 0$.

Examples.

1. The Euclidean vector space R^3 using vector cross product.
2. The set of all $n \times n$ real matrices $\mathfrak{M}_n$ with $[A,B] = AB - BA$.
3. The unique 1-dimensional Lie algebra R^1 with $[u,u] = 0$.

<u>Theorem 27.</u> Let M^n be a differential manifold. Let $\mathcal{L}(M^n)$ be the real linear space consisting of all differentiable (contravariant) tangent vector fields on M^n. For two such vector fields u, v define the Lie product, $[u,v]^i = \frac{\partial u^i}{\partial x^j} v^j - \frac{\partial v^i}{\partial x^j} u^j$ (in local coordinates). Then $\mathcal{L}(M^n)$ is a Lie algebra.

<u>Proof.</u>

We first must show that $\frac{\partial u^i}{\partial x^j} v^j - \frac{\partial v^i}{\partial x^j} u^j$ is a vector field (independent of the choice of local coordinates). Use any Riemann metric and the tensor covariant derivative $\frac{\delta u^i}{\delta x^j} = \frac{\partial u^i}{\partial x^j} + \left\{ {i \atop \ell\, j} \right\} u^\ell$. Then $[u,v]^i = v^j \frac{\delta u^i}{\delta x^j} - u^j \frac{\delta v^i}{\delta x^j}$. Also one can show that $[u,v]$ is the "Lie derivative" of u along v. That is, $\mathcal{L}_v u = \lim_{t \to 0} \frac{T_t u(P_t) - u(P_0)}{t}$ where P_t is the trajectory of the one-parameter group generated by v and T_t is the induced transformation of the tangent space at P_0 onto the tangent space at P_t. Thus $[u,v]$ is a well-defined bilinear product on the set $\mathcal{L}(M^n) \times \mathcal{L}(M^n)$ into $\mathcal{L}(M^n)$. Clearly $[u,v]' = -[v,u]$. Also the Jacoby identity is easily verified by a direct computation.

Q. E. D.

<u>Definition.</u> Let M^n be a differentiable manifold and $\mathcal{L}(M^n)$ the Lie algebra of all differentiable vector fields on M^n. A finite dimensional subalgebra of $\mathcal{L}(M^n)$ is called an infinitesimal transformation group on M^n.

<u>Definition.</u> Let L_1 and L_2 be real Lie algebras. A homomorphism $f: L_1 \to L_2$ is a linear transformation from L_1 into L_2 such that

$$f([u,v]) = [f(u), f(v)] .$$

If f is one-to-one onto L_2, then f is an isomorphism of L_1 onto L_2.

Remark. The direct product of two Lie algebras is a Lie algebra.

Definition. A subalgebra $K \subset L$, a Lie algebra, is a linear subspace which is closed under products. Further K is an ideal of L in case $[\ell K] \subset K$ for every $\ell \in L$.

Theorem 28. If K is an ideal in the Lie algebra L then the (additive) cosets of K form a Lie algebra L/K under

$$[(\ell_1 + K), (\ell_2 + K)] = [\ell_1, \ell_2] + K .$$

Moreover the natural projection $p: L \to L/K : \ell \to \ell + K$ is a homomorphism of L onto L/K. Also if $f: L_1 \to L_2$ is a homomorphism of the Lie algebra L_1 onto the Lie algebra L_2, the kernel

$$K_1 = f^{-1}(0) \subset L_1$$

is an ideal in L_1 and there exists an isomorphism $\varphi : L/K_1 \to L_2$, onto L_2, such that $f = \varphi p$.

Example. Let L be a Lie algebra. The smallest subalgebra containing all the commutators $[u, v]$ is an ideal $[L, L]$, the commutator subalgebra of L (or first derived algebra).

Theorem 29. Every 2-dimensional Lie algebra is isomorphic with

a.) $[u, v] = 0$

or

b.) $[u, v] = v$ for a basis of vectors u, v.

Every 3-dimensional Lie algebra is isomorphic with

a.) $[u_1, u_2] = 0, [u_1, u_3] = 0, [u_2, u_3] = 0$ commutator is 0.

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b.) $[u_1, u_2] = 0, [u_1, u_3] = 0, [u_2, u_3] = u_1$

c.) $[u_1, u_2] = 0, [u_1, u_3] = u_1, [u_2, u_3] = 0$

commutator has dimension 1.

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d.) $[u_1, u_2] = 0, [u_1, u_3] = u_1, [u_2, u_3] = c\, u_2$ $(c \neq 0)$

e.) $[u_1, u_2] = 0, [u_1, u_3] = u_1 + u_2, [u_2, u_3] = u_2$

commutator has dimension 2.

f.) $[u_1, u_2] = 0$, $[u_1, u_2] = \mu u_1 + u_2$, $[u_2, u_3] = -u_1 + \mu u_2$ (all real μ)

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g.) $[u_1, u_2] = u_1$, $[u_1, u_3] = 2u_2$, $[u_2, u_3] = u_3$ commutator is all L

h.) $[u_1, u_2] = u_3$, $[u_2, u_3] = u_1$, $[u_3, u_1] = u_2$

Proof.

Let L be a two dimensional Lie algebra and take a basis u, v . Then $[u,u] = 0$, $[v,v] = 0$ and we need only specify $[u, v] = au + bv$. If $a = b = 0$ we have the unique commutative Lie algebra. Suppose a or b is not zero, say $b \neq 0$. Replace v by $v_1 = \frac{a}{b} u + v$ so $[u, v_1] = bu$. Now replace u by $u_1 = \frac{1}{b} u$ so $[u_1, v_1] = v_1$ as required for the second case.

Next let L be 3-dimensional. First assume the commutator ideal $[L, L]$ is zero. Then, for a basis u_1, u_2, u_3 , we have

$[u_1, u_2] = 0$, $[u_1, u_3] = 0$, $[u_2, u_3] = 0$

Next assume that $[L, L]$ is 1-dimensional. Choose a basis of L so that $[u_1, u_2] = \alpha u_1$, $[u_1, u_3] = \beta u_1$, $[u_2, u_3] = \gamma u_1$ where $\alpha^2 + \beta^2 + \gamma^2 > 0$
If $\alpha = \beta = 0$, change the scale of u_2 so that $[u_2, u_3] = u_1$. Thus $[u_1, u_2] = 0$, $[u_1, u_3] = 0$, $[u_2, u_3] = u_1$. If $\beta \neq 0$ (otherwise interchange names of u_2 and u_3) let $\bar{u}_2 = u_2 - \frac{\alpha}{\beta} u_3$ to get $[u_1, \bar{u}_2] = 0$, $[u_1, u_3] = \beta u_1$, $[\bar{u}_2, u_3] = \gamma u_1$. Finally change the scale on $u_3' = \beta u_3$ to get $[u_1, u_2'] = 0$, $[u_1, u_3'] = u_1$, $[u_2', u_3] = 0$, where $u_2' = \bar{u}_2 - \frac{\gamma}{\beta} u_1$.
These two cases are distinct since u_1 is distinguished (up to a constant multiple) as the generator of $[L, L]$ and in the first case u_1 annihilates L by Lie products.

Now assume $[L, L]$ is 2-dimensional. Choose a basis so that u_1, u_2 generate $[L, L]$. Then we can require $[u_1, u_2] = 0$ or else $[u_1, u_2] = u_2$. We show that the case $[u_1, u_2] = u_2$ is impossible here. Write $[u_2, u_3] = c_{23}^1 u_1 + c_{23}^2 u_2$, $[u_3, u_1] = c_{31}^1 u_1 + c_{32}^2 u_2$.

Use the Jacobi identity to obtain $c_{23}^2 = 0$, $c_{31}^2 = 0$. But then $[L,L]$ is not 2-dimensional. Thus we can assume $[u_1,u_2] = 0$. Consider the matrix representing Lie multiplication of $[L,L]$ by any element not in $[L,L]$, say $v = au_1 + bu_2 + cu_3$, $c \neq 0$. This matrix is

$$[v,u_1] = c(c_{31}^1 u_1 + c_{31}^2 u_2)$$
$$[v,u_2] = c(c_{23}^1 u_1 + c_{23}^2 u_2).$$

Thus the matrix is

$$cA = c\begin{pmatrix} c_{31}^1 & c_{31}^2 \\ c_{23}^1 & c_{23}^2 \end{pmatrix}.$$

Case 1.) A has simple real eigenvalues. Choose a new basis in $[L,L]$ so that $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 \lambda_2 \neq 0$.

Then, take $c = 1$, and write $[u_1,u_2] = 0$, $[u_3,u_1] = \lambda_1 u_1$, $[u_3,u_2] = \lambda_2 u_2$.

Change scale of u_3 so $\lambda_1 = 1$ and $\lambda_2 \neq 0$. Thus

$[u_1,u_2] = 0$, $[u_3,u_1] = u_1$, $[u_3,u_2] = \lambda_2 u_2$, $\lambda_2 \neq 0$.

These are distinct for distinct values of $\lambda_2 \neq 0$ since the ratio λ_2/λ_1 is determined by the algebra. The normalization $\lambda_1 = 1$ fixes λ_2.

Case 2.) A has a multiple eigenvalue. Choose a new basis in $[L,L]$ so that $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda \neq 0$.

Then $[u_1,u_2] = 0$, $[u_3,u_1] = \lambda u_1 + u_2$, $[u_3,u_2] = \lambda u_2$.

Let $\bar{u}_3 = \frac{1}{\lambda} u_3$. Then $[u_1,u_2] = 0$, $[\bar{u}_3,u_1] = u_1 + \frac{1}{\lambda} u_2$, $[\bar{u}_3,u_2] = u_2$.

Then let $\bar{u}_2 = \frac{1}{\lambda} u_2$. Then $[u_1,\bar{u}_2] = 0$, $[\bar{u}_3,u_1] = u_1 + \bar{u}_2$, $[\bar{u}_3,\bar{u}_2] = \bar{u}_2$.

Case 3.) A has complex conjugate eigenvalues $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $\beta \neq 0$

Then $[u_1,u_2] = 0$, $[u_3,u_1] = \alpha u_1 + \beta u_2$, $[u_3,u_2] = -\beta u_1 + \alpha u_2$.

Change scale on u_3 so $\bar{u}_3 = \frac{1}{\beta} u_3$,

$$[u_1, u_2] = 0, \quad [\bar{u}_3, u_1] = \frac{\alpha}{\beta} u_1 + u_2, \quad [\bar{u}_3, u_2] = -u_1 + \frac{\alpha}{\beta} u_2 .$$

Thus

$[u_1, u_2] = 0, [u_3, u_1] = \mu u_1 + u_2, [u_3, u_2] = -u_1 + \mu u_2$, all real μ.

The complex number $\alpha + i\beta$, up to a real multiplier, is determined by the algebra. If we normalize $\beta = 1$, then α is fixed.

Finally assume $[L, L]$ has dimension 3. Then no independent elements commute — for then $[L, L]$ has dim less than 3. If there is a real subalgebra of dimension 2, take a basis so that

$[u_1, u_2] = u_1, [u_1, u_3] = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, [u_2, u_3] = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3$.
From the Jacobi identity we find $\alpha_3 = 0$, $\beta_2 + \beta_3 \alpha_1 = 0$, $\alpha_2(1 - \beta_3) = 0$.
Now $\alpha_2 \neq 0$ for otherwise $[u_1, u_3] = \alpha_1 u_1 = \alpha_1 [u_1, u_2]$ and $[L, L]$ does not have dimension 3. Thus $\beta_3 = 1$, $\beta_2 = -\alpha_1$. Thus

$[u_1, u_2] = u_1, [u_1, u_3] = \alpha_1 u_1 + \alpha_2 u_2, [u_2, u_3] = \beta_1 u_1 - \alpha_1 u_2 + u_3, \alpha_2 \neq 0$.
Let $u_3' = u_3 + \frac{\beta_1}{2} u_1 - \alpha_1 u_2$. Then $[u_1, u_3'] = \alpha_2 u_2$, $[u_2, u_3'] = u_3'$ (replace u_1 by $-u_1$ to make $\alpha_2 > 0$). Let $\bar{u}_1 = a u_1, \bar{u}_2 = u_2, \bar{u}_3 = a u_3'$ and take $a^2 \alpha_2 = 2$. Then

$$[\bar{u}_1, \bar{u}_2] = \bar{u}_1, \quad [\bar{u}_1, \bar{u}_3] = 2\bar{u}_2, \quad [\bar{u}_2, \bar{u}_3] = \bar{u}_3 .$$

Next assume $[L, L]$ is 3 dim and L contains no 2 dim subalgebra. Then one can show that the unique algebra is

$$[u_1, u_2] = u_3, \quad [u_2, u_3] = u_1, \quad [u_3, u_1] = u_2 .$$

Q. E. D.

<u>Theorem 30.</u> Let G be a n-dimensional Lie group. The right invariant vector fields of G, under the Lie product $[u, v]$, form a n-dimensional Lie algebra.

Proof.

Let u and v be right invariant vector fields. Let $(x^1, \cdots, x^n)$ be local coordinates in an open neighborhood W of e in G. For each point $g \in G$ consider the neighborhood W_g and use the local coordinates in W_g defined by the diffeomorphism $W \leftrightarrow W_g$. If the components of u in $(x^1, \cdots, x^n)$ are u^i then the components of u in W_g are also u^i. In fact a vector field is right invariant just in case it has the same numerical components in each such coordinate system in each such W_g.

Since $[u,v]^i = \frac{\partial u^i}{\partial x^j} v^j - \frac{\partial v^i}{\partial x^j} u^j$ we see that $[u,v]$ has the same components in W as in W_g. Thus $[u,v]$ is right invariant.

Thus the right invariant vector fields of G form a Lie algebra $\mathcal{L}_R(G)$. There is an isomorphism of the vector space T_e onto the vector space of $\mathcal{L}_R(G)$ since each tangent vector at e generates exactly one right invariant vector field. Thus $\dim \mathcal{L}_R(G) = n$.

Q. E. D.

Note. If right invariant vector fields u and v of $\mathcal{L}_R(G)$ are represented at the origin by curves $\varphi(t)$ and $\psi(t)$, then $c_1 u + c_2 v$ is represented at the origin by the curve $\varphi(c_1 t) \cdot \psi(c_2 t)$. Also $[u,v]_e$ is represented by the curve $\chi(s)$ where $\chi(t^2) = \varphi(t)\psi(t)\varphi(t)^{-1}\psi(t)^{-1}$. This is proved in Pontrjagin's text (Thm. 66) and is useful in computing the Lie product, or commutator, of vector fields u and v in $\mathcal{L}_R(G)$. Thus if G is commutative, so is the Lie algebra $\mathcal{L}_R(G)$ commutative.

The Lie algebra of $GL(n,R)$ is $\mathcal{M}_n$ with the Lie product $[A,B] = AB - BA$. Thus for any subgroup of $GL(n,R)$ the Lie algebra will be a subalgebra of $\mathcal{M}_n$.

Example. Compute the Lie algebra of the Lorentz group $L(4)$. Choose coordinates x_{ij} $(i,j = 1,2,3,4)$ near O , valid near I . Thus a matrix in $L(4)$ near I is

$$A = \begin{pmatrix} 1+x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & 1+x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & 1+x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & 1+x_{44} \end{pmatrix}.$$

These x_{ij} are subjected to the 10 defining equations of $L(4)$, namely $AJA^T = J$, where $J = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$. Choose a curve $x_{ij}(t)$, through I at $t = 0$, with initial tangent vector $a_{ij} = \dot{x}_{ij}(0)$. Differentiate the ten defining equations of $L(4)$, with respect to t , and set $t = 0$. In this way we obtain the subalgebra of $\mathcal{M}_4$ corresponding to $L(4) \subset GL(n, R)$

The Lie algebra of $L(4)$ consists of all real matrices of the form

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & a_{23} & a_{24} \\ a_{13} & -a_{23} & 0 & a_{34} \\ a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

__Theorem 31.__ Let G be a Lie group and $\mathcal{L}_R(G)$ be its Lie algebra. For each Lie subgroup N of G , the vector fields of $\mathcal{L}_R(G)$ which are tangent to N form a subalgebra $\mathcal{L}_R(N) \subset \mathcal{L}_R(G)$. Also each subalgebra of $\mathcal{L}_R(G)$ corresponds to exactly one such (connected) Lie subgroup of G . Thus there is a one-to-one correspondence between connected Lie subgroups of G and subalgebras of $\mathcal{L}_R(G)$. Furthermore a connected subgroup N is normal in G if and only if $\mathcal{L}_R(N)$ is an ideal in $\mathcal{L}_R(G)$.

__Theorem 32.__ Two Lie groups G_1 and G_2 are locally isomorphic if and only if their Le algebras are isomorphic. For each n-dimensional Lie algebra L there exists a unique (up to Lie isomorphism) simply-connected Lie group G

with $\mathcal{L}_R(G)$ isomorphic to L. If a connected Lie group H also has the Lie algebra L, then $H \cong G/N$ where N is a discrete normal central subgroup of G.

Example. Compute all connected Lie groups of dimension 2.

Case 1.) $[u,v] = 0$, commutative Lie algebra. Simply-connected commutative Lie group is plane R^2 with vector addition. Let N be a discrete normal subgroup of R^2. Then, after an automorphism of R^2, N consists in the integral multiples of one vector, or of two independent vectors. Thus there are just 3 groups, R^2, $S^1 \times R^1$, $S^1 \times S^1$. In general a n-dim. commutative Lie group is the product of circles and lines.

Case 2.) $[u,v] = v$. Here the simply-connected Lie group is $GA_+(1) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0$. The Lie algebra of $GA_+(1)$ is the subalgebra of $\mathcal{M}_2$ with basis $\bar{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\bar{v} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $[\bar{u}, \bar{v}] = -\bar{u}$. The center of $GA_+(1)$ consists only of the identity I. Thus $GA_+(1)$ is the only Lie group for this case.

Note. It A and B are endomorphisms of a vector space V (linear transformations of V into itself), then so is $[A,B] = AB - BA$ an endomorphism of V. This defines the Lie algebra of endomorphisms of V.

For $[A,B] = -[B,A]$ and $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$.

If a finite basis is designated in V, then this Lie algebra is a subalgebra of $\mathcal{M}_n$.

Definition. Let L be a Lie algebra. For each $\ell \in L$ consider the linear endomorphism of the vector space L into itself by $M_\ell : k \to [\ell, k]$. The map $\ell \to M_\ell$ is a homomorphism of L onto a Lie algebra of linear transformations of the vector space L into itself. If L is n-dimensional and we pick a basis in L, then $\ell \to M_\ell$ is a homomorphism of L into the matrix Lie algebra $\mathcal{M}_n$. This is the "adjoint representation" of L. We check that $\ell \to M_\ell$ is a homomorphism of L into the Lie algebra of endomorphisms of L.

$c_1\ell_1 + c_2\ell_2$ corresponds to the endomorphism

$$k \to [c_1\ell_1 + c_2\ell_2, k] = c_1[\ell_1, k] + c_2[\ell_2, k] = (c_1 M_{\ell_1} + c_2 M_{\ell_2})k.$$

Further $[\ell_1, \ell_2]$ corresponds to the endomorphism

$$k \to [[\ell_1,\ell_2], k] = [\ell_1,[\ell_2,k]] - [\ell_2,[\ell_1,k]] = (M_{\ell_1}M_{\ell_2} - M_{\ell_2}M_{\ell_1})k.$$

Thus the adjoint representation is a homomorphism $L \to \operatorname{End} L$.

<u>Remark.</u> The adjoint representation is an isomorphism of L into the Lie algebra of endomorphisms of L just in case L has no center.

For the kernel of $\operatorname{adj}: L \to \operatorname{End} L$ consists of those $\ell \in L$ for which $[\ell, k] = 0$ for all $k \in L$, that is, those $\ell \in L$ for which $[\ell, k] = [k, \ell]$ for all $k \in L$.

<u>Example.</u> Take $[u, v] = v$ as L. Here the center is empty. Use the basis u, v. Then $u \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $v \to \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. Thus the Lie group with L is the smallest subgroup of $GL(2, R)$ containing $\exp\left[a\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right]$. Here $\exp\begin{pmatrix} 0 & -b \\ 0 & a \end{pmatrix} = I + \begin{pmatrix} 0 & -b \\ 0 & a \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & -b \\ 0 & a \end{pmatrix}^2 + \dots$. Thus $\exp\begin{pmatrix} 0 & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & b\,\frac{1-e^a}{a} \\ 0 & e^a \end{pmatrix}$. This group is isomorphic with $GA_+(1)$.

10. <u>Infinitesimal Transformation Groups.</u>

<u>Definition.</u> Let $\varphi: G \times M^n \to M^n$ be a Lie transformation group, or only a local Lie transformation group. Let the Lie algebra $\mathcal{L}_R(G)$ of G be represented by the tangent space at e. For each vector $v \in \mathcal{L}_R(G)$ consider the one-parameter subgroup $g_v(t) \subset G$ with initial vector v. Then $g_v(t)$ acts on M^n by $\varphi(x, g_v(t))$. For each point $x_0 \in M^n$

consider the curve $\varphi(x_0, g_v(t))$ in M^n and let $V(x_0, v)$ be the tangent vector to this curve, that is $V^i(x_0, v) = \frac{\partial \varphi^i}{\partial t}(x_0, e)$.

The vector field $V(x, v)$ is differentiable on M^n (since $\varphi^i(x, g_v(t))$ is differentiable in $M^n \times R^1$). Thus we map $\mathcal{L}_R(G)$ into the linear space of differentiable vector fields on M^n. Call the image $\Lambda \subset \mathcal{L}(M^n)$. Then Λ is called the infinitesimal generator of $\varphi: G \times M^n \to M^n$.

<u>Theorem 33.</u> Let $\varphi: G \times M^n \to M^n$ be a local Lie transformation group and let $\Lambda \subset \mathcal{L}(M^n)$ be its infinitesimal generator. Then the map $\mathcal{L}_R(G) \to \Lambda$ is an homorphism of the Lie algebra $\mathcal{L}_R(G)$ onto the Lie algebra Λ. If $\varphi: G \times M^n \to M^n$ is locally effective, then $\mathcal{L}_R(G) \to \Lambda$ is an isomorphism.

<u>Proof.</u>

We first show that Λ is a linear space in $\mathcal{L}(M^n)$ and that $\mathcal{L}_R(G) \to \Lambda$ is a linear transformation onto Λ.

Consider the vector $c_1 v_1 + c_2 v_2 \in \mathcal{L}_R(G)$. Consider the 1-parameter local transformation group $\varphi(x, g_{c_1 v_1 + c_2 v_2}(t))$ and compute (in local coordinates) $V^i(x, c_1 v_1 + c_2 v_2) = \frac{\partial \varphi^i}{\partial t}(x, e)$.

Use canonical coordinates of first kind near e in G so $g^i_{c_1 v_1 + c_2 v_2}(t) = (c_1 v_1^i + c_2 v_2^i)t$. Then, writing $\varphi(x^1, \ldots, x^n, z^1, \ldots, z^n)$,

$$V^i(x, c_1 v_1 + c_2 v_2) = \frac{\partial \varphi^i}{\partial z^j}(x, 0)(c_1 v_1^j + c_2 v_2^j) .$$

Thus

$$V^i(x, c_1 v_1 + c_2 v_2) = c_1 V^i(x, v_1) + c_2 V^i(x, v_2) .$$

Thus Λ is a linear space, and $\mathcal{L}_R(G) \to \Lambda$ is a linear transformation.

Now assume $\varphi: G \times M^n \to M^n$ is locally effective (for a local transformation group this means that there is a neighborhood of e in which the only group element yielding the identity transformation is e — otherwise there is a 1-parameter subgroup which acts as the identity). If $v \to V(x,v) \equiv 0$ then for each x, $\varphi(x, g_v(t)) = x$ and so $\varphi: G \times M^n \to M^n$ is not effective. Thus the kernel of $\mathcal{L}_R(G) \to \Lambda$ is zero.

It is shown that $\mathcal{L}_R(G) \to \Lambda$ preserves the Lie product on p. 288, Pontrjagin. Q. E. D.

__Example__. Projective local transformation group on plane.

$$x_1 = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \qquad y_1 = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}.$$

This is a local transformation group. Coordinates in projective group are

$$\begin{pmatrix} a_1 = 1 + \alpha_1 & b_1 & c_1 \\ a_2 & b_2 = 1 + \beta_2 & c_2 \\ a_3 & b_3 & c_3 = 1 \end{pmatrix}.$$

Find basis for Λ, infinitesimal generator of projective transformation group. This consists of 8 vector fields in R^2.

$$x_1 = \frac{x + (\alpha_1 x + b_1 y + c_1)\delta t}{1 + (a_3 x + b_3 y)\delta t} = x + (\alpha_1 x + b_1 y + c_1)\delta t - x(a_3 x + b_3 y)\delta t + \cdots$$

and

$$\frac{x_1 - x}{\delta t} = \alpha_1 x + b_1 y + c_1 - a_3 x^2 - b_3 xy + O(\delta t).$$

$$\frac{y_1 - y}{\delta t} = a_2 x + \beta_2 y + c_2 - a_3 xy - b_3 y^2 + O(\delta t).$$

Thus consider the vector fields

$$\mathcal{U} = (\alpha_1 x + b_1 y + c_1 - a_3 x^2 - b_3 xy)\frac{\partial}{\partial x} + (a_2 x + \beta_2 y + c_2 - a_3 xy - b_3 y^2)\frac{\partial}{\partial y}.$$

Thus a basis for Λ consists of

$$u_1 = \frac{\partial}{\partial x}, \quad u_2 = \frac{\partial}{\partial y}, \quad u_3 = x\frac{\partial}{\partial x}, \quad u_4 = x\frac{\partial}{\partial y},$$

$$u_5 = y\frac{\partial}{\partial x}, \quad u_6 = y\frac{\partial}{\partial y}, \quad u_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad u_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

Or in Lie's notation:

$$p,\ q,\ xp,\ xq,\ yp,\ yq,\ x^2p + xyq,\ xyp + y^2q.$$

The affine transformation group on R^2 is generated by

$$p,\ q,\ xp,\ xq,\ yp,\ yq.$$

__Note.__ The one-parameter transformations generated by the members of Λ generate all $G \times M^n \to M^n$ (for a connected group G).

__Theorem 34.__ Let M^n be a differentiable manifold and Λ an infinitesimal transformation group on M^n. Then there exists exactly (up to isomorphism) locally effective, local Lie transformation group $\varphi: G \times M^n \to M^n$ for which the infinitesimal generator is Λ. If there exists a Lie transformation group generated by Λ, then there exists a unique effective Lie transformation group generated by Λ. This is always the case if M^n is compact.

__Corollary.__ If there exists a Lie transformation group generated by Λ, then Λ generates a unique locally effective $\varphi: \tilde{G} \times M^n \to M^n$ where $\tilde{G}$ is simply connected. The locally effective transformation groups generated by Λ are exactly $\tilde{G}/K \times M^n \to M^n$ where K is a discrete normal subgroup of $\tilde{G}$ which is contained in the discrete normal subgroup N yielding the identity transformation.

Theorem 35. Let $\varphi_1 : G_1 \times M_1^n \to M_1^n$ and

$$\varphi_2 : G_2 \times M_2^n \to M_2^n$$

be locally effective, local Lie transformation groups generated by Λ_1 and Λ_2, respectively. The local Lie transformation groups are isomorphic if and only if there exists a diffeomorphism of M_1^n onto M_2^n carrying Λ_1 onto Λ_2.

Corollary. If
$$\varphi_1 : G_1 \times M_1^n \to M_1^n$$
$$\varphi_2 : G_2 \times M_2^n \to M_2^n$$
are effective transformation groups generated by Λ_1 and Λ_2 respectively, then they are isomorphic if and only if there is a diffeomorphism of M_1^n onto M_2^n carrying Λ_1 onto Λ_2.

Thus all problems concerning Lie transformation groups can be referred to their infinitesimal generators.

Definition. Let Λ be an infinitesimal transformation group on a differentiable manifold M^n. For each point $P \in M^n$ let Λ_P be the subspace of the tangent space at P which is spanned by the vectors of Λ. Let $r = \max_{P \in M^n} \dim \Lambda_P$ and $0 \le r \le n$. If $\dim \Lambda_Q = r$, then Q is an ordinary point of Λ. The set of ordinary points is open in M^n. If $\dim \Lambda_Q < r$, then Q is a critical point of Λ. If $\dim \Lambda_Q = 0$, then Q is a fixed point of Λ. If $\dim \Lambda_Q = r$, then we say that Λ is locally transitive at Q.

Now let Λ be the infinitesimal generator of a 2-dimensional locally effective, local transformation group on R^2. In the neighborhood of P, an ordinary point of Λ, we shall choose local coordinates to display in a certain canonical form.

<u>Case 1.</u> Λ is commutative.

a.) Λ is locally transitive (map of tangent space of $e \in G$ is onto tangent space at P). Then there exists a basis for Λ of vector fields of the form $p, \varphi(x,y)q$ (in appropriate local coordinates near P) with $\varphi \neq 0$. But $[p, \varphi(x,y)q] = -\frac{\partial\varphi}{\partial x} p = 0$ so $\varphi = \varphi(y)$. Now change coordinates by $\bar{x} = x$, $\bar{y} = \int_0^y \frac{d\eta}{\varphi(\eta)}$ and then Λ has a basis $\boxed{p, q}$.

b.) Λ is commutative but nowhere locally transitive. Then take a basis for Λ in the form $p, \psi(x,y)p$ (we assume that Λ contains non-zero vectors). Then $[p, \psi(x,y)p] = -\frac{\partial\psi}{\partial x} p = 0$. Thus $\psi = \psi(y)$. Let $\bar{y} = \psi(y)$ (choose an open set near P in which $\psi'(y) \neq 0$ so Λ becomes $\boxed{p, yp}$.

<u>Case 2.</u> Λ is not commutative.

a.) Λ is locally transitive at P . Choose a basis for Λ of the form $q, \varphi_1(x,y)p + \varphi_2(x,y)q$ with $\varphi_1(x,y) \neq 0$. Then

$$[q, \varphi_1(x,y)p + \varphi_2 q] = -\frac{\partial\varphi_1}{\partial y} p - \frac{\partial\varphi_2}{\partial y} q = q .$$

Thus $\varphi_1 = \varphi_1(x)$ and $\varphi_2 = -y - a(x)$.

Let $\bar{x} = \int_0^x \frac{-d\xi}{\varphi_1(\xi)}$ to write generators for Λ as $q, -xp - (y + b(x))q$, or $q, xp + (y + b(x))q$, where $P = (x_0, y_0)$ and $x_0 \neq 0$. Again change variables by $\bar{x} = x, \bar{y} = y - x\int_{x_0}^x \frac{b(\xi)d\xi}{\xi^2}$.

Write Λ as $\boxed{q, xp + yq}$

b.) Λ not commutative and nowhere locally transitive. Choose a basis for Λ of the form $q, \psi(x,y)q$ with $[q, \psi(x,y)q] = -\frac{\partial\psi}{\partial y} q = q$. Thus $\psi = -y + h(x)$. Thus take a basis $q, (y - h(x))q$. Let $\bar{x} = x$, $\bar{y} = y - h(x)$ to get $\boxed{q, yq}$.

The problem of finding all transformation groups on the plane is very complicated as evidenced by the following examples.

Example. $q, xq, x^2q, \ldots, x^{r-4}q, yq, p, xp$ is a basis for a transitive infinitesimal transformation group on R^2. Yet the dimension is arbitrarily large.

Example. p, xp, x^2p, x^3p does not determine an infinitesimal transformation group on R^1 since the smallest Lie algebra in $\mathcal{L}(R^1)$ containing these four vector fields is infinite dimensional. For if $U_r = x^r p$ for $r \geq 0$, then $[U_r, U_s] = (s-r)U_{r+s-1}$.

Definition. Let V be a differentiable vector field on a connected differentiable M^n and choose local coordinates around a point $P \in M^n$ so

$$V^i(x) = (a^i + a^i_j x^j + a^i_{jk} x^j x^k + \ldots) \frac{\partial}{\partial x^i}.$$

The lowest order of the functions $a^i + a^i_j x^j + a^i_{jk} x^j x^k + \ldots$ at $P = 0$ is called the order of $V(x)$ at P. If $a^i = 0$, but some $a^i_j \neq 0$, then V has order 1 at P. A C^∞ vector field can have order ∞ at P but an analytic $(\neq 0)$ vector field (on a real analytic M^n) must have a finite order. The order of V at P is independent of the choice of local coordinates around P.

Theorem 36. Let V and U be infinitesimal transformations (that is, differentiable vector fields) on M^n. If V has order $\alpha \geq 0$ and U has order $\beta \geq 0$ at P, then $[U,V]$ has order $\geq \alpha + \beta - 1$.

Proof.

Write $V^i(x) = a^i_{j_1 \ldots j_\alpha} x^{j_1} \ldots x^{j_\alpha} + \ldots$ and $U^i(x) = b^i_{k_1 \ldots k_\beta} x^{k_1} \ldots x^{k_\beta} + \ldots$.

Then $[U,V]^i = \frac{\partial V^i}{\partial x^j} U^j - \frac{\partial U^i}{\partial x^j} V^j$

so the order of $[U,V]$ is $\geq \alpha + \beta - 1$.

Q. E. D.

Note: It might happen that $[u,v] \equiv 0$ and thus has infinite order.

Definition. Let M^n be a differentiable manifold and consider the tangent space T_P at a point P. The one-dimension subspace of T_P are called line elements and the set of all line elements at P is the line space L_P at P. Now L_P is a differentiable manifold since for each choice of local coordinates $(x^1, \ldots, x^n)$ on M^n around P there is a natural basis $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ for T_P and thus coordinates in T_P. These furnish homogeneous coordinates (still called $(x^1, \ldots, x^n)$) in L_P so that L_P is diffeomorphic with the real projective space P^{n-1}. A change in basis in T_P defines a projectivity of L_P onto itself.

Definition. Let Λ be an infinitesimal transformation group in a differentiable manifold M^n. For each point $P \in M^n$ consider the vector fields of Λ which have order ≥ 1 at P. This forms a subalgebra $\Lambda_1(P)$ of Λ. Each member of $\Lambda_1(P)$ defines a local one-parameter transformation group on M^n, with P fixed, and thus a one-parameter transformation group on L_P. Thereby we obtain a homomorphism of $\Lambda_1(P)$ onto an infinitesimal transformation group on L_P, called the direction transformation group D_P at P.

Theorem 37. Let Λ be an infinitesimal transformation group on a differentiable manifold M^n and let $P \in M^n$. Then the homomorphism $\Lambda_1(P) \to D_P$ can be expressed in local coordinates by

$$V^i(x) = (a^i_j x^j + a^i_{jk} x^j x^k + \ldots)\frac{\partial}{\partial x^i} \longrightarrow a^i_j x^j \frac{\partial}{\partial x^i} .$$

Therefore D_P is a subalgebra of the infinitesimal projective transformation group on P^{n-1}.

Note. A change of coordinates near $P \in M^n$ defines a projectivity of $L_P \cong P^{n-1}$ onto itself which defines an isomorphism between two representations of D_P as a subalgebra of the infinitesimal projective transformation group.

Note. The projective transformation group on P^{n-1} has dimension n^2-1.

Theorem 38. Let Λ be an infinitesimal transformation group, locally transitive at each point of a connected differentiable manifold M^n. Then the local transformation group generated by Λ is transitive on M^n.

Proof.

Let $P \in M^n$ and consider the set $K \subset M^n$ of all points which are images of P under the local transformation group. Then K is clearly open. If $Q \in \bar{K}$ then there is a neighborhood N of Q which consists of images of Q under Λ. Take $P_1 \in K \cap N$. Then move P to P_1 and thereafter to Q, under Λ. Thus $K = M^n$ and Λ is transitive on M^n. Q. E. D.

Theorem 39. Let Λ be a locally transitive (everywhere) infinitesimal transformation group on a connected differentiable manifold M^n. Let P and Q be points of M^n with corresponding direction transformation groups D_P and D_Q. Then there is a projectivity of L_P onto L_Q which carries D_P onto D_Q. Thus D_P on L_P and D_Q on L_Q are isomorphic transformation groups and, for a correct choice of local coordinates near P and Q, D_P and D_Q are represented by the same subalgebra of the infinitesimal projective transformation group on P^{n-1}.

Definition. Let Λ be an infinitesimal transformation group on a differentiable manifold M^n. If, at $P \in M^n$, D_P is all the infinitesimal projective transformation group, then we call Λ primitive at P. (The term "primitive" is slightly different in the works of Lie).

<u>**Theorem 41.**</u> Let Λ be an everywhere locally transitive and primitive, analytic infinitesimal transformation group on a real analytic manifold M^n. Then, for each $P \in M^n$, each member of Λ has an order ≤ 2 at P. Furthermore the dimension of Λ is $\leq B(n) \leq n + n^2 - 1 + \frac{n^2(n+1)}{2}$.

<u>**Proof.**</u>

Choose local coordinates near $P \in M^n$, say $(x^1, \cdots, x^n)$. Then Λ contains n vector fields of the form

$$p_1 + \cdots, \; p_2 + \cdots, \; \cdots, \; p_n + \cdots$$

where $p_i = \frac{\partial}{\partial x^i}$. The rest of a basis for Λ can be chosen from $\Lambda_1(P)$. There is an independent set of $n^2 - 1$ in $\Lambda_1(P)$ which maps onto D_P.

We display these for the case $n = 2$ as $xq + \cdots$, $yp + \cdots$, $xp - yq + \cdots$. Now let $u = \xi_s p + \eta_s q + \cdots$ for $s \geq 2$ be a member of Λ of highest order s at P. Then

$$[xq + \cdots, u] = x\frac{\partial \xi_s}{\partial y} p + \left(x\frac{\partial \eta_s}{\partial y} - \xi_s\right) q + \cdots$$

is of order s but of smaller degree in y in $x\frac{\partial \xi_s}{\partial y}$. Repeat this process so that we can assume that ξ_s does not contain y. Thus $\xi_s = x^s$. Now $[p, x^s p + \eta_s q + \cdots] = s x^{s-1} p + \eta_{s-1} q + \cdots$ is of order $\geq s - 1$. But

$$[s x^{s-1} p + \eta_{s-1} q + \cdots, x^s p + \eta_s q + \cdots] = -s x^{2s-2} p + \cdots$$

is of order $2s - 2$. Thus $2s - 2 \leq s$ and $s \leq 2$.

Now we complete our basis for Λ by vector fields which are of order 2 at P. But the number of symmetric linearly independent bilinear forms $a^i_{jk} x^j x^k$ $i, j, k = 1, 2, \ldots, n$ is just $n\left[\frac{n(n+1)}{2}\right] = \frac{n^2(n+1)}{2}$.

Thus

$$\dim \Lambda \leq B(n) \leq n + n^2 - 1 + \frac{n^2(n+1)}{2}.$$

Q. E. D.

<u>**Note.**</u> For the plane R^2 one can show that $B(2) = 8$ and this dimension is realized by the infinitesimal projective transformation group. For R^1,

$B(1)=3$ and all locally transitive analytic Λ are subalgebras of p, xp, x^2p.

For locally transitive Λ in the plane R^2 we can define the germ of the infinitesimal transformation group near a point $P \in R^2$ by restricting the open neighborhood of P in which we consider Λ. Such a germ is isomorphic with a local Lie group G acting on a quotient space G/N, where N is a closed abnormal local subgroup of G. In other words the germ of Λ near P is specified by a pair (R,S) of real Lie algebras where S is an abnormal subalgebra of R. The germ of Λ near P can be extended to a global transformation group on a manifold M^2 just in case the abnormal subgroup $\tilde{N}$ determined by S, is closed in the simply-connected Lie group $\tilde{G}$ determined by R. It is known that every such germ on R^2 can be extended to a global transformation group on M^2. However the corresponding statement is false in R^5.

The analytic, locally transitive, primitive, infintesimal transformation groups on R^2 each have a germ isomorphic with one of the following:

1. $p, q, yp, xq, xp-yq$;
2. p, q, xp, yp, xq, yq;
3. $p, q, xp, yp, xq, yq, x(xp+yq), y(xp+yq)$.

11. Differential Invariants of Transformation Groups.

Definition. A fiber bundle consists of three differentiable manifolds, the bundle space B, the base space M^n, the fiber F and a differentiable map $p: B \to M^n$ called projection onto M^n. For each point $P \in M^n$ there exists a local coordinate system $U(x)$ and a prescribed diffeomorphism of $p^{-1}(U)$ onto $U(x) \times F$. Using these "product coordinates" $U(x) \times F$ in $p^{-1}(U)$ the projection map is $p: U(x) \times F \to U(x)$.

The set $p^{-1}(P)$ is the fiber above P and it is diffeomorphic with F.

Definition. Let B be a fiber bundle over the base M^n. A cross-section is a differentiable map $\varphi: M^n \to B$ (into) such that $p\varphi =$ identity on M^n.

Remark. Assume there exists a Lie group G which acts effectively on F, $G \times F \to F$, and assume for each intersection of local coordinates $U_\alpha(x) \cap U_\beta(y)$ there exists a differentiable map of $U_\alpha \cap U_\beta \to G : Q \to g^\alpha_\beta(Q)$ Require that a point in the fiber above any $Q \in U_\alpha \cap U_\beta$ should have "product coordinates" (x_Q, f_α) and (y_Q, f_β) where $f_\beta = [g^\alpha_\beta(Q)] f_\alpha$. Then G is called the structure group of the bundle $\{B, M^n, F, p\}$.

Example. Let M^n be a differentiable manifold and consider the set of all (contravariant) tangent vectors at all points of M^n. Call this set of all tangent vectors $T(M^n)$. Coordinates defining the differentiable structure (and the topology) on $T(M^n)$ are defined for each coordinate system $U(x)$ in M^n as follows:

For a vector V in the tangent space at $Q \in U(x)$ write $V = v^i \frac{\partial}{\partial x^i}$ and take the $2n$ coordinates $(x^1_Q, \cdots, x^n_Q, v^1, \cdots, v^n)$. The projection is $V \in T_Q \Rightarrow p: V \to Q$. Thus we have defined the tangent bundle $T(M^n)$ over M^n. The fiber is R^n. The transition functions are $g^\alpha_\beta(Q) = \frac{\partial y^i}{\partial x^j} \in GL(n,R)$, and the structure group of the $T(M^n)$ is $GL(n,R)$.

Example. A line element at $P \in M^n$ is a one-dimensional subspace of the tangent space T_P. The set of all line elements at all points of M^n forms the line element bundle $L(M^n)$. The fiber is the real projective space P^{n-1} and $(x^1_Q, \cdots x^n_Q, v^1, \cdots, v^n)$, where the last coordinates are not all zero and are specified only up to a non-zero common multiple, yields the "product coordinates". The structure group of $L(M^n)$ is $PGL(n,R) = GL(n,R)/(cI)$. Note that there is no natural embedding of the base space M^n in $L(M^n)$. Write the projection map $p_1 : L(M^n) \to M^n$.

Example. A curvature line element is a class of non-singular differentiable maps of $f: R^1 \to M^n$ with $f(0) = P$. Let $(x^1, \ldots, x^n)$ be local coordinates centered at P and such that the tangent vector to f has a non-zero component along the x^1-axis. Say that f_1 and f_2 are equivalent (define the same curvature line element) in case the two curves can be written $x^2 = \varphi^2(x^1), \quad , x^n = \varphi^n(x^1)$ and $x^2 = \psi^2(x^1), \ldots, x^n = \psi^n(x^1)$ with $\varphi^{i\,\prime\prime}(0) = \psi^{i\,\prime\prime}(0)$ for $i = 2, 3, \ldots, n$. The set of all curvature line elements at all points of M^n is the bundle space $K(M^n)$. The $3n-2$ product coordinates in $K(M^n)$ are

$$(x^1, x^2, \ldots, x^n, \frac{v_2}{v_1}, \ldots, \frac{v_n}{v_1}, \varphi^{2\,\prime\prime}(0), \ldots, \varphi^{n\,\prime\prime}(0)).$$

Thus we have defined the bundle $K(M^n)$ over M^n . Write the projection map $p_2 : K(M^n) \to M^n$.

Remark. Note that there is a canonical projection

$p_{21} : K(M^n) \to L(M^n)$ (and $p_1 p_{21} = p_2$) so that $K(M^n)$ is a fiber bundle over $L(M^n)$ with fiber R^{n-1} .

If $f: M_1^n \to M_2^n$ is a diffeomorphism of M_1^n onto M_2^n , there is induced corresponding diffeomorphisms of $L(M_1^n)$ onto $L(M_2^n)$ and also $K(M_1^n)$ onto $K(M_2^n)$.

Definition. A first order differential equation, written $\frac{dy^i}{dx} = f^i(x,y)$ on a differentiable manifold M^n is a cross-section of M^n into $L(M^n)$. A second order differential equation, written $\frac{d^2y^i}{dx^2} = f^i(x,y,y')$, is a cross-section from an open set $\mathcal{O} \subset L(M^n)$ into $K(M^n)$. If $\mathcal{O} = p_1^{-1}(U)$, where U is open in M^n, then we say that $\frac{d^2y^i}{dx^2} = f^i(x, y, y')$ is defined over U.

Definition. A first order differential equation $\frac{dy^i}{dx} = f^i(x, y)$ on a differentiable manifold M^n is invariant under a local transformation group with infinitesimal generator Λ in case: for each diffeomorphism of an open set $U_1 \subset M^n$ onto an open set $U_2 \subset M^n$, defined by Λ, the induced map of $L(U_1)$ onto $L(U_2)$ carries the cross-section of $\frac{dy^i}{dx} = f^i(x, y)$ above U_1 onto the corresponding cross-section above U_2.

Definition. A second order differential equation $\frac{d^2y^i}{dx^2} = f^i(x, y, y')$ over an open set $\mathcal{O} \subset L(M^n)$ is invariant under a local transformation group with infinitesimal generator Λ in case: for each diffeomorphism of an open set $U_1 \subset M^n$ onto $U_2 \subset M^n$, defined by Λ, the induced diffeomorphism of $p_1^{-1}(U_1) \cap \mathcal{O} \to p_1^{-1}(U_2) \cap \mathcal{O}$ maps the cross-section of $\frac{d^2y^i}{dx^2} = f^i(x, y, y')$ above $p_1^{-1}(U_1) \cap \mathcal{O}$ in $K(p, \mathcal{O})$ onto the corresponding cross-section above $p_1^{-1}(U_2) \cap \mathcal{O}$.

Let V be a differentiable vector field (infinitesimal transformation) on M^n. Then V defines a vector field V' in $L(M^n)$ and also V'' in $K(M^n)$. For V generates a local one-parameter transformation group $\varphi: R^1 \times M^n \to M^n$. Each diffeomorphism (of open sets $U \to W \subset M^n$) of this transformation group induces a diffeomorphism of $p_1^{-1}(U) \to p_1^{-1}(W)$, and also of $p_2^{-1}(U) \to p_2^{-1}(W)$. Thus there is defined a local one-parameter transformation group

$$\varphi_1: R^1 \times L(M^n) \to L(M^n) \quad \text{and also}$$

$$\varphi_2: R^1 \times K(M^n) \to K(M^n).$$

The infinitesimal generators of these local transformation groups are V' and V'', respectively.

Definition. Let Λ be an infinitesimal transformation group on a differentiable manifold M^n. Then each member $V \in \Lambda$ lifts to a vector field V' in $L(M^n)$ and V'' in $K(M^n)$. Thus we map Λ onto Λ' in $\mathcal{L}(L(M^n))$, and also map Λ onto Λ'' in $\mathcal{L}(K(M^n))$. We call Λ' the first extension of Λ, and Λ'' the second extension of Λ.

Theorem 41. Let Λ be an infinitesimal transformation group on M^n. Then the extensions Λ' and Λ'' are infinitesimal transformation groups on $L(M^n)$ and $K(M^n)$, respectively. The maps

$$\Lambda \longrightarrow \Lambda'$$
$$\Lambda \longrightarrow \Lambda''$$

are abstract isomorphisms of these Lie algebras.

Proof. See Lie-Scheffers, Differentialgleichungen, p. 397.

Theorem 42. Let $\mathcal{D}: \frac{dy^i}{dx} = f^i(x,y)$ be a differential system on a differentiable manifold M^n. An infinitesimal transformation group Λ on M^n leaves $\mathcal{D}$ invariant if and only if each vector of Λ' is tangent to the cross-section $\mathcal{D} \subset L(M^n)$. This occurs if and only if a basis for Λ lifts to a basis for Λ' which is tangent to the cross-section $\mathcal{D}$.

Remark. $\mathcal{D}$ is invariant under Λ just in case each local one-parameter transformation group generated by a basis member of Λ leaves $\mathcal{D}$ invariant.

Theorem 43. Let $\mathcal{D}: \frac{d^2y^i}{dx^2} = f^i(x,y,y')$ be a differential equation over an open set $\mathcal{O} \subset L(M^n)$. Let Λ be an infinitesimal transformation group defined on the open set $p_1(\mathcal{O})$ of the differentiable manifold M^n. Then Λ leaves $\mathcal{D}$ invariant if and only if Λ'', defined on $p_2^{-1}(\mathcal{O})$, consists of vectors tangent to the cross-section $\mathcal{D} \subset K(M^n)$. This occurs if and only if a basis for Λ lifts to a basis for Λ' which is tangent to the cross-section $\mathcal{D}$.

<u>Remark.</u> $\mathcal{D}$ is invariant under Λ just in case each member of a basis of Λ leaves $\mathcal{D}$ invariant.

12. <u>The Complete Transformation Group of a Second Order Differential System.</u>

<u>Theorem 44</u>. Let $\mathcal{D}: \frac{d^2y^i}{dx^2} = f^i(x, y, y')$ be an analytic differential system defined on all $L(M^n)$ over a real analytic connected manifold M^n. The set $\Lambda_{\mathcal{D}}$ of all analytic infinitesimal transformations on M^n which leave $\mathcal{D}$ invariant form an infinitesimal transformation group of dimension $d \leq (n+1)^2 - 1$.

<u>Proof.</u>

A calculation in local coordinates (see Lie-Scheffers, <u>Differentialgleichungen</u>, p. 401) shows that if V_1 and V_2 are infinitesimal transformations which leave $\mathcal{D}$ invariant, then so do $c_1 V_1 + c_2 V_2$ and $[V_1, V_2]$. Thus $\Lambda_{\mathcal{D}}$ is a Lie algebra in $\mathcal{L}(M^n)$. By analyticity two vector fields of $\Lambda_{\mathcal{D}}$ which coincide on an open set of M^n are identical on M^n.

Suppose there are $(n+1)^2$ linearly independent vector fields $V_1, \cdots, V_{(n+1)^2}$ of Λ. Select n^2 points $P_1, \cdots, P_{n^2}$, near P on M^n, in general position (no 3 on same solution curve of $\mathcal{D}$) so that a linear combination of $V = c_1 V_1 + \cdots + c_{(n+1)^2} V_{(n+1)^2}$ vanishes at each of the n^2 points. This is possible since one need only solve n^2 linear equations in the $(n+1)^2$ unknowns $c_1, \cdots, c_{(n+1)^2}$. Thus the local transformation group generated by V holds each point $P_1, P_2, \cdots, P_{n^2}$ fixed. The direction group D_{P_1} is a subgroup of the infinitesimal projective transformation group on the line elements at P_1, that is, on the projective space P^{n-1}. But V induces an element of D_{P_1} which leaves n^2-1 directions fixed and thus V induces the identity transformation on the line elements at P_1. Similarly V induces the identity transformation on the line elements at $P_1, \ldots, P_{n^2}$.

. But every point on M^n near P is determined by the intersection of two solution curves radiating from P_1 and P_2. Thus each point of M^n near P is left fixed by the transformations generated by V. Thus V vanishes on an open neighborhood of P. Thus V vanishes on all M^n. But this contradicts the supposition that $V_1, \ldots, V_{(n+1)^2}$ were linearly independent. Therefore $\dim \Lambda \leq (n+1)^2-1$.

Q. E. D.

<u>Definition</u>. $\Lambda_{\mathcal{D}}$ is called the complete infinitesimal transformation group for $\mathcal{D}$.

<u>Theorem 45</u>. Let $\mathcal{D}: \dfrac{d^2y^i}{dx^2} = f^i(x,y,y')$ be an analytic differential system defined on all $L(M^n)$ over a real analytic connected manifold M^n. Consider the group G of all real analytic diffeomorphisms of M^n onto itself which preserves $\mathcal{D}$. Topologize G by the compact-open topology. Then G is a Lie group of dimension $\leq (n+1)^2-1$. Then the component of the identity G_e of G, using analytic coordinates on G, acts analytically as a transformation group $G_e \times M^n \to M^n$. Moreover $G_e \times M^n \to M^n$ is generated by a subalgebra of the complete infinitesimal transformation group $\Lambda_{\mathcal{D}}$. If M^n is compact, $G_e \times M^n \to M^n$ is generated by $\Lambda_{\mathcal{D}}$.

<u>Proof</u>.

A preliminary analysis shows that G is locally compact and acts continuously on M^n. Then see Montgomery-Zippin, p. 208 and p. 213. This is a very difficult theorem (has never been proved) and includes the results of Meyers-Steenrod that the group of isometries of a Riemannian space is a Lie group and also the theorem of Nomizu that the group of affinities of an affinely connected space is a Lie group.

Now consider analytic differential equations $\mathcal{D}: \frac{d^2y}{dx^2} = f(x,y,y')$ in the plane R^2. If $\mathcal{D}$ is invariant under q and xq, then (locally) $\mathcal{D}$ is of the form $y'' = f(x)$, as seen earlier. But such an equation is locally diffeomorphic with $y'' = 0$.

Theorem 46. If the analytic equation $\mathcal{D}: \frac{d^2y}{dx^2} = f(x,y,y')$ in R^2 is invariant under two linearly independent analytic infinitesimal transformations V_1 and V_2 which have the same path curves, then $\mathcal{D}$ is locally diffeomorphic with $y'' = 0$.

Proof.

The subset of $\Lambda_{\mathcal{D}}$ which consists of vector fields having the same path curves as V_1 is a subalgebra $\Lambda_{\mathcal{D}_{V_1}}$.

The locally intransitive groups on the plane are

1) $q, \psi_2(x)q, \ldots, \psi_n(x)q \quad (n \geq 3)$
2) $q, yq, \psi_3(x)q, \ldots, \psi_n(x)q \quad (n \geq 4)$
3) q, yq, y^2q
4) q, xq
5) q
6) q, xq, yq
7) q, yq.

Under the coordinate change $\bar{x} = \psi_2(x)$, $\bar{y} = y$ we note that 1) contains the infinitesimal transformations q, xq. Excepting the one-parameter group 5.), all the groups contain the pair q, xq or the pair q, yq. If $\mathcal{D}$ is invariant under q, xq then $\mathcal{D}$ is locally equivalent to $y'' = w(x)$. If $\mathcal{D}$ is invariant under q, yq then $\mathcal{D}$ is locally equivalent to $y'' = c(x)y'$. But each of these $y'' = w(x)$ or $y'' = c(x)y'$, is locally equivalent to $y'' = 0$ (see Kowalewski, p. 356).

<u>Theorem 47.</u> Let Λ be the complete infinitesimal transformation group for the analytic differential equation $\mathcal{D}: y'' = f(x,y,y')$ in the plane R^2. Then there is an open set in R^2 wherein Λ is isomorphic with exactly one of the following:

1.) q
2.) p, q
3.) $xp + yq, q$
4.) $p, q, xp + (x+y)q$
5.) $p, q + xp, 2xq + x^2p$
6.) $p, q, (c+1)xp + (c-1)yq$ real $c \neq \pm 1, c \neq \pm 3$
7.) $p, q, yp - xq + \lambda(xp + yq)$ all real λ
8.) $p, q, xp, yp, xq, yq, x(xp+yq), y(xp+yq)$.

<u>Proof.</u>

The only non-transitive infinitesimal group which is a complete group is q. From the list of all locally transitive infinitesimal groups in the plane we discard all those which contain either a two-parameter subgroup having a common set of path curves or else the subgroup $p, q, xp + yq$ (these are admitted only by $y'' = 0$, which further admits the infinitesimal projective group 8).

The two-parameter transitive groups p, q and $xp + yq, q$ are easily shown to be complete groups.

The seven remaining candidates are

$\alpha)$ $p,\ q,\ xp+(x+y)q$

$\beta)$ $p,\ q+xp,\ 2xq+x^2p$

$\gamma)$ $p,\ q,\ (c+1)xp+(c-1)yq$

$\delta)$ $p,\ q,\ yp-xq+\lambda(xp+yq)$

$\varepsilon)$ $p+y(xp+yq),\ q+x(xp+yq),\ xp-yq$

$\zeta)$ $p+x(xp+yq),\ q+y(xp+yq),\ yp-xq$

$\eta)$ $p-x(xp+yq),\ q-y(xp+yq),\ yp-xq$

and the infinitesimal projective group. A computation shows that if $y''=f(x,y,y')$ admits any one of $\varepsilon), \zeta), \eta)$ in the above list, then the only possibility is $y''=0$ and for this the complete group is the eight parameter infinitesimal projective group. The remaining candidates $\alpha), \beta), \gamma), \delta)$ are all complete groups except for $\gamma)$ when $c=\pm 1$ or $c=\pm 3$. For in these cases of $\gamma)$ the most general invariant differential equation is $y''=a y'^{\frac{3+c}{2}}$

But the condition of Tresse states that the most general equation equivalent to $y''=0$ is $y''=f(x,y,y')$, or

$$y''=A(x,y)y'^3+3B(x,y)y'^2+3C(x,y)y'+D(x,y)$$

where

$$D_{yy}-2C_{xy}+B_{xx}+2DA_x+AD_x-3DB_y-3BD_y-3CB_x+6CC_y=0$$

and

$$C_{yy}-2B_{xy}+A_{xx}-2AD_y-DA_y+3AC_x+3CA_x+3BC_y-6BB_x=0.$$

Q. E. D.

Corollary. If $\mathcal{D}: y''=f(x,y,y')$ admits a 2 parameter intransitive group, or a 4 parameter transitive group, then $\mathcal{D}$ is equivalent to $y''=0$ and has the infinitesimal projective group as its complete group.

Corollary. If $\mathcal{D}: y''=f(x,y,y')$ has a transitive three parameter group as its complete group, then $\mathcal{D}$ is locally equivalent to exactly one of the following

1. $y'' = e^{y'}$

2. $y'' = y'^2 + 4y$

3. $y'' = y'^{\alpha}$ with real $\alpha \neq 0, 1, 2, 3$

4. $y'' = (1+y'^2)^{3/2} e^{\lambda \tan^{-1} y'}$, real λ .

List of all locally transitive analytic infinitesimal transformation groups in the plane — up to local isometry (Mostow)

Groups with 0-dimensional direction groups.

1. p, q

2. $p, q + xp$

3. $p, q, xp + yq$

Groups with 1-dimensional direction groups

1. $p, q, xp + (x+y)q$

2. $p, q + xp, 2xq + x^2 p$

3. $p, w(x)q, \cdots, w^{(k-1)}(x)q \quad (k > 1)$ where $w^{(k)} = c_0 w^{(k-1)} + \cdots + c_k w,$ c_i real constant.

4. $p, q, xp + yq, x^2 p + 2xyq$

5. $p, q, xp + yq, xq, x^2 p, \ldots, x^5 p$

6. $p, q, xp, x^2 p$

7. $p, q, (c+1)xp + (c-1)yq$, all real c

8. $p + y(xp + yq), q + x(xp + yq), xp - yq$

9. $p, xp, x^2 p, q, yq, y^2 q$

10. $p, q, x^2 p, q, yq$

11. p, xp, q, yq

12. $p, q, yp - xq + \lambda(xp + yq)$, all real λ

13. $p, q, xp + yq, yp - xq$

14. $p, q, xp + yq, yp - xq, (x^2 - y^2)p + 2xyq, 2xyp + (y^2 - x^2)q$

15. $p + x(xp + yq),\ q + y(xp + yq),\ yp - xq$

? 16. $p - x(xp \overset{+}{-} yq),\ q - y(xp + yq),\ yp - xq$

Groups with 2-dimensional direction groups.

1. $p,\ q,\ xq,\ 2xp + yq,\ x(xp + yq)$

2. $p,\ q,\ xq,\ \ldots,\ x^s q,\ 2xp + syq,\ x(xp + syq) \quad (s > 2)$

3. $p,\ q,\ xq,\ \ldots,\ x^s q,\ xp + r yq \quad (r \neq 1,\ s > 0)$

4. $p,\ q,\ xq,\ \ldots,\ x^s q,\ xp + (s+1)yq + x^{s+1} q \quad (s > 0)$

5. $p,\ w(x)q,\ \ldots,\ w^{(r-1)}(x)q,\ yq$ where $w^{(r)} = c_0 w^{(r-1)} + \cdots + c_{r-1} w' + c_r w$,

$r > 1,\ c_i =$ real constant

6. $p,\ q,\ xp,\ xq + \tfrac{1}{2} x^2 p$

7. $p,\ q,\ xp,\ yq,\ xq,\ \ldots,\ x^s q$

8. $p,\ q,\ xp,\ yq,\ xq,\ \ldots,\ x^s q,\ x(xp + syq) \quad (s > 0)$

Groups with 3-dimensional direction groups — Primitive groups.

1. $p,\ q,\ yp,\ xq,\ xp - yq$

2. $p,\ q,\ xp,\ yp,\ xq,\ yq$

3. $p,\ q,\ xp,\ yp,\ xq,\ yq,\ x(xp + yq),\ y(xp + yq)$

13. Solvable Infinitesimal Transformation Groups and the Solution of Differential Equations by Quadrature.

Let L be a real Lie algebra. The commutator ideal $[L, L] = L_1$ is the smallest subalgebra of L which contains all the commutators of L. Similarly we can construct $[L_1, L_1] = L_2$, the commutator ideal of L_1, and $[L_k, L_k] = L_{k+1}$.

Definition. A finite dimensional real Lie algebra L is called solvable (or integrable) in case there exists an integer r such that $[L_r, L_r] = L_{r+1} = 0$ (the zero of L).

__Example.__ Every commutative finite dimensional Lie algebra is solvable. Both 2 dimensional Lie algebras are solvable. The Lie algebra of $O_+(3)$ (vectors in R^3 using vector cross product) is not solvable. It is shown in Pontrjagin, p. 277, that if L is solvable there exists a basis $V_1, V_2, \cdots, V_n$ such that $\{V_1, V_2, \cdots, V_{n-1}\}$ spans an ideal U_{n-1} in L. $\{V_1, \cdots, V_{n-2}\}$ spans an ideal U_{n-2} in U_{n-1}, and $\{V_1, \cdots, V_{n-s}\}$ spans an ideal V_{n-s} in U_{n-s+1} for $s = 1, 2, \cdots, n-1$.

__Theorem 48.__ Let $\mathcal{D}: \frac{dy^i}{dx} = f^i(x, y^1, \cdots, y^{n-1})$ be a first order differential system defined in an open set $\mathcal{O}$ of R^n. Let Λ be an analytic infinitesimal transformation group leaving $\mathcal{D}$ invariant. Assume

1.) Λ is solvable as an abstract Lie algebra

2.) $\dim \Lambda = n-1$

3.) the transitivity sets (integral manifolds) of Λ are each of dimension $n-1$.

4.) the line element of $\mathcal{D}$ is nowhere tangent to a transitivity set of Λ.

Then there exist analytic local coordinates (still called $(x, y^1, \cdots, y^{n-1})$) in an open set $\mathcal{O}_1 \subset \mathcal{O}$ such that Λ has a basis $V_1, \cdots, V_{n-1}$ where

$$V_s = V_s^i(x, y^1, \cdots, y^{n-1}) \frac{\partial}{\partial y^i}, \quad i, s = 1, 2, \cdots, n-2 \quad \text{and} \quad V_{n-1} = \frac{\partial}{\partial y^{n-1}}$$

In such coordinates we write

$$\mathcal{D}: \begin{aligned} \frac{dy^i}{dx} &= f^i_{(1)}(x, y^1, \cdots, y^{n-2}) \quad i = 1, 2, \cdots, n-2, \\ \frac{dy^{n-1}}{dx} &= f^{n-1}_{(1)}(x, y^1, \cdots, y^{n-2}). \end{aligned}$$

Thus in the manifold $y^{n-1} = const.$ (say $y^{n-1} = 0$) we have the infinitesimal transformation group $\Lambda(1)$ spanned by

$$V_s^i(x, y^1, \cdots, y^{n-2}, 0) \frac{\partial}{\partial y^i} \qquad i, s = 1, 2, \cdots, n-2.$$

Furthermore

1.') $\Lambda(1)$ is solvable as an abstract Lie algebra

2.') dim $\Lambda(1) = n-2$

3.') the transitivity sets of $\Lambda(1)$ are each of dimension $n-2$ in the R^{n-1} (defined by $y^{n-1}=0$).

4.') the line element of

$$\mathcal{D}(1): \quad \frac{dy^i}{dx} = f^i_{(1)}(x, y^1, \dots, y^{n-2}) \qquad i = 1, 2, \dots, n-2$$

is nowhere tangent to a transitivity set of $\Lambda(1)$. Also $\mathcal{D}(1)$ is invariant under $\Lambda(1)$.

Proof.

Choose a basis $V_1, V_2, \dots, V_{n-1}$ for Λ, none of which vanish in $\mathcal{O}_1 \subset \mathcal{O}$, and which yields the chain of ideals

$$\{V_1\},\ \{V_1, V_2\},\ \{V_1, V_2, V_3\},\ \dots\dots,\ \{V_1, V_2, \dots, V_{n-1}\}$$

described above for abstract solvable Lie algebras. Now each transitivity set of $\{V_1, \dots, V_{n-1}\}$ is an analytic $(n-1)$-submanifold of R^n. By a change of coordinates we assume that these transitivity sets are the hyperplanes x = constant. Then each member of Λ has a zero component along the x-axis. Further change coordinates so that $V_{n-1} = \frac{\partial}{\partial y^{n-1}}$.

Since $\{V_1, V_2, \dots, V_{n-2}\}$ is an ideal in Λ, coordinates in R^n can be chosen so that each transitivity set of $\{V_1, V_2, \dots, V_{n-2}\}$ lies in a hyperplane $y^{n-1} = const$. Then $\mathcal{D}(1)$ and $\Lambda(1)$ have the stated properties.

Q. E. D.

Remark. Using $\Lambda(1)$ and $\mathcal{D}(1)$ we can repeat the construction in the theorem to obtain $\Lambda(2)$ and $\mathcal{D}(2)$ in a R^{n-2}. Finally we obtain $\mathcal{D}(n-2): \frac{dy^1}{dx} = f_{(n-2)}(x, y^1)$ invariant under $\Lambda(n-2)$, say $\frac{\partial}{\partial y^1}$. Then $\mathcal{D}(n-2): \frac{dy^1}{dx} = f_{(n-2)}(x)$ can be integrated by a quadrature. Then a sequence of quadratures (interspersed by the coordinate changes specified in the above reduction) leads to the solution of ϑ in R^n.

This entire reduction is summarized by the statement: A differential system ϑ in R^n, which is invariant under a solvable ($n-1$)-dimensional infinitesimal transformation group Λ, can be solved by quadratures.

Note that locally ϑ can be written, after a change of local coordinates in R^n, as $\frac{dy^i}{dx} = 0$ for $i = 1, 2, \ldots, n-1$. Then the solutions are just the lines $y^i =$ constant. However, in the particular reduction specified in the above theorem, the coordinate changes are determined by the structure of the infinitesimal transformation group Λ. In practice, this involves solving systems of ordinary differential equations for the path curves of Λ. Sometimes the geometry of Λ is simpler than that of ϑ and, in such a case, the theorem might be of practical interest.

Lie Groups and Differential Equations — Problems.

1. The conformal local transformation group on the plane is

$$z \to z_1 = \frac{\alpha z + \beta}{\gamma z + \delta} \qquad \text{with complex } \alpha\delta - \beta\gamma \neq 0 .$$

Write this local transformation group in terms of the real Cartesian coordinates in R^2. Find a basis for the infinitesimal generator in R^2. Is the conformal local transformation group isomorphic with a subgroup of the projective local transformation group? Discuss the the nature of inversion maps with reference to the two local transformation groups.

2. In the number space R^3 show that the set of all differentiable vector fields is a Lie algebra by a direct computation based on the definition

$$[u, v]^i = \frac{\partial u^i}{\partial x^j} v^j - \frac{\partial v^i}{\partial x^j} u^j .$$

Find the right invariant vector fields for the Lie group R^3 and verify that they form a finite dimensional Lie algebra. Find a basis for the Lie algebra $\mathcal{L}_R(R^3)$.

3. Let G be a commutative connected Lie group and consider the effective, transitive, Lie transformation group $\varphi: G \times R' \to R'$. Prove that $G = R'$ and that the transformation group is isomorphic with the group of translations of $R' \times R' \to R'$.

4. Find the most general second order differential equation $y'' = f(x, y, y')$ in R^2 invariant under the infinitesimal transformation group $p, q, xp + (x+y)q$.

5. Verify that $q, xq, \cdots, x^{n-4}q, yq, p, xp, (n > 4)$ is a basis for an infinitesimal transformation group on R^2. For $n = 7$ is this isomorphic with a subgroup of the projective infinitesimal transformation group?

6. Show that

   a) q
   b) p, q
   c) $xp + yq, q$

   are complete infinitesimal transformation groups for some second order differential equations in R^2. Prove that the examples constructed are not qualitatively equivalent to $y'' = 0$.

BIBLIOGRAPHY

Classical Theory of Local Transformation Groups

1. L. Bianchi, Lezioni sulla Storia dei gruppi continui finiti, Pisa 1918.
2. A. Cohen, An Introduction to the Lie Theory of One-Parameter Groups, New York, 1911.
3. L. Eisenhardt, Continuous Groups of Transformations, Princeton 1933.
4. G. Kowalewski, Einführung in die Theorie der Kontinuierlichen Gruppen, Leipzig 1931.
5. S. Lie and F. Engel, Theorie der Transformationsgruppen, 3 vol. Leipzig 1888 (reprint 1930).
6. S. Lie and G. Scheffers, Vorlesungen über continuierliche Gruppen, Leipzig 1893.
7. S. Lie and G. Scheffers, Vorlesungen über Differentialgleichungen, Leipzig 1891.
8. S. Lie and G. Scheffers, Berührungstransformationen, Leipzig 1896.

Global Lie Groups

1. E. Cartan, La topologie des groupes de Lie, Paris 1936.
2. E. Cartan, La theorie des groupes finis et continus, Paris 1937.
3. C. Chevalley, Theory of Lie Groups, Princeton 1946.
4. P. Cohn, Lie Groups, Cambridge 1957.
5. K. Numizu, Lie Groups and Differential Geometry, Tokyo 1956.
6. L. Pontrjagin, Topological Groups, Princeton 1939.

Global Transformation Groups

1. D. Montgomery and L. Zippin, Topological Transformation Groups, New York 1955.
2. G. Mostow, "Extensibility of Local Lie Groups", Ann. of Math. 1950.
3. R. Palais, A Global Formulation of the Lie Theory of Transformation Groups, Memoir 22, Am. Math. Soc. 1957.