The Monodromy Group and Fuchsian Differential Equations.

1. Introduction. An Example of the Monodromy Group.

Consider the Legendre differential equation in the complex plane $\mathbb{C}$,

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + n(n+1)w = 0,$$

where n is a fixed positive integer. This is a special case of the hypergeometric equation (Gauss)

$$\frac{d^2w}{dz^2} + \left[\frac{1-\alpha_{11}-\alpha_{21}}{z-z_1} + \frac{1-\alpha_{12}-\alpha_{22}}{z-z_2}\right]\frac{dw}{dz}$$

$$+ \left[\frac{\alpha_{11}\alpha_{21}}{(z-z_1)^2} + \frac{\alpha_{12}\alpha_{22}}{(z-z_2)^2} + \frac{\alpha_{100}\alpha_{200} - \alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22}}{(z-z_1)(z-z_2)}\right] w = 0$$

Where the singularities of the coefficients are at $z_1 = +1$ and $z_2 = -1$ (and also $z_3 = \infty$). The hypergeometric equation displaying the three singularities on the complex sphere is

$$\frac{d^2w}{dz^2} + \left[\frac{1-a'-a''}{z-a} + \frac{1-b'-b''}{z-b} + \frac{1-c'-c''}{z-c}\right]\frac{dw}{dz}$$

$$+ \left[\frac{a'a''(a-b)(a-c)}{z-a} + \frac{b'b''(b-a)(b-c)}{z-b} + \frac{c'c''(c-a)(c-b)}{z-c}\right]\frac{w}{(z-a)(z-b)(z-c)} = 0$$

(Riemann-Papperitz).

If the singularities are in standard position $0, 1, \infty$ we write the hypergeometric equation

$$z(1-z)\frac{d^2w}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{dw}{dz} - \alpha\beta w = 0.$$

A solution analytic near $z = 0$ is the hypergeometric function (Euler-Gauss)

$$F(\alpha,\beta,\gamma,z) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+n)}\frac{z^n}{n!}$$

where $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}\,dt$ for $k > 0$ and $\Gamma(k+1) = k!$ for integer $k \geq 0$. If $\gamma \neq$ integer another solution near $z = 0$ is

$$z^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z).$$

After a linear fractional transformation of the complex sphere the three singularities can be placed at three prescribed points. In particular, for the points $1, -1, \infty$ we have Legendre's equation and

$$P_n(z) = F\left(-n, n+1, 1, \frac{1-z}{2}\right).$$

For the Legendre equation a basis of solutions is

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n[(z^2-1)^n]}{dz^n}$$ Legendre Polynomial

and

$$Q_n(z) = \frac{1}{2}P_n(z)\log\frac{z+1}{z-1} - \sum_{r=1}^{N}\frac{2n-4r+3}{(2r-1)(n-r+1)}P_{n-2r+1}(z)$$

where

$$N = \begin{cases} \frac{1}{2}n & \text{if } n \text{ even} \\ \frac{1}{2}(n+1) & \text{if } n \text{ odd} \end{cases}.$$

Note that $P_n(z)$ is entire and single-valued whereas $Q_n(z)$ has branch points at $z_1 = +1$ and $z_2 = -1$ and is hence multiple-valued in $\mathbb{C} - z_1 - z_2$.

By analytic continuation of the solutions around a small circuit $\mathcal{C}_1$, around z_1 counterclockwise, we define the linear transformation of the complex 2-dimensional vector space of solutions into itself

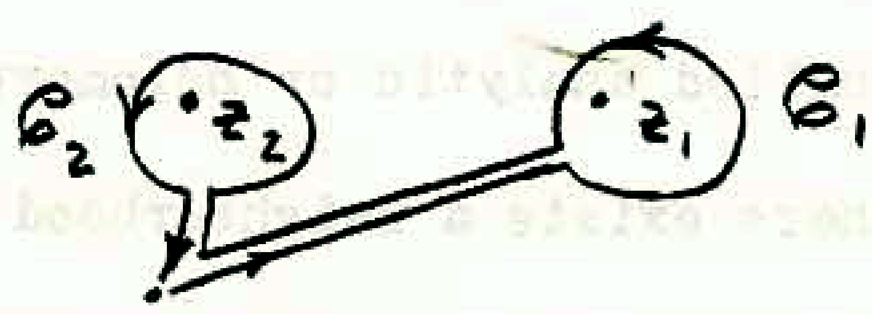
$$\mathcal{C}_1 : \begin{array}{l} P_n \to P_n \\ Q_n \to Q_n - \pi i P_n \end{array} : \begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix}.$$

Around a counterclockwise loop $\mathcal{C}_2$ encircling the singularity $z_2 = -1$ we have the linear transformation

$$\mathcal{C}_2 : \begin{array}{l} P_n \to P_n \\ Q_n \to Q_n + \pi i P_n \end{array} : \begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix}.$$

Around a loop $\mathcal{C}_3$ encircling both $z_1 = 1$, $z_2 = -1$ positively (and hence negatively around ∞) we have the transformation

$$\mathcal{C}_3 : \begin{array}{l} P_n \to P_n \\ Q_n \to Q_n \end{array} : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix}$$

as indicated in the figure 

The multiplicative group of 2×2 complex matrices corresponding to $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ and all closed paths in $\mathbb{C}$ define the monodromy group of the Legendre differential equation (and the qualitative behavior near the singularity).

The monodromy group determines the linear differential equation (under appropriate hypotheses) uniquely and describes the properties of its solutions. For example, all solutions are single valued if and only if the monodromy group is just I. Also the solutions are algebraic functions just in case the monodromy group is finite.

We shall deal with functions of several complex variables, for example, $F(\alpha, \beta, \gamma, z)$ or

$$\frac{dw}{dz} = v$$

$$\frac{dv}{dz} = \frac{2zv}{1-z^2} - \frac{n(n+1)}{1-z^2} w .$$

2. Survey of Properties of Analytic Functions of Several Variables.

The space $\mathbb{C}^n$ of n-complex variables is the set of ordered n-tuples of complex numbers with the topology defined by the homeomorphism

$$\mathbb{C}^n \to R^{2n} : (z^1 = x^1 + iy^1, \ldots, z^n = x^n + iy^n) \to (x^1, \ldots, x^n, y^1, \ldots, y^n).$$

Then $\mathbb{C}^n$ is an n-dimensional complex linear vector space and it is also a Lie group.

An ε-neighborhood of $(z_0^1, \ldots, z_0^n)$ is the set $\sum_{j=1}^{n} |z^j - z_0^j|^2 < \varepsilon^2$ for some $\varepsilon > 0$. A metric yielding this topology is

$$\|z - w\| = \left[\sum_{j=1}^{n} |z^j - w^j|^2 \right]^{1/2}$$

, which is the Euclidean metric on R^{2n}. Thus if a real or complex valued function $w = u + iv$ is defined on an open set $\mathcal{O}$ in $\mathbb{C}^n \times R^m$ it is continuous just in case u and v are continuous real valued functions. Also w is called analytic or holomorphic on $\mathcal{O}$ in case for each point $P \in \mathcal{O}$ there exists a neighborhood N_P and a convergent power series in n complex and m real variables (with complex coefficients) which is absolutely convergent in N_p and therein converges to w.

A complex n-sequence is a function from the set of all non-negative integral lattice points $\mathbb{Z}^n$ into $\mathbb{C}$. A n-fold power series consists of a n-sequence of terms $a_J(z-c)^J = a_{j_1j_2\cdots j_n}(z^1-c^1)^{j_1}\cdots(z^n-c^n)^{j_n}$ together with the directed system of partial sums. Consider the directed set $\mathcal{S}$ of all finite subsets of $\mathbb{Z}^n$ (partially ordered by inclusion). For each $N \in \mathcal{S}$ assign the complex number (for fixed z)

$$\sum_{J\in N} a_J(z-c)^J$$, the partial sum of the series corresponding to the index set N.

<u>Definition</u>. Let $\sum A_J$ be a complex n-series whose terms are the n-sequence A_J. We say $\sum\limits_{J=0}^{\infty} A_J = B$ (or $\sum A_J$ converges unconditionally to B) in case for each $\varepsilon > 0$, there exists a set $N(\varepsilon) \in \mathcal{S}$ such that $\left|\sum\limits_{J\in N} A_J - B\right| < \varepsilon$ for each $N \supset N(\varepsilon)$, $N \in \mathcal{S}$.

<u>Theorem 1</u>. Let $\sum A_J$ be a complex n-series. Then $\sum A_J$ converges to a complex number if and only if the Cauchy criterion holds:

given $\varepsilon > 0$ $\exists\, N(\varepsilon) \in \mathcal{S}$ such that $\left|\sum\limits_{J\in M} A_J\right| < \varepsilon$ for each set $M \in \mathcal{S}$ with $M \cap N(\varepsilon) = \emptyset$.

<u>Proof</u>. Assume $\sum\limits_{J=0}^{\infty} A_J = B$. Let $\varepsilon > 0$ and take $N(\varepsilon)$ so that $\left|\sum\limits_{J\in N} A_J - B\right| < \varepsilon/2$ for each $N \supset N(\varepsilon)$. Then for $M \in \mathcal{S}$ with $M \cap N(\varepsilon) = \emptyset$ we have $\sum\limits_{J\in N(\varepsilon)\cup M} A_J = \sum\limits_{J\in N(\varepsilon)} A_J + \sum\limits_{J\in M} A_J$

Thus

$$\left|\sum_{J\in M} A_J\right| \le \left|\sum_{J\in N(\varepsilon)\cup M} A_J - B - \left[\sum_{J\in N(\varepsilon)} A_J - B\right]\right|.$$

Thus $\left|\sum\limits_{J\in M} A_J\right| < \varepsilon$ so the Cauchy criterion holds.

Conversely assume the Cauchy criterion for the n-series $\sum A_J$. Let $K_m \in \mathcal{S}$ be the n-cubes with diagonal corners of $(0,\dots,0)$ and $(m,\dots,m)$. Then the 1-sequence $\sum_{J \in K_m} A_J$ is Cauchy and thus converges to a complex number B. There exists $N(\varepsilon) \in \mathcal{S}$ such that

$$\left|\sum_{J\in M} A_J\right| < \varepsilon/2 \quad \text{for} \quad M \cap N(\varepsilon) = \emptyset.$$

Take $\hat{N}(\varepsilon)$ a n-cube K_m which contains $N(\varepsilon)$ and

$$\left|\sum_{J\in K_{\hat{N}(\varepsilon)}} A_J - B\right| < \varepsilon/2$$

Thus for each $N \supset K_{\hat{N}(\varepsilon)}$

$$\left|\sum_{J\in N} A_J - B\right| < \varepsilon \qquad \text{Q. E. D.}$$

Corollary. Assume $\sum_{J=0}^{\infty} A_J = B$. Let K_m be the n-cube $0 \le j_1 \le m, \dots, 0 \le j_n \le m$ and let T_m be the n-triangle $j_1 + \cdots + j_n \le m$. Then the 1-series converge $\sum_{K_m} A_J = B$ and $\sum_{T_m} A_J = B$.

Proof.

The n-cubes or n-triangles contain a prescribed $N(\varepsilon)$, for sufficiently large m. Q. E. D.

Corollary. The n-series $\sum_{J=0}^{\infty} A_J$ converges if and only if the n-series $\sum_{J=0}^{\infty} |A_J|$ converges.

Proof.

If $\sum_{J=0}^{\infty} |A_J|$ converges then $\sum_{J\in M} |A_J| < \varepsilon$ for each finite index set M outside some $N(\varepsilon)$. But then $\left|\sum_{J\in M} A_J\right| < \varepsilon$ and $\sum_{J=0} A_J$ satisfies the Cauchy convergence criterion.

Conversely assume $\sum_{J=0}^{\infty} A_J$ converges. Given $\varepsilon > 0$ $\exists$ $N(\varepsilon)$ such that $|\sum_{J \in M} A_J| < \varepsilon/4$ for each finite set M outside $N(\varepsilon)$. If $\sum_{J \in M} |A_J| \geq \varepsilon$ then either the positive or negative real or the positive or negative imaginary components of the terms A_J, $J \in M$, must total $\geq \varepsilon/4$. For

$$\sum_{J \in M} |A_J| \leq \sum [(Re\ A_J)^2 + (Im\ A_J)^2]^{1/2} \leq \sum |Re\ A_J| + \sum |Im\ A_J|.$$

Hence there is a subset $M' \subset M$ for which, say, $\sum_{J \in M'} Re\ A_J \geq \varepsilon/4$ and $Re\ A_J > 0$ for $J \in M'$. But then $|\sum_{J \in M'} A_J| \geq \varepsilon/4$ which is not the case. Therefore $\sum_{J \in M} |A_J| < \varepsilon$ and $\sum_{J=0}^{\infty} |A_J|$ converges.

Q. E. D.

__Corollary.__ If $\sum_{J=0}^{\infty} A_J$ converges, then for each ε there exists a $N(\varepsilon) \in \mathcal{S}$ such that $|A_J| < \varepsilon$ for each index $J \notin N(\varepsilon)$.

__Proof.__

For each ε there exists $N(\varepsilon) \in \mathcal{S}$ such that $\sum_{J \in M} |A_J| < \varepsilon$ for each finite M outside $N(\varepsilon)$. In particular let M be a set containing just one point of $\mathbb{Z}^n$. Q. E. D.

__Corollary.__ If $\sum_{J=0}^{\infty} A_J$ converges, then there exists a finite bound M such that $|A_J| < M$ for each $J \in \mathbb{Z}^n$.

__Corollary.__ If $\sum_{J=0}^{\infty} A_J$ contains non-negative real terms A_J and if the 1-series over n-cubes $\sum_{J \in K_m} A_J = B$, then $\sum_{J=0}^{\infty} A_J = B$.

__Proof.__

The partial sums of the 1-series $\sum_{m=0}^{\infty} c_m = \sum_{J \in K_m} A_J$ are monotonic non-decreasing and bounded above as they approach the limit B.

But, given $\varepsilon > 0$, take $N(\varepsilon)$ to be a n-cube such that $B - \varepsilon \le \sum_{J \in N(\varepsilon)} A_J \le B$. Then for each finite set $N \supset N(\varepsilon)$ we have $B - \varepsilon \le \sum_{J \in N} A_J \le B$ and hence $\sum_{J=0}^{\infty} A_J = B$.

Q. E. D.

Example. $\sum_{J=0}^{\infty} r_1^{j_1} \cdots r_n^{j_n} = \frac{1}{(1-r_1)\cdots(1-r_n)}$, $0 < r < 1$. Since all terms are positive we sum over n-cubes

$$\sum_{j_1,\dots,j_n=0}^{m} r_1^{j_1} \cdots r_n^{j_n} = \sum_{j_1,\dots,j_{n-1}=0}^{m} r_1^{j_1} \cdots r_{n-1}^{j_{n-1}} \left(\sum_{j_n=0}^{m} r_n^{j_n} \right) = \prod_{\lambda=1}^{n} \frac{1 - r_\lambda^{m+1}}{1 - r_\lambda}$$

Thus

$$\sum_{J=0}^{\infty} r_1^{j_1} \cdots r_n^{j_n} = \lim_{m \to \infty} \prod_{\lambda=1}^{n} \frac{1 - r_\lambda^{m+1}}{1 - r_\lambda} = \frac{1}{(1-r_1)\cdots(1-r_m)} .$$

Theorem 2. If a power series (say, about $0, 0, \dots, 0$) converges at $\hat{z} = (\hat{z}^1, \dots, \hat{z}^n)$, then it converges uniformly in a polycylinder $|z^j| < \rho^j < |\hat{z}^j|$ (assume all $\rho^j > 0$).

Proof.

Consider $\sum_{J=0}^{\infty} a_J z^J$ and

$$|a_J z^J| \le |a_J \rho^J| \le |a_J \hat{z}^J (\rho/z)^J| \le M r^J$$

for $r = (r^1, \dots, r^n)$ on $0 < r < 1$ and z in the polycylinder. By the Cauchy criterion, and comparison with $\sum_{J=0}^{\infty} r^J$, we see that $\sum_{J=0}^{\infty} a_J z^J = f(z)$ converges for each $|z| < \rho$.

Now given $\varepsilon > 0$ take $N(\varepsilon)$ such that $\left|\sum_{J \in M} r^J\right| < \varepsilon/2$ for each finite set M outside $N(\varepsilon)$. Then

$$\left|f(z) - \sum_{J \in N+M} a_J z^J\right| \ge \left|f(z) - \sum_{J \in N} a_J z^J\right| - \sum_{J \in M} |a_J z^J|$$

$$\ge \left|f(z) - \sum_{J \in N} a_J z^J\right| - \varepsilon/2 .$$

At $|z^*| < \rho$ we can choose M so large that $\left|f(z^*) - \sum_{J \in N+M} a_J z^{*J}\right| < \varepsilon/2$.

Then $\quad |f(z^*) - \sum_{J \in N} a_J z^{*J}| < \varepsilon \,, \quad N \geq N(\varepsilon)$.

Thus the convergence is uniform in the polycylinder $|z| < \rho$, since $N(\varepsilon)$ is independent of z^*.

Q. E. D.

Now let $w(z_1, \ldots, z_n)$ be a holomorphic (complex-valued) function on an open set $\mathcal{O} \subset \mathbb{C}^n$. Then

$$w(z_1, \ldots, z_n) = u(z_1, \ldots, z_n) + i v(z_1, \ldots, z_n)$$
$$= u(x_1, \ldots, x_n, y_1, \ldots, y_n) + i v(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

and $u(x,y)$, $v(x,y)$ are real analytic on the corresponding open set in R^{2n}. Also

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \qquad \text{for all } 1 \leq j \leq n$$
$$\frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}$$

Also the complex derivatives $\frac{\partial w}{\partial z_j}$ exist, are analytic and can be computed by differentiation of the power series for w term-by-term. In particular,

$$\frac{\partial^K w}{\partial z^K}(0) = \frac{\partial^K w}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}(0) = k_1! \, k_2! \ldots k_n! \, a_{k_1 \ldots k_n}.$$

Thus if two holomorphic functions have the same power series near a point P, then they coincide on an open neighborhood of P and thus they coincide on any connected open set which contains P.

Also we have the Cauchy integral formula

$$a_{k_1 \ldots k_n} = \frac{1}{(2\pi i)^n} \int_{C_1} \cdots \int_{C_n} \frac{w(\zeta)\, d\zeta_1 \ldots d\zeta_n}{\zeta_1^{k_1+1} \ldots \zeta_n^{k_n+1}}$$

where C_j is the circle $|\zeta_j| = R_j$ positively oriented in the ζ_j-plane. Thus the integration is only over the distinguished boundary of the polycylinder, not the entire topological boundary. Also from

$$w(z_1,\ldots,z_n) = \frac{1}{(2\pi i)^n}\int_{C_1}\frac{d\rho_1}{\rho_1 - z_1}\int_{C_2}\frac{d\rho_2}{\rho_2 - z_2}\cdots\int_{C_n}\frac{w(\rho_1,\ldots\rho_n)\,d\rho_n}{(\rho_n - z_n)}$$

we have

$$w(z) = \frac{1}{(2\pi i)^n}\int_{C_1}\cdots\int_{C_n}\sum_{J=0}^{\infty}\left(\frac{z}{\rho}\right)^J \frac{w(\rho)\,d\rho_1\cdots d\rho_n}{\rho_1\cdots\rho_n} \qquad \text{for} \qquad |z| < |\rho| \ .$$

Then the power series for $w(z)$ converges in the largest polycylinder within the domain of analyticity of $w(z)$. If $|w(z)| \le M$ in the polycylinder $|z| \le \rho$, then

$$|a_J| \le \frac{1}{(2\pi)^n}\int_{C_1}\cdots\int_{C_n}\frac{M|d\rho_1|\cdots|d\rho_n|}{\rho_1^{j_1+1}\cdots\rho_n^{j_n+1}} \le \frac{M}{\rho^J}\ .$$

Conversely if $w(z_1,\ldots,z_n)$ is continuous and fulfills the Cauchy formula, then it is holomorphic. Also if $w = u + iv$ with $u, v \in C^1$ satisfying the Cauchy-Riemann equations, then w is holomorphic. Further if $w(z_1,\ldots,z_n)$ is defined in an open set and therein $\frac{\partial w}{\partial z_j}$ exist everywhere, then w is holomorphic.

Linear combinations, products, and compositions of holomorphic functions are holomorphic. Two important results hold for holomorphic functions on $\mathcal{O} \subset \mathbb{C}^n$ (but not in general for real analytic functions).

1. Let $w_1(z_1,\ldots,z_n), w_2(z),\ldots, w_m(z),\ldots$ be a sequence of holomorphic functions in $\mathcal{O} \subset \mathbb{C}^n$ and $\lim_{m\to\infty} w_m(z) = w(z)$ uniformly in $\mathcal{O}$, then $w(z)$ is holomorphic in $\mathcal{O}$.

2. Consider $w_1(z_1,\ldots,z_n),\ldots, w_n(z_1,\ldots,z_n)$ holomorphic near $\hat{z}$ with values $(\hat{w}_1,\ldots,\hat{w}_n) = \hat{w}$. Then the map $z \to w$ of a neighborhood of $\hat{z}$ onto a neighborhood of $\hat{w}$ is one-to-one if and only if the complex Jacobian $\left(\frac{\partial w_i}{\partial z_k}\right)_{\hat{z}}$ is non-singular. Note $\left|\frac{\partial w}{\partial z}\right|^2 = \det \frac{\partial(u,v)}{\partial(x,y)}$. The inverse map $w \to z$ is also holomorphic and this defines a holomorphic isomorphism.

3. Existence, Uniqueness, Continuation, and Singularities of Solutions.

Theorem 3. Consider the differential system

$$\frac{dw^i}{dz} = f^i(z, w^1, \ldots, w^n, \mu^1, \ldots, \mu^m) \qquad i = 1, 2, \ldots, n$$

where f^i are holomorphic in a polycylinder $\mathcal{C}: |z| \le a,\ |w^i| \le b,\ |\mu^j| \le r$ (say, centered at $z_0 = 0,\ w_0 = 0,\ \mu_0 = 0$). Let $|f^i| \le M$ in $\mathcal{C}$ and set $M' = (n+1)M(1 - r'/r)^{-m}$ for $0 < r' < r$. Then there exists a solution $w^i(z, \mu)$ holomorphic in $m+1$ complex variables in $|\mu^j| < r',\ |z| < \rho = a(1 - e^{-b/aM'})$, (wherein the power series converges) and satisfying the initial conditions $w^i(z_0) = w_0^i$ for each μ. Furthermore, for each $|\mu^j| < r'$, any solution $\hat{w}^i(z, \mu)$, satisfying the initial conditions (z_0, w_0), must coincide with $w^i(z, \mu)$ for z near z_0.

Proof.

Since $w^i(z_0)$ is given (fix μ throughout), $\frac{dw^i}{dz}(z_0) = f^i(z_0, w(z_0), \mu)$ is known. Also $\frac{d^2w^i}{dz^2}(z_0) = \frac{\partial f^i}{\partial z} + \frac{\partial f^i}{\partial w^j}\frac{\partial w^j}{\partial z}$ and also subsequent derivatives of $w^i(z, \mu)$ at $z = z_0$ are determined. Thus the uniqueness required in the last conclusion of the theorem holds. We proceed with the existence proof.

We first compute a formal series solution

$$w^i(z, \mu) = a_1^i(\mu) z + a_2^i(\mu) z^2 + \cdots + a_k^i(\mu) z^k + \cdots$$

where each $a_k^i(\mu)$ is a formal power series. To do this write $f^i(z, w, \mu) = \sum_{j,k=0}^{\infty} A_{jk}^i(\mu) z^j w^k$ where each $A_{jk}^i(\mu)$ is an analytic function. Then the coefficients a_k^i are certain polynomials (with positive coefficients) in the indeterminates A_{jk}^i and hence $a_k^i(\mu)$ are analytic.

By the method of majorants, if $f^i \ll g^i = \sum B^i_{jk} z^j w^k$ (that is $|A^i_{jk}| \le B^i_{jk}$) and if the formal series for $\frac{dw^i}{dz} = g^i$ converges, then so does the formal series for $\frac{dw^i}{dz} = f^i$.

Now

$$f^i(z,w,\mu) \ll \frac{M}{(1-z/a)(1-w^1/b)\cdots(1-w^n/b)(1-\mu^1/r)\cdots(1-\mu^m/r)}$$

$$= M \sum_{J=0}^{\infty} (z/a)^h (w^1/b)^{j_1} \cdots (w^n/b)^{j_n} (\mu^1/r)^{k_1} \cdots (\mu^m/r)^{k_m}$$

for $|z|<a$, $|w|<b$, $|\mu|<r$.

Thus we must produce an analytic solution $W^i(z,\mu)$ of

$$\frac{dW^i}{dz} = \frac{M}{(1-z/a)(1-W^1/b)\cdots(1-W^n/b)(1-\mu^1/r)\cdots(1-\mu^m/r)}$$

This will be accomplished by finding $W^i(z,\mu) = W(z,\mu)$ which satisfies

$$\frac{dW}{dz} = \frac{M}{(1-z/a)(1-W/b)^n(1-\mu^1/r)\cdots(1-\mu^m/r)}$$

This is solved by

$$W = b - b\left\{1 + \frac{(n+1)a}{b}\,\frac{M}{(1-\mu^1/r)\cdots(1-\mu^m/r)}\,\log(1-z/a)\right\}^{\frac{1}{n+1}}.$$

Use power series for $\log(1-z/a)$ and then binomial series to obtain formal power series for $W(z)$. Verify convergence at $z=\rho=a(1-e^{-b/aM'})$ and $\mu = r' < r$. Q. E. D.

<u>Theorem 4.</u> Consider $\frac{dw^i}{dz} = f^i(z,w,\mu)$ holomorphic in an open set $\mathcal{O} \subset \mathbb{C}^{n+m+1}$. Let $w^i(z,\mu_0)$ be the holomorphic solution, for a fixed μ_0 with $w^i(z_0,\mu_0) = w^i_0$ in $\mathcal{O}$. Let C be a curve (piecewise differentiable image of $[0,1]$) in z-space initiating at z_0 and lying in the (projected) domain of holomorphy of $f^i(z,w,\mu)$. Then the solution $w^i(z,\mu_0)$ can be analytically continued along C provided the points $(z, w(z,\mu_0), \mu_0)$ lie in a compact subset of $\mathcal{O}$ along C . Moreover, (in this case) for each initial point $\hat{z}_0, \hat{w}_0,$ and $\hat{\mu}_0$ near (z_0, w_0, μ_0) , the corresponding solution $w^i(z, \hat{\mu}_0, \hat{z}_0, \hat{w}_0)$ can be continued analytically along C and, for z near $C(1)$, $\hat{\mu}_0$ near μ_0, $\hat{z}_0, \hat{w}_0$ near z_0, w_0 this is a holomorphic function of $2+n+m$ complex variables.

Proof.

Let $C(z,w,\mu)$ lie in a compact subset $K \subset \mathcal{O} \subset \mathbb{C}^{n+m+1}$. Fix the initial data z_0, w_0, μ_0 and take the uniform bounds a, b, M so that each point on C is the center of a polycylinder of radii a, b and $|f^i(z,w,\mu)| < M$ in K. Then, from each point on C, the solution $w(z,\mu_0)$ can be analytically continued for a radius of $\rho = a(1 - e^{-b/2aM'})$. Therefore $w(z,\mu_0)$ can be continued to the end of C.

For the analytic dependence on the initial conditions and the parameters see Coddington and Levinson, p. 36.

Q. E. D.

<u>Theorem 5.</u> Consider $\dfrac{dw^i}{dz} = f^i(z,w)$ $i = 1, 2, \cdots, n$ where $f^i(z,w)$ are holomorphic in $\mathcal{O}_z \times \mathcal{O}_w = \mathcal{O}$ where $\mathcal{O}_z$ is simply-connected. Assume, for each initial condition in $\mathcal{O}$ and each path in $\mathcal{O}_z$ leading from this point, the solution can be analytically continued. Then each solution $w^i(z)$ is a single-valued holomorphic function.

Proof.

Monodromy theorem. Q. E. D.

<u>Corollary</u>. Consider the linear system

$$\frac{dw^i}{dz} = A^i_j(z)\, w^j + B^i(z)$$

where $A^i_j(z)$ and $B^i(z)$ are holomorphic in a simply-connected region $\mathcal{O}_z \subset \mathbb{C}^1$. Then each solution $w^i(z)$, for each $i = 1, 2, \ldots, n$, is a single-valued holomorphic function in $\mathcal{O}_z$.

Proof.

Each path $C \subset \mathcal{O}_z$ lies in a compact subdomain of $\mathcal{O}_z$ wherein

$|A^i_j(z)|$ and $|B^i|$ have finite bounds. The usual exponential estimates show that $w^i(z)$ is bounded along C and so continuation is possible.

Q. E. D.

Corollary. Consider

$$\frac{d^n w}{dz^n} + a_{n-1}(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots + a_0(z)w = b(z)$$

where all the coefficients are holomorphic in a simply-connected domain $\mathcal{O}_z$. Then each solution $w(z)$ is single-valued, holomorphic in $\mathcal{O}_z$. Thus the set of all solutions is a n-complex vector space plus a particular holomorphic solution.

Examples of singularities.

1. $\frac{dw}{dz} = \frac{w}{z}$ apparent singularity at $z=0$.
   singularity of first kind for coefficients.
   $w = kz$

2. $\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} = 0$ singularity of first kind for coefficients.
   $w_1 = \log z$, $w_2 = 1$ regular singular point for solutions.

3. $w'' + w'/z - w/4z^2 = 0$ singularity of first kind for coefficients.
   $w_1 = \sqrt{z}$, $w_2 = 1/\sqrt{z}$ regular singular point for solutions.

4. $w' = w/z^2$ singularity of second kind for coefficients.
   $w = e^{-1/z}$ irregular singular point for solutions.

5. $\frac{dw^1}{dz} = w^1$, $\frac{dw^2}{dz} = \frac{-3}{16z^2}w^1$ singularity of second kind for coefficients.
   $\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} z^{1/4} \\ \frac{1}{4}z^{-3/4} \end{pmatrix}$ or $\begin{pmatrix} z^{3/4} \\ \frac{3}{4}z^{-1/4} \end{pmatrix}$
   regular singular point for solutions.

6. $\frac{dw}{dz} = -w^2$ regular coefficients everywhere
   $w = \dfrac{1}{z - z_0 + 1/w_0}$ moveable poles for solutions.

7. $\dfrac{dw}{dz} = -w(\log w)^2$

$w = e^{\frac{1}{z-c}}$ moveable essential singularities.

8. $w'' + (w')^3 = 0$ regular coefficients everywhere

$w = \sqrt{z - z_0} + c$ moveable branch points.

9. $\dfrac{dw}{dz} = -w^3$ regular coefficients everywhere.

$w = \dfrac{1}{\sqrt{2(z-z_0) + 1/2w_0^2}}$ moveable branch points.

10. $\dfrac{d^2w}{dz^2} = \left(\dfrac{dw}{dz}\right)^2 \cdot \dfrac{2w-1}{w^2+1}$ poles for coefficients.

$w = \tan[\log(Az-B)]$ moveable essential singularities.

4. Complex Manifolds.

Definition. Let M^{2n} be a differentiable manifold and let $\mathcal{H}$ be a family of the local coordinates of M^{2n} such that:

1. $\mathcal{H}$ covers M^{2n}
2. each homeomorphism in $\mathcal{H}$ is from an open set on M^{2n} onto an open set in $\mathbb{C}^n \cong R^{2n}$.
3. the change of coordinates between overlapping systems in $\mathcal{H}$ is a holomorphic isomorphism between open subsets of $\mathbb{C}^n$.
4. $\mathcal{H}$ is maximal with regard to 1) 2) 3).

Then $\mathcal{H}$ is called the complex structure for the complex (holomorphic) manifold M^{2n}. Also $\mathcal{H}$ is subordinate to the differentiable structure of M^{2n} . For $n=1$, M^{2n} is called a Riemann surface.

Remarks. Clearly a complex manifold M^{2n} has even dimension $2n$ or "complex dimension n". Also M^{2n} is orientable.

Example 1. $M^{2n} = \mathbb{C}^n$ or

2. M^{2n} is open subset of a complex manifold.

3. T^{2n}: $\mathbb{C}^n$ modulo a group generated by 2n independent translations:

$z_j \to z_j + \alpha_1,\ z_j \to z_j + \alpha_2, \ldots, z_j \to z_j + \alpha_{2n}$ where

$\alpha_1, \ldots, \alpha_{2n}$ are linearly independent in real vector space.

4. T^2: $z \to z+1$ and $z \to z + \tau$ where

$I_m\ \tau \neq 0.$

5. S^2 (Riemann sphere) with coordinates z and $\rho = \frac{1}{z}$ near ∞.

__Definition.__ If M_1^{2n} and M_2^{2m} are complex manifolds, the product manifold $M_1^{2n} \times M_2^{2m}$ is also a complex manifold, using the product holomorphic coordinate functions.

__Definition.__ Holomorphic function, map, isomorphism are clear.

__Remark.__ A holomorphic function on a compact complex manifold is a constant. This follows from the maximum modulus theorem.

__Remarks.__ Among all spheres only S^2 and possibly S^6 can be complex manifolds. The product of two odd dimensional spheres is a complex manifold. The complex projective spaces are P_c^{2n} : $(z_1, z_2, \ldots, z_n)$ with identification of 1-complex lines in $\mathbb{C}^n - 0$. Now $P_c^2 = S^2$.

__Definition.__ A "differentiable" covering space of a differentiable manifold (connected) M^n is a differentiable manifold $\hat{M}^n$ and a differentiable map π, called projection, such that

1. $\pi : \hat{M}^n \to M^n$ onto

and

2. for each point $P \in M^n$ there exists an open neighborhood N_P such that each component of $\pi^{-1}(N_P) \subset \hat{M}^n$ is mapped diffeomorphically onto N_P by π.

Remark. If $\pi : \hat{M}^{2n} \to M^{2n}$ is a covering space of a complex manifold M^{2n}, then the complex structure can be lifted to $\hat{M}^{2n}$.

Definition. Let M^n be a differentiable manifold and take $P \in M^n$. Consider the set Σ of all differentiable curves $(0 \le t \le 1)$ in M^n initiating from P, and ending at points of M^n. Call two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in Σ equivalent if $\mathcal{C}_1(1) = \mathcal{C}_2(1)$ and the closed curve $\mathcal{C}_1\mathcal{C}_2^{-1}$ is homotopic to P, with P fixed, in M^n. The set $\tilde{M}^n$ of all such equivalence classes of curves can be topologized and given a differentiable structure in a standard way so that $\pi : \tilde{M}^n \to M^n$ is a covering space. Here $\pi(\mathcal{C}_1) = \mathcal{C}_1(1) \in M^n$. This is called the universal or simply-connected covering space of M^n since $\tilde{M}^n$ is simply-connected and it covers each covering space of M^n; moreover either of these two properties characterizes $\tilde{M}^n$.

Remark. Let $w(z^1, \dots, z^n)$ be defined as a holomorphic function on an open neighborhood $P \in \mathbb{C}^n$ (or of any complex manifold). Assume $w(z^1, \dots, z^n)$ can be continued analytically along each curve in $\mathcal{O} \subset \mathbb{C}^n$, say initiating from P. Then $w(z^1, \dots, z^n)$ defines a single-valued holomorphic function on the universal covering space $\tilde{M}^n$. However, we can call two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{O}$, initiating at P and with the same endpoint Q, equivalent in case the analytic continuation of $w(z^1, \dots, z^n)$ along $\mathcal{C}_1$ and $\mathcal{C}_2$ leads to the same power series function element in a neighborhood of $\mathcal{C}_1(1) = \mathcal{C}_2(1) = Q$. Then these equivalence classes of curves define a covering space $M_w^n \to \mathcal{O}$ on which $w(z^1, \dots, z^n)$ is holomorphic and single-valued. Then $\tilde{M}^n \to M_w^n \to \mathcal{O}$ and M_w^n is the complex manifold generated by the function element $w(z^1, \dots, z^n)$ near P. If M_w^n has complex dimension 1, it is the Riemann surface of $w(z)$. The projection function z and the generating function $w(z)$ are each holomorphic (single-valued) on M_w^n.

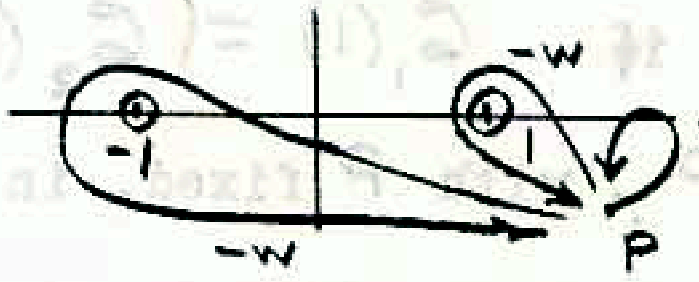
Example. $\log z$ infinite branch point at $z = 0$.

$\sqrt{z}$ 2-branch point (algebraic)

$z^i = e^{i \log z}$ infinite branch point

$z^{\sqrt{z}} = e^{\sqrt{z} \log z}$ infinite branch point

$w = \sqrt{(z-1)^3(z+1)}$



$w = \sqrt{-1+2z-2z^3+z^4}$
$= i + iz + \cdots$

The description of the Riemann surface consists of indicating the power series upon continuation along every path in $\mathbb{C} - \{1\} - \{-1\}$.

Definition. Let M be a Riemann surface. A function $w(z)$ holomorphic on $M - \{D\}$ where D consists of isolated points, and having poles at the points of D is meromorphic on M.

Note. If $w(z) = \frac{a_{-n}}{z^n} + \frac{a_{-n+1}}{z^{n-1}} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots$ (with $a_{-n} \neq 0$), then in other local coordinates (or uniformizing parameter) $z = z(\rho)$ we have

$$w(z(\rho)) = \frac{a_{-n}}{z(\rho)^n} + \frac{a_{-n+1}}{z(\rho)^{n-1}} + \cdots + \frac{a_{-1}}{z(\rho)} + a_0 + a_1 z(\rho) + \cdots .$$

Since $\frac{dz}{d\rho}(0) \neq 0$, $w(\rho)$ is meromorphic at P and with the same order pole. Thus the order of a pole, or a zero, of a meromorphic function on M is invariant and does not depend on the local coordinate system. Similarly the order of a branch point (multiplicity of covering space over punctured neighborhood of P of Riemann surface generated by a power series element) is an invariant of a "multiple valued" function on M.

Theorem 6. The set of all meromorphic functions on a Riemann surface M is a field under the usual addition and multiplication of functions.

Proof.

The zeros of a meromorphic function are isolated and each is of finite order. If $w(z)$ is meromorphic, $\frac{1}{w(z)}$ is zero at the poles of $w(z)$ and is also meromorphic. Q. E. D.

<u>Notation</u>. On an open set $\mathcal{O} \subset \mathbb{C}^1$ consider the function

$$\frac{a_n(z)w^n + \cdots + a_0(z)}{b_m(z)w^m + \cdots + b_0(z)}, \text{ in } \mathcal{O} \times \mathbb{C}^1, \text{ where}$$

$a_n(z) \neq 0$, $b_m(z) \neq 0$ and $a_k(z)$, $b_k(z)$ are meromorphic in $\mathcal{O}$. Two such functions are considered the same if they coincide in the algebraic sense of rational functions in an indeterminant w, over the coefficient field comprised of the meromorphic functions on $\mathcal{O}$. This means:

a.) over the field of meromorphic functions of z factor into primes the numerator and denominator polynomials, cancel the common factors.

b.) Divide numerator and denominator by $b_m(z)$ to achieve the standard (unique) form.

The domain of definition, and holomorphy, of the given function is understood to be the subset of $\mathcal{O} \times \mathbb{C}^1$ where the standard form has a non-zero denominator and none of the coefficients have poles.

If
$$\frac{a_n(z)w^n + \cdots + a_0(z)}{b_m(z)w^m + \cdots + b_0(z)} = \frac{\hat{a}_n(z)w^n + \cdots + \hat{a}_0(z)}{\hat{b}_s(z)w^s + \cdots + \hat{b}_0(z)}$$
on an open subset of $\mathcal{O} \times \mathbb{C}^1$ (where they are both holomorphic), then they have the same standard form and define the same function. For the standard forms are equal on an open set in $\mathcal{O} \times \mathbb{C}^1$ and thus we can assume $b_m(z) \equiv 1$, $\hat{b}_s(z) \equiv 1$ and the rational functions are reduced to lowest terms. Then, by standard arguments in the algebra of polynomials, the numerators must agree, and the denominators must agree — up to a multiple by a meromorphic function of z, which is evidently 1.

5. First Order Nonlinear Differential Equations.

Theorem (Painlevé) 7. Consider

$$\mathcal{D})\quad \frac{dw}{dz} = \frac{g(z,w)}{h(z,w)} = \frac{a_n(z)w^n + \cdots + a_0(z)}{w^m + b_{m-1}(z)w^{m-1} + \cdots + b_0(z)}$$

where the coefficients $a_k(z)$, $b_k(z)$ are meromorphic in $\mathcal{O}$, and g/h is in standard form (irreducible). Let $w(z)$ be a solution of $\mathcal{D})$ (possibly multiple-valued) in a deleted neighborhood N_P of a point $P \in \mathcal{O}$. Assume $(z, w(z))$, for $z \in N_P$, lies in the domain of holomorphy of $\frac{g(z,w)}{h(z,w)}$ in $\mathcal{O} \times \mathbb{C}'_w$. Then, along each curve approaching P, $\lim_{z \to P} w(z)$ exists (possibly infinite).

Proof.

For each $\delta > 0$ consider the domain Δ in $\mathbb{C}'_w$ consisting of the union of disjoint discs $|w - w_1| < \delta, \ldots, |w - w_r| < \delta, |w| > 1/\delta$ where $w_1, \ldots, w_r$ are the roots of $h(P, w) = 0$. If, for each $\delta > 0$, there exists $\varepsilon > 0$ such that $|z - P| < \varepsilon$ implies $w(z) \in \Delta$, then $\lim_{z \to P} w(z)$ exists.

Thus assume there exist points in N_P

$z_\alpha \to P$, $|z - z_\alpha| \le \varepsilon$, and $w(z_\alpha)$ in $\mathbb{C}'_w - \Delta$, which is compact. But for each point $(z_\alpha, w(z_\alpha))$ there is a uniform radius of convergence of $w(z)$ since g/h is holomorphic in the compact set $|z - z_\alpha| \le \varepsilon, w \in \mathbb{C}'_w - \Delta$ and $|w| \le 1/\delta$. Therefore $w(z)$ is analytic at P and $\lim_{z \to P} w(z)$ exists.

Q. E. D.

Painlevé defined the fixed or intrinsic singularities of $\mathcal{D}$ as points z_0 of $\mathcal{O}$ at which

1.) at least one coefficient $a_k(z)$, $b_k(z)$ has a pole

2.) $g(z_0, w)$ and $h(z_0, w)$ have a common root w.

Then allow solutions $w(z)$ with values near $w = \infty$ and also solutions $z(w)$, when $h(z_0, w_0) = 0$. Painlevé then showed that if $w(z)$ is a solution in the neighborhood of a point P which is not a fixed singularity, then $\lim_{z \to P} w(z)$ exists.

Thus P is a point of determinancy of $w(z)$ or is a regular singularity. This is summarized by stating that the first order differential equation $\mathcal{D})$ has no moveable essential singularities.

Theorem 8. Let

$$\mathcal{D})\quad \frac{dw}{dz} = \frac{g(z,w)}{h(z,w)} = \frac{a_n(z)w^n + \dots + a_0(z)}{w^m + b_{m-1}(z)w^{m-1} + \dots + b_0(z)}$$

be given for $z \in \mathbb{C}'$, $w \in \mathbb{C}'$ as above. If $\mathcal{D})$ has no moveable branch points, then $\mathcal{D})$ is the Riccati equation

$$\frac{dw}{dz} = a_0(z) + a_1(z)w + a_2(z)w^2 .$$

If no solution of $\mathcal{D})$ has a moveable pole, then $\mathcal{D})$ is linear so $a_2(z) \equiv 0$.

Proof.

If z_0 is not a fixed singularity, and if $h(z_0, w) = 0$ has a root w_0, then $\left(\frac{dz}{dw}\right)_0 = 0$. Then $z = a_2 w^2 + \dots$ so $w(z)$ has a branch point at z_0. Thus demand $h(z,w) = h(z)$ and so $\frac{dw}{dz} = a_n(z)w^n + \dots + a_0(z)$. Now consider solutions assuming the value ∞, that is, let $W = 1/w$. Then

$$\frac{dW}{dz} = -\frac{1}{w^2}\left[a_n(z)w^n + \dots + a_0(z)\right] = -\frac{a_n}{W^{n-2}} - \dots - a_2 - a_1 W - a_0 W^2.$$

But $\frac{dW}{dz}$ must equal a polynomial in W and hence $a_n(z) \equiv 0, \dots, a_3(z) \equiv 0$.

Finally consider special case where $a_0(z), a_1(z), a_2(z)$ are real for real z, for convenience. Then $w(x)$ is also real, for real initial value $\frac{dw}{dx} = a_0(x) + a_1(x)w + a_2(x)w^2$. Take an interval I on x-axis where $|a_2(x)| > 2\alpha > 0$. Say $a_2(x) > 0$. Then for sufficiently large initial condition w_0 at x_0, $\frac{dw}{dx} > 0$ and $\frac{dw}{dx} > \alpha w^2$. But then $w(x)$ has a pole on I where $\frac{1}{w_0} + \alpha(x - x_0) = 0$ or $x - x_0 = -1/\alpha w_0$. By choosing w_0 sufficiently large, at x_0, we get the pole on I and not at a fixed singularity of $\mathcal{D})$. Thus if $\mathcal{D})$ has no moveable poles, $\mathcal{D})$ is linear.

Q. E. D.

6. Linear Differential Equations and the Monodromy Group.

Definition. A linear differential equation, with meromorphic coefficients,

$$\mathcal{D}) \quad w^{(n)} + a_{n-1}(z) w^{(n-1)} + \dots + a_0(z) w = 0$$

is well-defined on a Riemann surface M in case:

1. for each local coordinate system (z) on M there is prescribed such a linear differential equation with meromorphic coefficients in (z),
2. on the overlap of two coordinate systems (z) and (ρ), the prescribed n-th order linear differential equations have the same holomorphic solutions on each open subset of $(z) \cap (\rho)$.

Remark. If $\mathcal{D})$ is well-defined on M and has the above form in (z), then in (ρ) where $z = z(\rho)$, we can write the equation of $\mathcal{D})$ as (say $n = 2$)

$$\frac{d^2 w}{d\rho^2} + \frac{\frac{d^2\rho}{dz^2} + a_1(z(\rho))\,\rho'}{(d\rho/dz)^2}\frac{dw}{d\rho} + \frac{a_0(z(\rho))}{(d\rho/dz)^2} w = 0.$$

This follows since an independent set of solutions $w_1(z), \dots, w_n(z)$ completely determines a monic homogeneous n-th order linear differential equation; namely

$$w^{(n)} = c_1 w_1^{(n)} + \dots + c_n w_n^{(n)} \quad \text{or} \quad \begin{vmatrix} w^{(n)} & \dots & w \\ w_1^{(n)} & \dots & w_1 \\ \vdots & & \\ w_n^{(n)} & \dots & w_n \end{vmatrix} = 0$$

where $c_1, \dots, c_n$ are defined by $\begin{matrix} w = c_1 w_1 + \dots + c_n w_n \\ \vdots \\ w^{(n-1)} = c_1 w_1^{(n-1)} + \dots + c_n w_n^{(n-1)} \end{matrix}$.

Note that the poles of the coefficients are intrinsic.

Definition. A linear differential equation

$$\mathcal{D})\qquad w^{(n)} + a_{n-1}(z)\,w^{(n-1)} + \cdots + a_0(z)\,w = 0$$

with $a_k(z)$ having poles at $z=0$ is of the first kind (Fuchsian) at $z=0$ in case $b_{n-k}(z) = z^k a_{n-k}(z)$ $k=1,2,\dots,n$ are all holomorphic at $z=0$. Otherwise $\mathcal{D})$ has a singularity of the second kind at $z=0$. The highest order pole among the $b_{n-k}(z) = z^k a_{n-k}(z)$ is the rank of $\mathcal{D})$.

Note. In local coordinates (ρ), with $z=z(\rho)$, the coefficient of $d^{n-k}w/d\rho^{n-k}$ is a polynomial in the derivatives of $\rho(z)$ and the coefficients $a_{n-1}(z(\rho)),\dots,a_{n-k+1}(z(\rho))$ plus $a_{n-k}(z(\rho))\left(\frac{d\rho}{dz}\right)^{n-k}$, all divided by $\left(d\rho/dz\right)^n$. Thus the property of $\mathcal{D})$ having a singularity of the first kind at a point is independent of the local coordinates. Also the rank of $\mathcal{D})$ at a point P is intrinsic.

Definition. A linear homogeneous system, with meromorphic coefficients, is defined on a Riemann surface M in case

$$\mathcal{S})\qquad \frac{dw^i}{dz} = A^i_j(z)\,w^j \qquad\qquad i,j = 1,2,\dots,n$$

1. for each local coordinate system (z) a meromorphic matrix $A^i_j(z)$ is prescribed.
2. on the overlap of two coordinate systems $(z)\cap(\rho)$, the prescribed n-th order homogeneous linear differential systems have the same holomorphic solution functions.

Remark. If $\mathcal{S})$ is given in (z), as above, then in (ρ) with $z=z(\rho)$ we can write $\mathcal{S})\ \frac{dw^i}{d\rho} = A^i_j(z(\rho))\frac{dz(\rho)}{d\rho}w^j$.
For a fundamental matrix solution $W(z) = \begin{pmatrix} w^1_1 \dots w^1_n \\ \vdots \\ w^n_1 \dots w^n_n \end{pmatrix}$ determines the differential system in (z) by

$$\frac{dW}{dz} = A(z)W \quad \text{or} \quad A(z) = \frac{dW}{dz}W^{-1}.$$

Note if we take $\hat{W} = WC$ then $\frac{d\hat{W}}{dz}\hat{W}^{-1} = \frac{dW}{dz}W$. The poles, and their orders, for the coefficients $A^i_j(z)$ are intrinsic.

__Definition.__ If $A(z)$ has a pole of first order at $z=0$ then $\mathcal{S})$ has a singularity of the first kind there. Otherwise $\mathcal{S})$ has a singularity of second kind. The highest order pole in $zA(z)$ is the rank of $\mathcal{S})$. Thus, at a point P, $\mathcal{S})$ is analytic, or has a singularity of rank $\mu = 0, 1, 2, \ldots$. If $\mu = 0$, then P is a first kind singularity, if $\mu \geq 1$ then P is a second kind singularity. The rank of the singularity is independent of the coordinate system around P.

__Remark.__ Consider

$$\mathcal{D}) \quad \frac{d^2w}{dz^2} + a_1(z)\frac{dw}{dz} + a_0(z)w = 0$$

and the corresponding system

$$\mathcal{S}) \quad \begin{aligned} \frac{dw}{dz} &= v \\ \frac{dv}{dz} &= -a_0(z)w - a_1(z)v \end{aligned}$$

Here a singularity of the first kind for $\mathcal{D})$ may yield a singularity of the second kind for $\mathcal{S})$.

Also change local coordinates $z = z(\rho)$ and we have

$$\mathcal{D}') \quad \frac{d^2w}{d\rho^2} + \frac{\rho'' + a_1\cdot\rho'}{\rho'^2}\frac{dw}{d\rho} + \frac{a_0}{\rho'^2}w = 0$$

and

$$\mathcal{S}') \quad \begin{aligned} \frac{dw}{d\rho} &= v\,\frac{dz}{d\rho} \\ \frac{dv}{d\rho} &= -a_0\cdot\frac{dz}{d\rho}\,w - a_1\cdot\frac{dz}{d\rho}\,v. \end{aligned}$$

Thus the correspondence between $\mathcal{D})$ and $\mathcal{S})$ depends on the choice of local coordinates.

Now consider

$$\mathcal{D}) \quad w^{(n)} + a_{n-1}(z)w^{(n-1)} + \cdots + a_0(z) = 1$$

where $a_k(z)$ are meromorphic $a_n(z) \equiv 1$ and let $b_{n-k}(z) = z^k a_{n-k}(z)$ for $k = 0, 1, 2, \ldots, n$.

The corresponding system is $\mathcal{S})\ \dfrac{dW}{dz} = A(z) W$

where

$$A(z) = \frac{1}{z}\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 2 & 1 & & \\ 0 & 0 & 0 & 3 & \ddots & \\ & & & & & 1 \\ -b_0 & -b_1 & & & & (n-1)-b_{n-1} \end{pmatrix}$$

Then if $w(z)$ is a solution of $\mathcal{D})$ define the vector solution of $\mathcal{S})$ by

$$W(z) = \begin{pmatrix} w_1(z) \\ w_2(z) \\ \vdots \\ w_n(z) \end{pmatrix} \qquad \text{with } w_k(z) = z^{k-1}\frac{d^{k-1}w}{dz^{k-1}} \quad \text{for } k = 1, 2, \ldots, n.$$

Then it is easy to verify

$$z w_k' = (k-1)\, w_k + w_{k+1} \qquad \text{for} \quad k = 1, 2, \ldots, n-1$$

$$z w_n' = (n-1)\, w_n - [z^n a_{n-1} w^{(n-1)} + \cdots + z^n a_0 w]$$

or

$$z w_n' = (n-1)\, w_n - [b_{n-1} w_n + \cdots + b_1 w_2 + b_0 w_1].$$

Vice versa, if $w_1 = w$ in a solution of $\mathcal{S})$ then
$w_2 = z\dfrac{dw}{dz}$, $w_3 = z^2\dfrac{d^2w}{dz^2} + \cancel{z\dfrac{dw}{dz}} - \cancel{z\dfrac{dw}{dz}}$, etc. Thus the solutions of $\mathcal{D})$ and $\mathcal{S})$ correspond. Note that $\mathcal{D})$ has a singularity of the first or second kind, say at $z = 0$, just in case $\mathcal{S})$ has the same kind of singularity. Also the rank of the singularity is the same for $\mathcal{D})$ and for $\mathcal{S})$.

Again it is easy to check that the correspondence $\mathcal{D} \Leftrightarrow \mathcal{S}$ depends on the choice of local coordinates. Thus, for local analysis in one coordinate patch we can replace $\mathcal{D})$ by $\mathcal{S})$. But for global analysis on a Riemann surface the theories of $\mathcal{D})$ and of $\mathcal{S})$ are distinct.

<u>Definition</u>. Let $\mathcal{D})$ (or let $\mathcal{L})$) be a linear homogeneous differential equation (or first order system), with meromorphic coefficients on a Riemann surface M. Let X be the (isolated) singularities of $\mathcal{D})$ (or of $\mathcal{L})$) in M and at each ordinary, or nonsingular, point P the solution family is holomorphic in a neighborhood of P and forms an n-complex vector space. Consider the set of all closed loops in $M - X$, based at P. Each such loop defines a linear transformation of the solution vector space at P, upon analytic continuation of the solutions around the loop. Call two such loops equivalent, in the sense of monodromy, in case they produce the same linear transformation of the solution space onto itself. The monodromy equivalence classes of loops define the monodromy group of $\mathcal{D})$ (or of $\mathcal{L})$) based at P. The multiplication of two monodromy classes of loops is defined by following one representative loop after the other, as in the fundamental group of $\pi_1(M - X)$.

<u>Remark</u>. The monodromy group $\Upsilon(\mathcal{D})$ (or of $\mathcal{L}$), based at P is represented by a subgroup of matrices of $GL(n, \mathbb{C})$ once a basis has been chosen for the solution space near P. For the solutions of $\mathcal{L})$ there is a natural basis at P, with initial values I, the identity matrix. A change of solution basis changes the representation $\mu \subset GL(n, \mathbb{C})$ of the monodromy group of $\mathcal{D})$ or $\mathcal{L})$ to a conjugate subgroup $C \mu C^{-1}$, for a fixed $C \in GL(n, \mathbb{C})$.

For a change in the base point P to P' in $M - X$, the monodromy group is (abstractly) isomorphic (but not in a distinguished way). For a fixed basis of solutions at P and P', and a fixed isomorphism between the abstract monodromy groups affected by a curve in $M - X$ joining P' to P, the representations of the monodromy group are conjugate in $GL(n, \mathbb{C})$.

Theorem 9. Let $\mathcal{D})$ or $\mathcal{S})$ be a linear homogeneous differential equation or system on a Riemann surface M, and let Σ be the isolated singular points. For each base point $P \in M-\Sigma$ there is a natural homomorphism

$$\pi_1(M-\Sigma) \rightarrow \Upsilon(\mathcal{D}) \qquad \text{or} \qquad \Upsilon(\mathcal{S}).$$

Proof.

Two homotopic paths representing the same element of $\pi_1(M-\Sigma)$ define the same monodromy for $\mathcal{D})$ or $\mathcal{S})$. Also the multiplication of paths in Υ is defined so that the map $\pi_1 \rightarrow \Upsilon$ is a homomorphism.

Q. E. D.

Example.

$$\frac{dw}{dz} + \frac{w}{z} = 0, \qquad w = \frac{c}{z}$$

$$\frac{dw}{dz} + \frac{2w}{z} = 0, \qquad w = \frac{c}{z^2}.$$

These two equations each have the trivial monodromy group I in $\mathbb{C}-\{0\}$. However the equations are not the same, under a conformal change of independent variable since the solutions have different function-theoretic behavior near the singularity.

Remarks. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be linear homogeneous differential systems with meromorphic coefficients on a Riemann surface M. Assume $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same singularity set Σ. Let $P \in M-\Sigma$ and consider the natural bases for $\mathcal{S}_1$ and $\mathcal{S}_2$ which are the identity at I. Let $W_1(z)$ and $W_2(z)$ be the analytic continuations of these fundamental solution matrices. Suppose that for each point $Q \in \Sigma$ there exists a matrix $V_Q(z)$ holomorphic near Q except for possible poles at Q such that $W_1(z) = V_Q(z) W_2(z)$ (say, for the analytic continuation from P to Q along a specified path, and thus for every path). Then $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same monodromy group for $W_1(z)C_1 = V_Q(z)W_2(z)C_2$ implies $C_1 = C_2$.

If, in addition, $V_Q(z)$ is holomorphic at Q, for each $Q \in \Sigma$, and if M is compact, then $\mathcal{L}_1 = \mathcal{L}_2$. For consider $W_1(z)W_2(z)^{-1} = \Delta(z)$ which is single-valued and holomorphic on M. Thus $\Delta(z) = I$ and so $W_1(z) = W_2(z)$ on M.

7. Fuchsian Differential Equations on Riemann Surfaces.

Example. $\frac{dw}{dz} = \frac{A}{z} w$ for a constant matrix A. Solution matrix is $w = z^A = \exp(A \log z) = I + A \log z + \frac{A^2(\log z)^2}{2!} + \ldots$ which is I at $z = 1$.

For $\frac{dw}{dz} = e^{A \log z}\,\frac{A}{z} = \frac{A}{z} w$.

If

$$A = \operatorname{diag}\{A_1, A_2, \ldots, A_n\}$$

, then

$$w = z^A = \operatorname{diag}\{z^{A_1}, z^{A_2}, \ldots, z^{A_n}\}.$$

If

$$A = \begin{pmatrix} \lambda & 1 & & & 0 \\ 0 & \lambda & 1 & & \\ 0 & 0 & \lambda & 1 & \\ & & & \ddots & 1 \\ 0 & 0 & & & \lambda \end{pmatrix}$$

, then

$$z^A = e^{A \log z} = e^{\lambda \log z} \begin{pmatrix} 1 & \log z & & \frac{(\log z)^{r-1}}{(r-1)!} \\ 0 & 1 & \log z & \\ 0 & 0 & 1 & \\ & & \ddots & \\ 0 & \cdot & \cdot & 1 \end{pmatrix}$$

Assume all eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are simple, or (in the general case) let $y = Pw$ for constant P so

$$\frac{dy}{dz} = P \frac{dw}{dz} = P \frac{A}{z} w = \frac{PAP^{-1}}{z} y$$

and assume

$$PAP^{-1} = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

(or Jordan canonical form).

Then a fundamental matrix solution is

$y = \operatorname{diag}\{z^{\lambda_1}, \ldots, z^{\lambda_n}\}$ or

$w = P^{-1}\operatorname{diag}\{z^{\lambda_1}, \ldots, z^{\lambda_n}\}$, and

$w = P^{-1}\operatorname{diag}\{z^{\lambda_1}, \ldots, z^{\lambda_n}\} P = z^A$ is the fundamental matrix for which $w = I$ at $z = 1$.

Theorem 10. Consider

$$\frac{dw}{dz} = A(z)\, w$$

where $A(z)$ is holomorphic (single-valued) in a deleted neighborhood of $z = 0$. Then every fundamental matrix $\Phi(z)$ has the form

$$\Phi(z) = S(z)\, z^P \quad , \text{ near } z = 0 .$$

Here $S(z)$ is single-valued holomorphic on $0 < |z| < a$ and P is a constant matrix.

Proof.

Note $\Phi'(z) = A(z)\Phi(z)$ and $A(z e^{2\pi i}) = A(z)$

Then $\Phi'(z e^{2\pi i}) = A(z)\, \Phi(z e^{2\pi i})$, upon a circuit analytic continuation. Thus $\Phi(z e^{2\pi i}) = \Phi(z)\, C$ for a nonsingular constant matrix C. Take P such that $C = e^{2\pi i P}$ and then $\Phi(z e^{2\pi i}) = \Phi(z)\, e^{2\pi i P}$.

Define $S(z)$, possibly multivalued, by $\Phi(z) = S(z)\, z^P$ on $0 < |z| < a$.

But

$$\Phi(z e^{2\pi i}) = S(z e^{2\pi i})(z e^{2\pi i})^P = S(z e^{2\pi i})\, z^P e^{2\pi i P}$$

Then

$$S(z)\, z^P e^{2\pi i P} = S(z e^{2\pi i})\, z^P e^{2\pi i P}$$

Thus $S(z) = S(z e^{2\pi i})$ and $S(z)$ is single-valued and holomorphic in $0 < |z| < a$. Q. E. D.

Note. Assume $T^{-1}PT = J$ is in Jordan canonical form. Then

$$S(z)\,z^{P}\,T = S(z)\,T\,T^{-1}z^{P}\,T = S(z)\,T\,z^{J}$$

is a fundamental solution matrix. Note $S(z)T$ is single-valued and holomorphic in $0<|z|<a$. Denote the column vectors of $U = S(z)T$ by $u_j(z)$. Then the column vectors of $S(z)\,T\,z^{J}$ are

$$\begin{aligned}
\varphi_j(z) &= z^{\lambda_j}\,u_j(z) \qquad j=1,2,\dots,q\\
\varphi_{q+1}(z) &= z^{\lambda_{q+1}}\,u_{q+1}(z)\\
\varphi_{q+2}(z) &= z^{\lambda_{q+1}}\left[u_{q+1}(z)\log z + u_{q+2}(z)\right]\\
&\;\vdots\\
\varphi_{q+r_1}(z) &= z^{\lambda_{q+1}}\left[\frac{u_{q+1}(z)}{(r_1-1)!}(\log z)^{r_1-1} + \dots + u_{q+r_1}(z)\right]\\
\varphi_{q+r_1+1}(z) &= z^{\lambda_{q+2}}\,u_{q+r_1+1}(z)\\
&\;\vdots\\
\varphi_n(z) &= z^{\lambda_{q+s}}\left[\frac{u_{n-r_s+1}}{(r_s-1)!}(\log z)^{r_s-1} + \dots + u_n(z)\right].
\end{aligned}$$

For each eigenvalue λ of P there is always at least one vector solution $z^{\lambda}u(z)$, where $u(z)$ is single-valued holomorphic in $0<|z|<a$.

Note. The representation

$$\Phi(z) = S(z)\,z^{P}$$

was given without reference to the local coordinates. But P is not unique. For instance if we replace P by $P+kI$ for an integer k, then we replace $S(z)$ by $S(z)\,z^{-k}$.

Note. Replacing the fundamental matrix $\Phi(z)$ by $\Psi(z) = \Phi(z)K$ merely replaces C by $K^{-1}CK$ and thus P by $K^{-1}PK$. Thus the eigenvalues of P (mod 1) are invariants of the differential system $\mathcal{S}$) at $z=0$, provided k can be standardized.

Definition. If

$$\mathcal{S})\quad \frac{dw}{dz} = A(z)\,w$$

where $A(z)$ has an isolated pole at $z=0$, has a solution matrix

$\Phi(z) = S(z)\,z^P$ where $S(z)$ is single-valued and meromorphic at $z=0$, then $\mathcal{S})$ has a regular singularity at $z=0$. Otherwise $\mathcal{S})$ has an irregular singularity at $z=0$.

Note. If $\Phi(z) = S(z)\,z^P$ has a regular singular point at $z=0$, then so has each other fundamental solution $\Phi(z)K$. In this case we can write $\Phi(z) = \tilde{S}(z)\,z^{P-kI}$ for an integer k, and $\tilde{S}(z)$ is analytic at $z=0$ with $\tilde{S}(0)\neq 0$.

Definition. A homogeneous linear differential system $\mathcal{S})\ \frac{dw}{dz} = A(z)\,w$, with meromorphic coefficients, on a Riemann surface M is called Fuchsian in case every singularity is a regular singularity.

Theorem 11. Let

$$\mathcal{S})\quad \frac{dw}{dz} = A(z)\,w$$

have a singularity of the first kind at $z=0$, that is, $A(z)$ has a pole of order one at $z=0$. Then $\mathcal{S})$ has a regular singularity at $z=0$.

Proof.

Write $w' = \frac{1}{z}\tilde{A}(z)\,w$ where $\tilde{A}(z)$ is analytic for $0\le |z| < a$ and $\tilde{A}(0)\neq 0$. Let $\Phi(z) = S(z)\,z^P$ be a fundamental solution matrix. We show that $S(z)$ has, at worst, a pole at $z=0$.

Let $w = \varphi(z)$ be a non-zero solution and write

$\tilde{\Phi}(\rho, \theta) = \Phi(\rho e^{i\theta})$, $r = \|\tilde{\Phi}\|$ (use norm as sum of absolute value of components). Then

$$\left\|\frac{\partial\tilde{\Phi}}{\partial\rho}\right\| = \left\|\frac{d\Phi}{dz}(\rho e^{i\theta})\right\| \le \|\tilde{A}(\rho e^{i\theta})\| \frac{r(\rho,\theta)}{\rho}.$$

But $\left|\frac{\partial r}{\partial\rho}\right| \le \left\|\frac{\partial\tilde{\Phi}}{\partial\rho}\right\|$ and since $\|\tilde{A}(z)\| \le c$

$$\frac{\partial r}{\partial\rho} + \frac{cr}{\rho} \ge 0 \quad \text{on} \quad 0 < \rho \le \rho_1 < a .$$

Hence $\rho_1^c r(\rho_1,\theta) - \rho^c r(\rho,\theta) \ge 0$. Let $M = \max r(\rho_1, \theta)$ on $0 \le \theta \le 2\pi$ so $\|\Phi(\rho e^{i\theta})\| = r(\rho,\theta) \le \frac{\rho_1^c r(\rho_1,\theta)}{\rho^c} \le \frac{M\rho_1^c}{\rho^c}$.

Thus there exists $d > 0$ so that

$$|\Phi(z)| \le \frac{d}{\rho^c} \quad (0 \le \theta \le 2\pi, \ 0 < |z| \le \rho_1).$$

But

$$|z^{-P}| \le |e^{-P\ln\rho}||e^{-i\theta P}| \le \left[(n-1) + e^{|\ln\rho|\cdot|P|}\right]|e^{-i\theta P}|$$

Also

$$|e^{-i\theta P}| \le (n-1) + e^{2\pi|P|}, \quad \text{since} \quad |e^A| = |I + A + \frac{A^2}{2!} + \dots| \le n + e^{|A|} - 1.$$

Thus $|z^{-P}| \le n\rho^{-|P|}\left[(n-1) + e^{2\pi|P|}\right]$ on $0 < \rho < 1$, $0 \le \theta \le 2\pi$.

Thus $\rho^{c+|P|}|S(z)| \le \tilde{d}$ on $0 < \rho < \min(1, \rho_1)$, all θ.

Therefore $S(z)$ is meromorphic at $z = 0$.

Q. E. D.

Definition. The differential equation

$$\mathcal{D}) \quad w^{(n)} + a_{n-1}(z)w^{(n-1)} + \dots + a_0(z)w = 0,$$

with meromorphic coefficients at $z = 0$ has a regular singularity in case every solution near $z = 0$ can be expressed as a finite (constant) linear combination of terms $z^r(\log z)^k p(z)$, where r is a complex number, k is an integer $0 \le k \le n-1$ and $p(z)$ is analytic at $z=0$ with $p(0) \neq 0$. Otherwise $\mathcal{D})$ has an irregular singularity.

<u>Theorem 12.</u> If

$$\mathcal{D}) \quad w^{(n)} + a_{n-1}(z) w^{(n-1)} + \cdots + a_0(z) w = 0$$

has a singularity of the first kind at $z=0$ (that is $b_{n-k}(z) = z^k a_{n-k}(z)$ is analytic at $z=0$), then $\mathcal{D})$ has a regular singularity at $z=0$.

<u>Proof.</u>

The corresponding first order system

$$\mathcal{S}) \quad w' = A(z)\, w \qquad \text{with} \qquad A(z) = \frac{1}{z}\begin{pmatrix} 0 & 1 & & & 0 \\ & 1 & 1 & & \\ & & 2 & \ddots & \\ & & & \ddots & 1 \\ -b_0 & -b_1 & \cdots & \cdots & (n-1)-b_{n-1} \end{pmatrix}$$

has a singularity of the first kind at $z=0$ and thus has a regular singularity at $z=0$. Now each solution $w(z)$ of $\mathcal{D})$ is the first component of a solution of $\mathcal{S})$. But our analysis of the matrix solution $S(z)\, z^P$, where $S(z)$ is meromorphic at $z=0$, shows that $w(z)$ has the required form. Q. E. D.

<u>Definition.</u> A linear homogeneous differential equation

$$\mathcal{D}) \quad w^{(n)} + a_{n-1}(z) w^{(n-1)} + \cdots + a_0(z)\, w = 0$$

with meromorphic coefficients on a Riemann surface M is called Fuchsian in case each singularity is of the first kind.

<u>Remark.</u> The main theorem of Fuchs is that $\mathcal{D})$ has a singularity of the first kind at a point of M if and only if $\mathcal{D})$ has a regular singularity there.

<u>Theorem 13.</u> A linear homogeneous differential equation

$$\mathcal{D}) \quad w^{(n)} + a_{n-1}(z) w^{(n-1)} + \cdots + a_0(z)\, w = 0,$$

with meromorphic coefficients at $z = 0$, has a singularity of the first kind at $z = 0$ if and only if $\mathcal{D})$ has a regular singularity at $z = 0$.

Proof.

See Coddington-Levinson, p. 125. If $\mathcal{D})$ has a regular singularity then it has atmost a singularity of the first kind at $z = 0$.

Definition. Consider the differential system, with singularity of the first kind at $z = 0$,

$$\mathcal{S})\quad \frac{dw}{dz} = \left(\frac{A}{z} + \sum_{m=0}^{\infty} A_m z^m\right) w \qquad \text{, constant } A, A_m.$$

The eigenvalues of A are called the exponents of $\mathcal{S})$ at $z = 0$, and the sum of these exponents is $\operatorname{Tr} A$. For a Fuchsian equation, with holomorphic $b_{n-k}(z) = z^k a_{n-k}(z)$, the exponents at $z = 0$ are the roots of the indicial equation:

$$\rho(\rho-1)\cdots(\rho-n+1) + b_{n-1}(0)\rho(\rho-1)\cdots(\rho-n+2) + \cdots + b_0(0) = 0.$$

Also the sum of the exponents is $-b_{n-1}(0) + n(n-1)/2$.

Note. The matrix A, and thus the eigenvalues of $\mathcal{S})$ at $z = 0$, do not depend on the choice of local coordinates. Similarly each of the exponents $\rho_1, \dots, \rho_n$ (possibly multiple) is function-theoretically determined as the lowest order exponent of a solution $w = z^{\rho}(c_0 + c_1 z + \cdots)$. Also direct computations show that $b_{n-k}(0)$ are invariant.

Theorem 14. Let $\mathcal{S})\ \frac{dw}{dz} = A(z)\,w$ be a differential system on a compact Riemann surface M. Assume the coefficients are meromorphic and each singularity is of the first kind. Let $E_Q = \lambda_1^Q + \cdots + \lambda_n^Q$ be the sum of the exponents for each singular point $Q \in M$. Then $\sum_{Q \in M} E_Q = 0$.

Proof.

Consider a fundamental solution matrix $W(z)$ and let $\Delta(z)$ be its determinant. Then
$$\frac{d}{dz}(\log \Delta) = \frac{\Delta'(z)}{\Delta(z)} = \text{Trace } A(z)$$
is a meromorphic differential (single-valued) on M. Moreover $\Delta'(z)/\Delta(z)$ is holomorphic except at the singular points Q of $\mathcal{S})$ where it has the residue E_Q. But the sum of all the residues of a meromorphic differential on M is $\sum_{Q \in M} E_Q = 0$.

Q. E. D.

Remark. A similar analysis with a Fuchsian equation $\mathcal{D})$ on M proves
$$\sum_{Q \in M} E_Q = \frac{n(n-1)}{2}\left[N + 2g - 2\right]$$
, where n is the order of $\mathcal{D})$ and there are N singularities, and M has genus $g \geq 1$. The case of the Fuchs relations on the sphere $g = 0$ will occur later.

8. Local Theory of Fuchsian Differential Equations.

Theorem 15. If z_0 is a regular singularity for the equation
$$(E): \quad w^{(n)} + a_1(z)\, w^{(n-1)} + a_2(z)\, w^{(n-2)} + \cdots + a_n(z)\, w = 0,$$
then (E) has at most a singularity of the first kind at z_0.

Indication of the Proof.*

First observe that in any case (E) has a solution of the form
$$\Phi_1 = (z - z_0)^{\lambda}\, p(z),$$

---

* A complete proof, as given in Coddington and Levinson, p. 125. Theorem 15, and the report on the Frobenius method were prepared by Mr. H. Radjavi. The definition of the t-rank of a meromorphic matrix and the "Fuchs theorem" for systems were presented by Dr. W. Harris.

where $p(z)$ is analytic in $0<|z-z_0|<a$ for some a. But by assumption ϕ_1 must be of the form $\phi_1=(z-z_0)^{\mu}q(z)$, where $q(z)$ is analytic in $0\le|z-z_0|<a$ and $q(z_0)\ne 0$.

We now proceed by induction. For $n=1$ we have, using ϕ_1, that $a_1(z)=-\dfrac{\mu}{z-z_0}+\dfrac{q'(z)}{q(z)}$ so that the theorem is true. For the nth order equation let $w=\phi_1 v$ (change of the dependent variable). Then $u=v'$ satisfies an equation of order $(n-1)$:

$$u^{(n-1)}+c_1u^{(n-2)}+\cdots+c_{n-1}u=0,$$

where $c_{n-m}=\binom{n}{m}\dfrac{\phi_1^{(n-m)}}{\phi_1}+a_1\binom{n-1}{m}\dfrac{\phi_1^{(n-m-1)}}{\phi_1}+\cdots+a_{n-m}$.

Since the $(\phi_i/\phi_1)'$, $i=1,2,\ldots,n-1$ form a fundamental set of solutions and since the $(\phi_i/\phi_1)'$ are again linear combinations of terms of the form

$$(z-z_0)^{\alpha}(\log(z-z_0))^{\beta}p(z)$$

with p analytic in $0<|z-z_0|<a$, it follows from the induction hypothesis that c_{n-m} has at most a pole of order $n-m$.

Now take $$C_1=n\frac{\phi_1'}{\phi_1}+a_1.$$

Since ϕ_1'/ϕ_1 and C_1 have at most poles of order 1, a_1 has at most a pole of order 1. In general $\phi_1^{(k)}/\phi_1$ has at most a pole of order k and we use induction to prove that c_k has at most a pole of order k and this completes the proof.

The Frobenius Method.*

If the origin is a regular point for the nth order equation then the equation has the form $L(w)=z^nw^{(n)}+z^{n-1}a_1(z)w^{(n-1)}+\cdots+a_n(z)w=0$ with analytic a_j, $j=1,2,\ldots,n$.

---

* An indication of the method is given in Coddington and Levinson, page 132.

Then $a_j(z) = a_{j0} + a_{j1} z + a_{j2} z^2 + \cdots$.

If $a_{jk} = 0$ for all j and all $k \geq 1$, then we have the Euler-Cauchy equation with corresponding indicial polynomial

$$f(\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1) + \lambda(\lambda-1)\cdots(\lambda-n+2)a_{10} + \cdots + a_{n0}.$$

Then we have $L(z^\lambda) = f(\lambda) z^\lambda$.

The Frobenius method is a generalization of the method used for the Euler-Cauchy equation: We try to find a formal series solution of the form

$$\phi(z) = z^\lambda + c_1 z^{\lambda+1} + c_2 z^{\lambda+2} + \cdots ,$$

such that $L(\phi(z)) = f(\lambda) z^\lambda$. If we form $L(\phi(z))$ and collect terms we will have

$$L(\phi(z)) = f(\lambda) z^\lambda + [c_1 f(\lambda+1) + b_1] z^{\lambda+1} + [c_2 f(\lambda+2) + b_2] z^{\lambda+2} + \cdots$$

b_j is a polynomial in λ and is linear in $c_1, c_2, \ldots$ and c_{j-1}, so that the recursive system

$$c_j = \frac{-b_j(\lambda, c_1, c_2, \ldots, c_{j-1})}{f(\lambda+j)}$$

could be solved to give c_j if $f(\lambda+j) \neq 0$ $j = 1, 2, \ldots$

The c_j are rational functions of λ. If λ_1 is a root of the indicial equation for which $f(\lambda+j) \neq 0$, $j = 1, 2, \ldots$, then from $L(\phi_1(z)) = f(\lambda_1) z^{\lambda_1} = 0$ it follows that $z^{\lambda_1} + c_1 z^{\lambda_1+1} + c_2 z^{\lambda_1+2} + \cdots$ is a solution.

If λ_1 is a double root of $f(\lambda) = 0$, we can find another solution by differentiating both sides of $L(\phi(z)) = f(\lambda) z^\lambda$ with respect to λ which gives

$$\left[\frac{\partial}{\partial\lambda} L\,\phi(t)\right]_{\lambda=\lambda_1} = L\left[\left(\frac{\partial}{\partial\lambda}\phi(z)\right)_{\lambda=\lambda_1}\right] = \left[f(\lambda_1)\log z + f'(\lambda_1)\right] z^\lambda$$

so that $\left[\frac{\partial}{\partial\lambda}\phi(z)\right]_{\lambda=\lambda_1}$ is a solution. If λ_1 has multiplicity m we can, by $m-1$ differentiations, obtain m independent solutions.

We now consider the distinct roots of $f(\lambda)=0$:

$$\lambda_j = r_j + i s_j \ , \qquad j = 1, 2, \ldots, p ,$$

and arrange them in the descending order of the real parts, i.e., $r_1 \geq r_2 \geq r_3 \geq \cdots$. The above method is certainly applicable for λ_1 and gives m_1 different solutions, where m_1 is the multiplicity of λ_1 . Let m_j be the multiplicity of λ_j . If $\lambda_1 - \lambda_2$ is not a positive integer, the above method is applied to give m_2 solutions corresponding to λ_r . If $\lambda_1 - \lambda_2 = k$ is a positive integer we consider a function of the form

$$\phi(z) = (\lambda - \lambda_2)^{m_1} z^{\lambda} + c_1 z^{\lambda+1} + c_2 z^{\lambda+2} + \cdots ,$$

and find the c_j so that $L(\phi(z)) = f(\lambda)(\lambda - \lambda_2)^{m_1} z^{\lambda}$.

The c_j thus formed have $(\lambda - \lambda_2)^{m_1}$ as a factor for $j \leq k-1$. The c_j are obtained as rational functions of λ with no poles at $\lambda = \lambda_2$. We now differentiate both sides of the last relation m_1 times

$$L\left(\frac{\partial^{m_1}\phi}{\partial\lambda^{m_1}}\right) = m_1!\, f(\lambda) z^{\lambda} + g ,$$

where g has a factor $(\lambda - \lambda_2)$. Hence $\left(\frac{\partial^{m}\phi}{\partial\lambda^{m_1}}\right)_{\lambda=\lambda_2}$ is a solution which is easily seen to be different from the solutions corresponding to the root λ_1 . Further differentiations are used to give different solutions if $m_2 > 1 \ldots$.

Having formed independent solutions corresponding to the roots $\lambda_1, \lambda_2, \cdots,$ and λ_k we proceed as follows:

We use a function of the form

$$\phi(z) = (\lambda - \lambda_{k+1})^{\sum_1^k \varepsilon_j m_j} z^{\lambda} + c_1 z^{\lambda+1} + c_2 z^{\lambda+2} + \cdots ,$$

where ε_j is 1 or 0 according as $\lambda_j - \lambda_{k+1}$ is or is not a positive integer. We then find the c_j so that

$$L(\phi(z)) = (\lambda - \lambda_{k+1})^{\sum_1^k \varepsilon_j m_j} f(\lambda) z^{\lambda}$$

Now differentiating with respect to λ, $\sum_1^k \varepsilon_j m_j$ times we obtain a solution

$$\left(\frac{\partial^{\sum_1^k \varepsilon_j m_j} \phi(z)}{\partial \lambda^{\sum_1^k \varepsilon_j m_j}} \right)_{\lambda = \lambda_{k+1}}$$

which is readily seen to be different from the solutions obtained for λ_j, $j \le k$. If $m_{k+1} > 1$ we find other solutions by further differentiations.

The convergence of the formal series solution of

$$L(w) = z^n w^{(n)} + z^{n-1} a_1(z) w^{(n-1)} + \dots + a_n(z) w = 0$$

is guaranteed by the existence theorem for the regular singular solution and the uniqueness theorem for formal series solutions, cf. Coddington-Levinson, p. 117. In this reference one also finds a corresponding algorithm for solving differential systems near a singularity of the first kind.

<u>Theorem 16.</u> Consider the differential system

$$w' = \left(z^{-1} R + \sum_{m=0}^{\infty} z^m A_m \right) w$$

with $R \neq 0$ and A_m constant matrices, having a singularity of the first kind at $z = 0$. If R has characteristic roots which do not differ by positive integers, then there exists a fundamental solution matrix Φ of the form

$$\Phi(z) = P z^R \qquad (0 < |z| < c \quad \text{for} \quad c > 0),$$

where P is the convergent power series $P(z) = \sum_{m=0}^{\infty} z^m P_m$ with $P_0 = I$.

<u>Definition.</u> Let $A(z)$, $B(z)$ be $n \times n$ matrices of holomorphic functions in a punctured vicinity of z_0, $0 < |z - z_0| < a$, $a > 0$. We say A is

equivalent to B, $A \sim B$, if there exists a $n \times n$ matrix of meromorphic functions in a punctured vicinity of z_0, $0<|z-z_0|<b$, $b>0$, with $\det T(z) \neq 0$, such that

$$B(z) = T^{-1}(z)A(z)T(z) - T^{-1}(z)T'(z)$$

Definition. Let $A(z)$ be a n-by-n matrix of meromorphic functions and $A(z) = (z-z_0)^{-K}\{A_0+(z-z_0)A_1+\cdots\}$, $A_0 \neq 0$, $0<|z-z_0|<a$, $m(A)=\max\{0,K\}$. The t-rank of $\mu(A(z_0))$ of A at the point z_0 is defined to be

$$\mu(A(z_0)) = \min_{B\sim A}\{m(B)\}$$

We note that a singular point of the first kind at z_0 corresponds to a t-rank at z_0 of zero or one. Clearly, if z_0 is a regular singular point for a system $w' = A(z)w$ and $B \sim A$, then z_0 is a regular singular point for the system $w' = B(z)w$.

Theorem 17. The point z_0 is at most a regular singular point for the system $w' = A(z)w$ if and only if the t-rank, $\mu(A(z_0))$, is at most one.

Proof.

Without loss of generality assume $z_0 = 0$. Let $\mu(A(0)) \leq 1$, then $A \sim B = z^{-K}\{B_0+zB_1+\dots\}$ where $K \leq 1$. If $\Psi(z)$ is a fundamental matrix for $w' = B(z)w$, and $A \sim B$, i.e. $B=T^{-1}AT - T^{-1}T'$, then $\Phi(z) = T(z)\Psi(z)$ is a fundamental matrix for $w' = Aw$. If $K \leq 0$, $\Psi(z)$ is analytic and $T\Psi$ has at most a pole at $z=0$. If $K=1$, $\Psi(z) = S(z)z^P$ where $S(z)$ is holomorphic in a punctured vicinity of $z=0$ and has at most a pole at $z=0$. Thus $T(z)S(z)$ has at most a pole at $z=0$, and $\Phi = (TS)z^P$, and hence $z=0$ is a regular singular point.

Assume $z=0$ is a regular singular point. A fundamental matrix has the form $\Phi(z)=S(z)z^P$, where $S(z)$ is holomorphic in a punctured vicinity of $z=0$ and has at most a pole at $z=0$, and P is a constant matrix. $\Phi'=A\Phi$, hence

$$A(z)=\Phi'(z)\Phi^{-1}(z)=S'(z)S^{-1}(z)+\frac{1}{z}S(z)PS^{-1}(z).$$

Let $B(z)\sim A(z)$ with the matrix $T(z)$, then

$$B(z)=T^{-1}AT-T^{-1}T'=T^{-1}(S'S^{-1})T+\frac{1}{z}T^{-1}SPS^{-1}T-T^{-1}T'.$$

But $T^{-1}T=I$, hence $T^{-1}T'=-(T^{-1})'T$, and

$$B(z)=(T^{-1}S)'(T^{-1}S)^{-1}+\frac{1}{z}(T^{-1}S)P(T^{-1}S)^{-1}.$$

Thus $S(z)$ has the required properties for a $T(z)$ matrix and using S as a T matrix we get a matrix equivalent to A, i.e. $A(z)\sim\frac{1}{z}P$.

Thus $\mu(A(0))\leq 1$. Q. E. D.

9. Fuchsian Differential Equations on the Riemann Sphere.

Consider the differential equation

$$\mathcal{D})\quad w^{(n)}+a_{n-1}(z)w^{(n-1)}+\dots+a_0(z)w=0.$$

Define the differential operator $\delta=z\frac{d}{dz}$ and $D=\frac{d}{dz}$. Then

$$z^mD^mw=\delta(\delta-1)(\delta-2)\dots(\delta-m+1)w\quad\text{for}\quad m\geq 1$$

as seen by induction. Then write

$$\mathcal{D})\quad z^nw^{(n)}+z^{n-1}b_{n-1}(z)w^{(n-1)}+\dots+b_0(z)w=0$$

for $z^ka_{n-k}(z)=b_{n-k}(z)$.

In terms of the operator δ we can write

$$\hat{\mathcal{D}})\quad [(\delta)(\delta-1)\cdots(\delta-n+1)+b_{n-1}(z)\delta(\delta-1)\dots(\delta-n+2)+\dots+b_0(z)]w=0.$$

Now collect terms to write

$$\hat{\mathcal{D}})\quad \delta^{(n)}w+Q_{n-1}(z)\delta^{(n-1)}w+\dots+Q_0(z)w=0.$$

Note that the $b_{n-k}(z)$ and the $Q_{n-\ell}(z)$ are linearly related and so $\mathcal{D})$ has a regular singularity (or is analytic) at $z=0$ if and only if all $Q_{n-\ell}(z)$ are holomorphic at $z=0$. Also if all $b_{n-k}(z)=b_{n-k}$ are constants, then $\hat{\mathcal{D}})$ is linear with constant coefficients — this is the case for the Euler-Cauchy equation.

Theorem 18. Let

$$\mathcal{D})\quad w^{(n)}+a_{n-1}(z)w^{(n-1)}+\cdots+a_0(z)w=0$$

be a differential equation in the complex plane $\mathbb{C}$. Then $\mathcal{D})$ defines a Fuchsian differential equation on the Riemann sphere $S^2=\mathbb{C}+\infty$ if and only if each coefficient $a_{n-k}(z)$ is a rational function (with poles of order $\le k$) and furthermore

$$|z^k a_{n-k}(z)|=|b_{n-k}(z)|<B \qquad \text{as} \quad z\to\infty$$

for some bound B, that is,

$$a_{n-k}(z)=O\left(\frac{1}{z^k}\right) \qquad \text{as} \quad z\to\infty .$$

Proof.

Use $\delta=z\dfrac{d}{dz}$ to write $\mathcal{D})$ as

$$\hat{\mathcal{D}})\quad \delta^{(n)}w+Q_{n-1}(z)\delta^{(n-1)}w+\cdots+Q_0(z)w=0.$$

Now consider local coordinates near $z=\infty$, $\rho=\frac{1}{z}$, near $\rho=0$. Define $\delta^*=\rho\dfrac{d}{d\rho}=-z\dfrac{d}{dz}=-\delta$ in the overlap of coordinates. Thus write $\mathcal{D})$ near $z=\infty$ in the form

$$\hat{\mathcal{D}}^*)\quad \delta^{*(n)}w+Q^*_{n-1}(\rho)\delta^{*(n-1)}w+\cdots+Q^*_0(\rho)w=0$$

where

$$Q^*_{n-k}(u)=(-1)^k Q_{n-k}\left(\frac{1}{u}\right).$$

Thus $\mathcal{D})$ has a regular singularity (or is analytic) at $z=\infty$ just in case all $Q^*_{n-\ell}(\rho)$ are holomorphic near $\rho=0$. Thus if $\mathcal{D})$ is Fuchsian on S^2, all $Q_{n-\ell}(z)$ are rational functions. Thus if

$\mathcal{D})$ is Fuchsian on S^2, then all $b_{n-k}(z)$ are rational functions on S^2, and also all $b_{n-k}(z)$ are holomorphic at $z = \infty$. This means $|z^k a_{n-k}(z)| < B$ as $z \to \infty$.

Conversely if all $a_{n-k}(z)$ are rational functions and if

$$a_{n-k}(z) = O\left(\frac{1}{z^k}\right) \quad \text{as} \quad z \to \infty,$$

then $\mathcal{D})$ is Fuchsian on S^2.

Q. E. D.

<u>Corollary</u>. Let

$$\mathcal{D}) \quad w^{(n)} + a_{n-1}(z)\, w^{(n)} + \dots + a_0(z)\, w = 0$$

be Fuchsian on the sphere S^2. Then (except $w' = 0$) $\mathcal{D})$ has atleast one singularity of the first kind, say at ∞, on S^2. If $\mathcal{D})$ has exactly one singularity, say at ∞, then $\mathcal{D})$ is $w^{(n)} = 0$. If $\mathcal{D})$ has exactly two singularities, say ∞ and 0, then $\mathcal{D})$ is the Euler-Cauchy equation

$$z^n w^{(n)} + b_{n-1} z^{n-1} w^{(n-1)} + \dots + b_0 w = 0$$

for constant coefficients b_{n-k}.

<u>Proof.</u>

If $\mathcal{D})$ has no singularity in $\mathbb{C} = S^2 - \infty$ then each $a_{n-k}(z)$ is a polynomial and is thereby zero identically. Thus $\mathcal{D})$ must be $w^{(n)} = 0$. But if $n \geq 2$, even this equation has solutions which are not analytic at ∞ and so $\mathcal{D})$ must have ∞ as a singularity of the first kind. If $\mathcal{D})$ has a singularity at ∞ and at 0, then

$$a_{n-k}(z) = \frac{b_{n-k}(z)}{z^k} \quad \text{for} \quad k = 1, 2, \dots, n$$

and $b_{n-k}(z)$ is a polynomial which is holomorphic at ∞. Thus all $b_{n-k}(z) = b_{n-k}$ are constant.

Q. E. D.

Theorem 19. Let

$$\mathcal{D})\quad w^{(n)} + a_{n-1}(z)\,w^{(n-1)} + \cdots + a_0(z)\,w = 0$$

be Fuchsian on S^2 with exactly N singular points of the first kind. At each singular point Q let $\rho_1^{(Q)} + \cdots + \rho_n^{(Q)} = E_Q$ be the sum of the (multiple) exponents of $\mathcal{D})$ at Q. Then

$$\sum_{Q\in S^2} E_Q = (N-2)\,\frac{n(n-1)}{2}.$$

Proof.

Now $a_{n-1}(z) = \sum_{j=1}^{N-1} \frac{A_j}{(z-a_j)}$ for the $N-1$ finite singularities $a_1,\ldots,a_{N-1}$ (we assume coordinates have been chosen so that ∞ is a singularity of $\mathcal{D})$). The indicial equation at $z = a_j$ is

$$\rho(\rho-1)\cdots(\rho-n+1) + A_j\,\rho(\rho-1)\cdots(\rho-n+2) + \cdots = 0.$$

Thus the sum of the exponents at a_j is

$$E_j = \frac{n(n-1)}{2} - A_j.$$

Now the indicial equation at ∞ is

$$\rho^n + Q^*_{n-1}(0)\,\rho^{n-1} + \cdots + Q^*_0(0) = 0,$$

or

$$\rho^n - Q_{n-1}(\infty)\,\rho^{n-1} + \cdots + (-1)^n Q_0(\infty) = 0.$$

Thus at ∞

$$E_N = Q_{n-1}(\infty) = \frac{-n(n-1)}{2} + b_{n-1}(\infty)$$

or

$$E_N = \frac{-n(n-1)}{2} + \sum_{j=1}^{N-1} A_j.$$

Therefore

$$\sum_{j=1}^{N} E_j = (N-1)\,\frac{n(n-1)}{2} - \sum_{j=1}^{N-1} A_j - \frac{n(n-1)}{2} + \sum_{j=1}^{N-1} A_j$$

or

$$\sum_{j=1}^{N} E_j = (N-2)\,\frac{n(n-1)}{2}.$$

Q. E. D.

10. Remarks on the Second Order Fuchsian Equation on S^2.

The first order Fuchsian equation is trivial

$$\frac{dw}{dz} + a_0(z)w = 0 \qquad \text{so} \qquad w = w_0 e^{\int_{z_0}^{z} -a_0(s)\,ds}.$$

For second order Fuchsian equations the cases of one or two singularities are trivial and the first interesting case is that of three singularities, the hypergeometric equation.

The most general second order Fuchsian equation on S^2, say with singularity at ∞, is

$$\mathcal{D})\quad w'' + \left\{\sum_{j=1}^{N-1} \frac{A_j}{(z-a_j)}\right\} w' + \left\{\sum_{j=1}^{N-1} \frac{B_j}{(z-a_j)^2} + \sum_{j=1}^{N-1} \frac{C_j}{(z-a_j)}\right\} w = 0.$$

The condition $|z^2 a_0(z)| < B$ as $z \to \infty$ requires $\sum_{j=1}^{N-1} C_j = 0$.

No other conditions are needed and every choice of the constants A_j, B_j, C_j with $\sum_{j=1}^{N-1} C_j = 0$ yields a Fuchsian differential equation on S^2.

At $z = a_j$ the indicial equation is $\rho(\rho-1) + A_j\rho + B_j = 0$ so the exponents ρ_j, ρ_j' satisfy $\rho_j + \rho_j' = 1 - A_j$ and $\rho_j\rho_j' = B_j$.

Thus we can write

$$\mathcal{D})\quad w'' + \left\{\sum_{j=1}^{N-1} \left(\frac{1-\rho_j-\rho_j'}{z-a_j}\right)\right\} w' + \left\{\sum_{j=1}^{N-1} \frac{\rho_j\rho_j'}{(z-a_j)^2} + \sum_{j=1}^{N-1} \frac{C_j}{(z-a_j)}\right\} w = 0.$$

(There are $N-3$ excessive coefficients beyond number of exponents).

At ∞ the indicial equation is

$$\rho(\rho+1) - \left\{\sum_{j=1}^{N-1} A_j\right\}\rho + \sum_{j=1}^{N-1} (B_j + C_j a_j) = 0.$$

Thus

$$\rho_\infty + \rho_\infty' = -1 + \sum_{j=1}^{N-1} A_j.$$

Thus Fuch's relation for the exponents is

$$\sum_{j=1}^{N-1} (1-\rho_j-\rho_j') = \rho_\infty + \rho_\infty' + 1$$

or $$\sum_{j=1}^{N-1}(1-\rho_j-\rho_j') + (1-\rho_\infty-\rho_\infty') = 2\,.$$

11. The Hypergeometric Differential Equation on the Riemann Sphere.*

A. Introduction.

The hypergeometric equation is a special case of the general second order Fuchsian equation with three singularities. In the Euler form the singularities are at $z = 0, 1, \infty$ and the equation is:

$$z(1-z)\frac{d^2w}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{dw}{dz} - \alpha\beta w = 0.$$

The complex constants α, β, γ are related to the exponential behavior of the solutions at the singularities. The exact relationship is displayed in the Riemann-Papperitz equation:

$$\frac{d^2w}{dz^2} + \left[\frac{1-a'-a''}{z-a} + \frac{1-b'-b''}{z-b} + \frac{1-c'-c''}{z-c}\right]\frac{dw}{dz}$$
$$+\left[\frac{a'a''(a-b)(a-c)}{z-a} + \frac{b'b''(b-a)(b-c)}{z-b} + \frac{c'c''(c-a)(c-b)}{z-c}\right]\frac{w}{(z-a)(z-b)(z-c)} = 0$$

which is the general second order Fuchsian equation with singularities at $z = a, b, c$ and with exponents $a', a'', b', b'', c', c''$ at these points.

To avoid logarithmic solutions, the following development assumes that all sums and differences involving the α, β, γ are non-integral.

B. Power Series Solution near $z = 0$.

In the power series solution $w = z^p \sum a_k z^k$ of the hypergeometric equation, the indicial equation is:

$$(\gamma + p - 1)\,p\,a_0 = 0$$

---

* This section is a report prepared by Mr. R. Larson.

and the recursion relation is

$$a_{k+1} = \frac{(k+p)(k+p+\alpha+\beta)+\alpha\beta}{(k+p+1)(k+p+\gamma)}\, a_k, \quad k = 0, 1, 2, \ldots .$$

For the root $p = 0$ of the indicial equation the recursion relation is

$$a_{k+1} = \frac{(\alpha+k)(\beta+k)}{(\gamma+k)(k+1)}\, a_k$$

and this yields the power series

$$w = a_0 \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)\, k!} z^k .$$

If γ is not a negative integer this series is absolutely convergent for $|z| < 1$ and converges everywhere on the unit circle except possibly at $z = \pm 1$. At $z = +1$ the series converges if $Re(\gamma - \alpha - \beta) > 0$ and at $z = -1$ it converges if $Re(\gamma - \alpha - \beta + 1) > 0$.

Convention dictates that the function described by this series for $a_0 = 1$ plus its analytic extension into the plane, slit along the real axis from $+1$ to ∞, be called $F(\alpha, \beta, \gamma, z)$.

For the other root of the indicial equation, $p = 1 - \gamma$, the recursion relation is

$$a_{k+1} = \frac{(k+\alpha-\gamma+1)(k+\beta-\gamma+1)}{(k+1)(k+2-\gamma)}\, a_k .$$

Therefore a second solution in $|z| < 1$ is

$$w = z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z).$$

The two functions found in this section form a basis of solutions in $|\arg(1-z)| < \pi$.

C. <u>Kummer's 24 Solutions</u>.

There are six linear fractional transformations of the sphere that carry the points $z = 0, 1, \infty$ onto the points $t = 0, 1, \infty$ (for instance $t = \frac{z}{z-1}$ carries $z = 0, 1$ and ∞ into $t = 0, \infty$ and 1, respectively). If we make one of these changes of variable and at the same time change the dependent variable from w to $z^s(1-z)^r v$ then the constants s and r can be determined so that the resulting differential equation is also hypergeometric. In this manner the basis in $|z| < 1$ can be used to find basis in various neighborhoods of $0, 1$ and ∞.

If we let $t = 1 - z$ then $s = r = 0$ and the equation transforms to

$$t(1-t)\frac{d^2w}{dt^2} + [(\alpha + \beta - \gamma + 1) - (\alpha + \beta + 1)t]\frac{dw}{dt} - \alpha\beta w = 0.$$

This is a hypergeometric equation with $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \alpha + \beta - \gamma + 1$. The power series solutions that we have are valid in $|t| = |1 - z| < 1$, thus in the unit circle about $z = 1$ we have a basis:

$$F(\alpha, \beta, \alpha + \beta - \gamma + 1, 1 - z)$$

and $$(1-z)^{\gamma - \alpha - \beta} F(\gamma - \beta, \gamma - \alpha, \gamma - \alpha - \beta + 1, 1 - z).$$

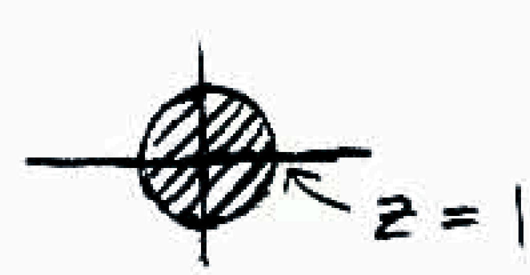
In this manner we can obtain twelve functions that are solutions of the hypergeometric equation and by using the property that $F(\alpha, \beta, \gamma, z) = F(\beta, \alpha, \gamma, z)$ we obtain twelve more functions. These are Kummer's twenty-four solutions; their complete derivation can be found in Volume 1 of the Bateman Manuscript Project.

Kummer's 24 Solutions of the Hypergeometric Equation

(with power series regions of convergence).

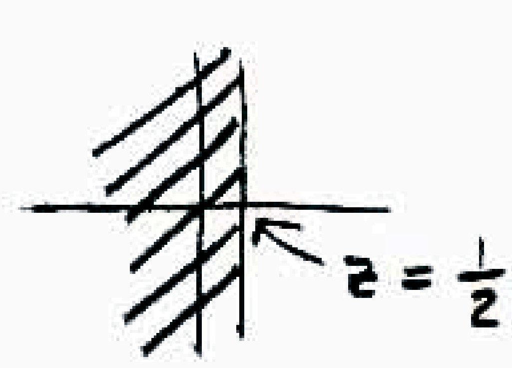
1) $u_1 = F(a, b, c, z)$

2) $(1-z)^{c-a-b} F(c-a, c-b, c, z)$



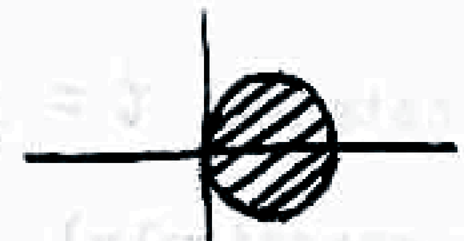
3) $(1-z)^{-a} F\left(a, c-b, c, \frac{z}{z-1}\right)$

4) $(1-z)^{-b} F\left(c-a, b, c, \frac{z}{z-1}\right)$



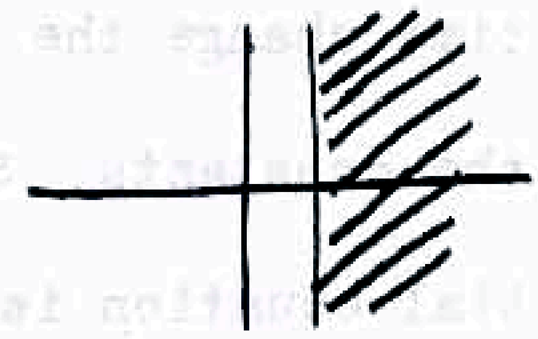
5) $u_2 = F(a, b, a+b+1-c, 1-z)$

6) $z^{1-c} F(a+1-c, b+1-c, a+b+1-c, 1-z)$



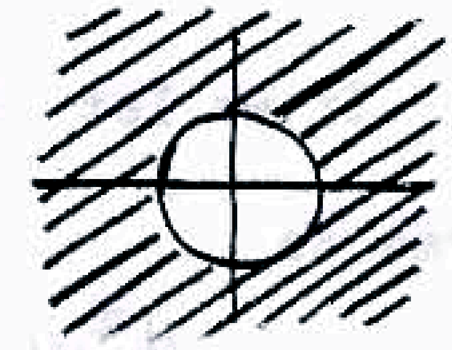
7) $z^{-a} F\left(a, a+1-c, a+b+1-c, \frac{z-1}{z}\right)$

8) $z^{-b} F\left(b+1-c, b, a+b+1-c, \frac{z-1}{z}\right)$



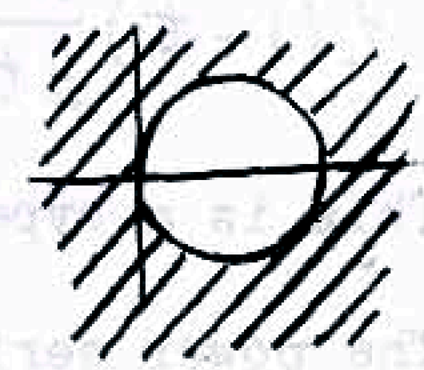
9) $u_3 = (-z)^{-a} F\left(a, a+1-c, a+1-b, \frac{1}{z}\right)$

10) $(-z)^{b-c}(1-z)^{c-a-b} F\left(1-b, c-b, a+1-b, \frac{1}{z}\right)$



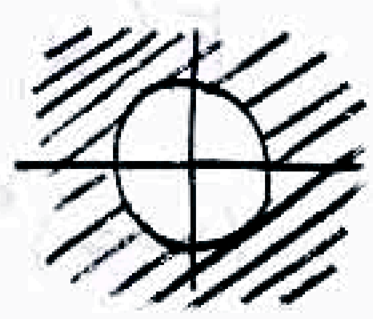
11) $(1-z)^{-a} F\left(a, c-b, a+1-b, \frac{1}{1-z}\right)$

12) $(-z)^{1-c}(1-z)^{c-a-1} F\left(a+1-c, 1-b, a+1-b, \frac{1}{1-z}\right)$



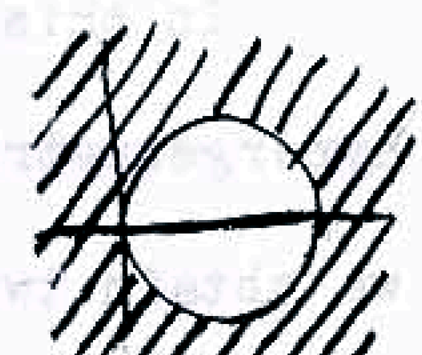
13) $u_4 = (-z)^{-b} F\left(b+1-c, b, b+1-a, \frac{1}{z}\right)$

14) $(-z)^{a-c}(1-z)^{c-a-b} F\left(1-a, c-a, b+1-a, \frac{1}{z}\right)$



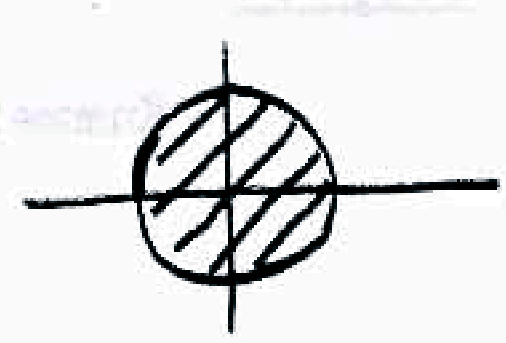
15) $(1-z)^{-b} F\left(b, c-a, b+1-a, \frac{1}{1-z}\right)$

16) $(-z)^{1-c}(1-z)^{c-b-1} F\left(b+1-c, 1-a, b+1-a, \frac{1}{1-z}\right)$



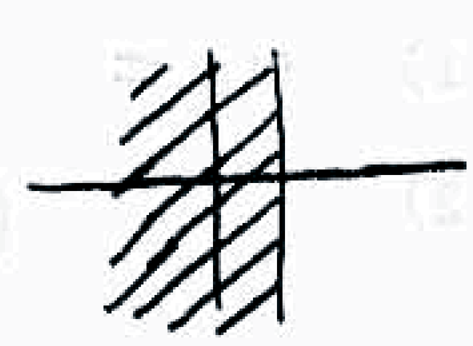
17) $u_5 = z^{1-c} F(a+1-c, b+1-c, 2-c, z)$

18) $z^{1-c}(1-z)^{c-a-b} F(1-a, 1-b, 2-c, z)$



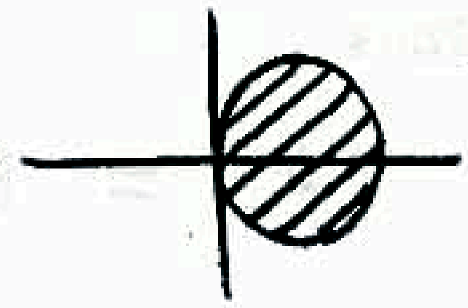
19) $z^{1-c}(1-z)^{c-a-1} F\left(a+1-c, 1-b, 2-c, \frac{z}{z-1}\right)$

20) $z^{1-c}(1-z)^{c-b-1} F\left(b+1-c, 1-a, 2-c, \frac{z}{z-1}\right)$



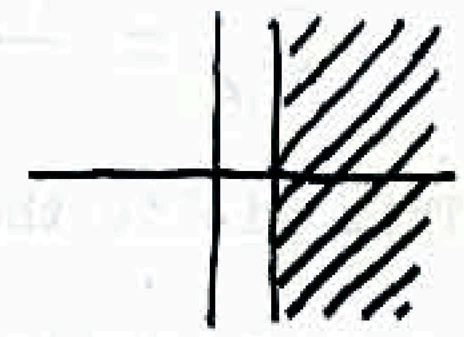
21) $u_6 = (1-z)^{c-a-b} F(c-a, c-b, c+1-a-b, 1-z)$

22) $z^{1-c}(1-z)^{c-a-b} F(1-a, 1-b, c+1-a-b, 1-z)$



23) $z^{a-c}(1-z)^{c-a-b} F\left(c-a, 1-a, c+1-a-b, \frac{z-1}{z}\right)$

24) $z^{b-c}(1-z)^{c-a-b} F\left(c-b, 1-b, c+1-a-b, \frac{z-1}{z}\right)$



References:

Bateman Manuscript Project: Vol. 1

Forsyth, Differential Equations.

Whittaker and Watson, Modern Analysis

Rainville, Intermediate Differential Equations.

D. Analytic Extension of the Solutions.

By comparing the 24 power series pairwise in their common region of convergence it is seen that there are only six different functions, $u_1, u_2, \ldots, u_6$. Since they are all solutions of the same differential equation there is a set of linear relationships connecting any three of them. If it is written in the form

$$u_i = A_{ij} u_j + B_{ik} u_k$$

then by considering the three corresponding power series that have a common region of convergence and working in this region, we can determine the values of the constants.

For example, $u_1 = A_{12} u_2 + B_{16} u_6$. Use power series 1), 5) and 21) within $|z| \leq 1$, $|z-1| \leq 1$. Thus

$$F(\alpha, \beta, \gamma, z) = A_{12} F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-z) + B_{16}(1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma+1-\alpha-\beta, 1-z),$$

Evaluating these functions for $z = 0$ and $z = 1$ we get two linear equations. From these equations we can express A_{12} and B_{16} in terms of gamma functions by using $F(\alpha, \beta, \gamma, 1) = \dfrac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\beta)\,\Gamma(\gamma-\alpha)}$.

Thus

$$A_{12} = F(\alpha, \beta, \gamma, 1)$$

$$B_{16} = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} .$$

This yields the complete (multiple valued) analytic extension of $F(\alpha, \beta, \gamma, z)$ into the entire plane.

Series 17) was the other basis element in $|z| < 1$ and its extension is

$$u_5 = \frac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} u_2 + \frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)} u_6 .$$

E. <u>The Monodromy Group for the Hypergeometric Equation.</u>

Denote the 24 solutions of Kummer by $y_1, y_2, \ldots, y_{24}$. Then in $|z| < 1$ we have y_1 and y_{17} (i.e. u_1, u_5) as a basis. Also in $|z-1| < 1$ we have y_5 and y_{21} (u_2, u_6) as a basis. On the overlap of these two regions the bases are related by:

$$\begin{pmatrix} y_1 \\ y_{17} \end{pmatrix} = \begin{pmatrix} \dfrac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)} & \dfrac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \\[2ex] \dfrac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} & \dfrac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)} \end{pmatrix} \begin{pmatrix} y_5 \\ y_{21} \end{pmatrix} = \Phi \begin{pmatrix} y_5 \\ y_{21} \end{pmatrix} .$$

If we do the analytic extension around $z = 0$ we get

$$\begin{pmatrix} y_1 \\ y_{17} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i \gamma} \end{pmatrix} \begin{pmatrix} y_1 \\ y_{17} \end{pmatrix} = \Psi_0 \begin{pmatrix} y_1 \\ y_{17} \end{pmatrix}$$

and around $z = 1$

$$\begin{pmatrix} y_5 \\ y_{21} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i 2\pi(\gamma-\alpha-\beta)} \end{pmatrix} \begin{pmatrix} y_5 \\ y_{21} \end{pmatrix} = \Psi_1 \begin{pmatrix} y_5 \\ y_{21} \end{pmatrix} .$$

Thus if we have a solution in $|z|<1$ we can get its analytic continuation around $z=1$ by the following:

$$\begin{pmatrix} y_1 \\ y_{17} \end{pmatrix} = \Phi \begin{pmatrix} y_5 \\ y_{21} \end{pmatrix} \to \Phi \Psi_1 \begin{pmatrix} y_5 \\ y_{21} \end{pmatrix} = \Phi \Psi_1 \Phi^{-1} \begin{pmatrix} y_1 \\ y_{17} \end{pmatrix}.$$

Therefore the monodromy group is generated by Ψ_0 and $\Phi \Psi_1 \Phi^{-1}$.

F. <u>An Alternate Basis near $z=0$.</u>

Consider the basis y_1, y_{17} in $|z|<1$. Do the analytic extension of y_1 around $z=1$. This yields $y_1 \to y_1' = ay_1 + by_{17}$. Now y_1 and y_1' are independent and can be used as a new basis in $|z|<1$. This basis has the interesting property that one of its elements is obtained from the other by analytic continuation. With this basis the monodromy group takes the following form:

<u>around $z=1$</u>

$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 \\ -p & 1+p \end{pmatrix} \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$$

<u>around $z=0$</u>

$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ q\cdot r & r \end{pmatrix} \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$$

where

$$p = e^{i2\pi(\gamma-\alpha-\beta)} , \quad q = 1 - e^{-i2\pi\gamma} ,$$

$$r = 1 + \frac{[1-e^{i2\pi(\gamma-\alpha-\beta)}]\sin\pi\alpha \sin\pi\beta}{\sin\pi(\alpha-\gamma)\sin\pi(\beta-\gamma) - \sin\pi\alpha \sin\pi\beta} .$$

G. <u>Uniqueness of the Riemann-Papperitz Equation.</u>

A necessary and sufficient condition for

$$w'' + \left\{ \sum_{j=1}^{N-1} \frac{1-\rho_j-\rho_j'}{z-a_j} \right\} w' + \left\{ \sum_{j=1}^{N-1} \frac{\rho_j \rho_j'}{(z-a_j)^2} + \sum_{j=1}^{N-1} \frac{c_j}{z-a_j} \right\} w = 0$$

to be a Fuchsian differential equation with singularities at $a_1, a_2, \ldots, a_{N-1}, \infty$ and with exponents $\rho_1, \rho_1', \rho_2, \rho_2', \ldots, \rho_{N-1}, \rho_{N-1}', \rho_\infty, \rho_\infty'$ is that $\sum_{j=1}^{N-1} (\rho_j + \rho_j') + \rho_\infty + \rho_\infty' = N-2$ and $\sum_{j=1}^{N-1} c_j = 0$.

If $N=3$ it will be noted that this is the Riemann-Papperitz equation with one singularity at infinity. Also for $N=3$ the conditions become

$$\rho_1 + \rho_1' + \rho_2 + \rho_2' + \rho_\infty + \rho_\infty' = 1$$

and

$$c_1 + c_2 = 0.$$

By computing the indicial equation at $z = \infty$ we find that

$$\rho_\infty \rho_\infty' = \rho_1 \rho_1' + \rho_2 \rho_2' + c_1 a_1 + c_2 a_2 .$$

The last two equations can be used to determine the values of c_1 and c_2

$$c_1 = \frac{\rho_1 \rho_1' + \rho_2 \rho_2' - \rho_\infty \rho_\infty'}{a_2 - a_1} = -c_2 .$$

Thus, given six arbitrary numbers, subject to the condition that their sum is unity, there is a unique Riemann-Papperitz equation (with $z = \infty$ as one of its singular points) having these six numbers as its exponents.

Since this observation is stated for the sphere, the Riemann-Papperitz equation with three arbitrary singularities can be obtained from the equation having a singularity at $z = \infty$ by a linear fractional transformation. Therefore the general equation is also determined uniquely by its exponents.

Recall that in the general Fuchsian theory it is possible to have an analytic basis for the solutions at one of the indicated singularities. Likewise, in the above observation the singularities may be regular points. For example, if $\rho_1 = 0$, $\rho_1' = 1$ then $z = a_1$ is a singularity in name only.

H. <u>Relation between the Riemann-Papperitz Equation and the Hypergeometric Equation.</u>

Let the symbol $P\begin{pmatrix} a & b & c & \\ \rho_1 & \rho_2 & \rho_3 & z \\ \rho_1' & \rho_2' & \rho_3' & \end{pmatrix}$ denote the Riemann-Papperitz equation with singularities at $z = a, b, c$ and with exponents $\rho_1, \rho_1', \ldots$ at these points (this is called the Riemann-Papperitz symbol or R-P symbol). The R-P symbol has two very useful properties.

i) if $z = \dfrac{At+B}{ct+D}$ transforms $z = a \to t = a_1$, $z = b \to t = b_1$, $z = c \to t = c_1$ then

$$P\begin{pmatrix} a & b & c & \\ \rho_1 & \rho_2 & \rho_3 & z \\ \rho_1' & \rho_2' & \rho_3' & \end{pmatrix} = P\begin{pmatrix} a_1 & b_1 & c_1 & \\ \rho_1 & \rho_2 & \rho_3 & t \\ \rho_1' & \rho_2' & \rho_3' & \end{pmatrix}.$$

(See Whittaker and Watson, 4th edition, p. 207).

ii) $$P\begin{pmatrix} a & b & c & \\ \rho_1 & \rho_2 & \rho_3 & z \\ \rho_1' & \rho_2' & \rho_3' & \end{pmatrix} = \left(\frac{z-a}{z-b}\right)^{K} P\begin{pmatrix} a & b & c & \\ \rho_1 - K & \rho_2 + K & \rho_3 & z \\ \rho_1' - K & \rho_2' + K & \rho_3' & \end{pmatrix}$$

i.e. if $w(z)$ is any solution of the equation represented by the R-P symbol on the left and if $w(z) = \left(\frac{z-a}{z-b}\right)^{K} w_1(z)$ then $w_1(z)$ satisfies the equation represented by the R-P symbol on the right. This can be shown by expanding the solutions in power series about $z = a$.

By property ii) we have:

$$(*)\quad P\begin{pmatrix} a & b & c & \\ \rho_1 & \rho_2 & \rho_3 & z \\ \rho_1' & \rho_2' & \rho_3' & \end{pmatrix} = \frac{(z-a)^{\rho_1}(z-b)^{\rho_2}}{(z-c)^{\rho_1+\rho_2}} P\begin{pmatrix} a & b & c & \\ 0 & 0 & \rho_3+\rho_2+\rho_1 & z \\ \rho_1'-\rho_1 & \rho_2'-\rho_2 & \rho_3'+\rho_2+\rho_1 & \end{pmatrix}.$$

Now let $\alpha = \rho_3+\rho_2+\rho_1$, $\beta = \rho_3'+\rho_2+\rho_1$, $1-\gamma = \rho_1'-\rho_1$ and make the transformation $t = \dfrac{(z-a)(b-c)}{(z-c)(b-a)}$, then the right side of (*) becomes

$$(**)\quad \left[\frac{(b-a)}{(b-c)}t\right]^{\rho_1}\left[\frac{(a-b)(1-t)}{(a-c)}\right]^{\rho_2} P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & t \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{pmatrix}.$$

The R-P symbol in (**) represents the hypergeometric equation. Thus we can express the solution of the Riemann-Papperitz equation in terms of $F(\alpha, \beta, \gamma, t)$. Also, it is now possible to write down the monodromy group for the general equation. Let $\binom{X}{Y}$ be a basis, near $\underline{a}$, for the solutions of the general Riemann-Papperitz equation. The monodromy group for the R-P symbol in (**) is generated by Ψ_0 and $\Phi\Psi_1\Phi^{-1}$, based at $t=0$ and $t=1$ respectively. The R-P symbol on the right side of (*) has this same monodromy group except that the generating elements refer to loops around $z=a$ and $z=b$. Thus in going around $z=a$,

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{(z-a)^{\rho_1}(z-b)^{\rho_2}}{(z-c)^{\rho_1+\rho_2}}\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow e^{i2\pi\rho_1}\frac{(z-a)^{\rho_1}(z-b)^{\rho_2}}{(z-c)^{\rho_1+\rho_2}}\cdot\Psi_0\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= e^{i2\pi\rho_1}\Psi_0\begin{pmatrix} X \\ Y \end{pmatrix}$$

and around $z=b$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow e^{i2\pi\rho_2}\,\Phi\Psi_1\Phi\begin{pmatrix} X \\ Y \end{pmatrix}.$$

12. Hypergeometric Equations With a Finite Monodromy Group.

Consider the hypergeometric differential equation

$$w'' + \frac{-\gamma+(1+\alpha+\beta)z}{z(z-1)}w' + \frac{\alpha\beta}{z(z-1)}w = 0$$

with the Riemann symbol

$$P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{pmatrix}.$$

We seek all α, β, γ or equally well all $\lambda = 1-\gamma$, $\mu = \gamma-\alpha-\beta$, $\nu = \alpha-\beta$ for which all solutions are algebraic functions. This occurs if and only if the monodromy group is finite.

Consider two independent solutions $w_1(z)$ and $w_2(z)$ and form the multi-valued Schwarz function

$$s(z) = w_1(z)/w_2(z).$$

Upon analytic continuation around a singularity $\mathcal{C}$

$$s(z) = \frac{a\,w_1(z)+b\,w_2(z)}{c\,w_1(z)+d\,w_2(z)}.$$

This defines a homomorphism of the monodromy group Ψ into the group of conformal automorphisms of the Riemann sphere:

$$\Psi(\mathcal{E}) \to PGL(1, \mathbb{C}).$$

Moreover, from the general theory of the hypergeometric equation, the transformation

$$w_1(z) \to a w_1(z)$$
$$w_2(z) \to a w_2(z)$$

corresponds to both exponents being the same at one singularity; and this is impossible for then logarithmic terms occur in the general solution. Thus the homomorphism of Ψ into the group of conformal automorphisms is an isomorphism.

Therefore the problem of finding all finite monodromy groups is reduced to finding all finite groups of conformal automorphisms of the Riemann sphere.

Let G be a finite group of conformal automorphisms of the sphere S^2. If an element $g \in G$ had just one fixed point in S^2, then g would be conjugate in $PGL(1, \mathbb{C})$ to a translation of the complex plane $S^2 - \infty$. But this is impossible for then g generates an infinite cyclic subgroup of G. Thus g has two fixed points in S^2 and is conjugate to a rigid rotation of S^2. A careful study of $PGL(1, \mathbb{C})$ shows that G is a group of isometries of the metric sphere S^2.

Thus the only finite groups which can occur as the monodromy group of a hypergeometric equation are

1. the identity group
2. the cyclic group, generated by a rotation through $2\pi/n$
3. the regular bipyramid group
4. tetrahedral group
5. octahedral group
6. icosahedral group.

Each of 3, 4, 5, and 6 can occur as the monodromy group of a hypergeometric equation with three singular points, cf. Poole and Bieberbach.

Schwarz described these groups by considering the map of the upper half z-plane onto a fundamental triangle bounded by circular arcs in the S-sphere. The interior angles in this triangle are $\pi\lambda, \pi\mu, \pi\nu$. Then $\lambda+\mu+\nu > 1$. Moreover the reflections of the fundamental triangle, over its circular boundary edges, and such successive reflections, must cover the S-sphere just once.

Using the finiteness of the monodromy group Ψ, we see that λ, μ, ν must be rational. Also a preliminary normalization enables us to consider only hypergeoemtric equations for which $0 < \lambda, \mu, \nu < 1$. Schwarz ennumerated fifteen solutions for λ, μ, ν which yield all possible monodromy groups (some several times).

13. <u>Existence and Uniqueness Theorems for Fuchsian Differential Equations With a Prescribed Monodromy Group.</u>

<u>Theorem 20.</u> Let

$$\mathcal{S}_1)\ \frac{dw^i}{dz} = A^i_{1j}(z)w^j, \qquad \mathcal{S}_2)\ \frac{dw^i}{dz} = A^i_{2j}(z)w^j$$

be meromorphic differential systems on a compact Riemann surface M, with the same singular points which are all of the first kind. Assume that at each singularity $\mathcal{S}_1)$ and $\mathcal{S}_2)$ have the same exponents, no two of which differ by an integer. Assume that $\mathcal{S}_1)$ and $\mathcal{S}_2)$ have the same monodromy group in $GL(n, \mathbb{C})$, relative to a solution basis which reduces to the identity I at $P \in M$. Then $\mathcal{S}_1)$ is the same differential system as $\mathcal{S}_2)$.

<u>Proof.</u>

Let $W_1(z)$ and $W_2(z)$ be the solution matrices of $\mathcal{S}_1)$ and $\mathcal{S}_2)$.

respectively, which reduce to I at the base point $P \in M$. Consider $w_1(z)\, w_2(z)^{-1}$ as continued analytically along all curves in $M - \Sigma Q_\alpha$ which initiate at P.

Around a closed loop in $M - \Sigma Q_\alpha$ we find $w_1(z) \to w_1(z) C_1$ and $w_2(z) \to w_2(z) C_2$. But the monodromy is the same for $\mathcal{S}_1)$ and $\mathcal{S}_2)$ around this loop and hence $C_1 = C_2$.

Thus $w_1(z) C_1 C_1^{-1} w_2(z)^{-1} = w_1(z)\, w_2(z)^{-1}$

and hence $w_1(z)\, w_2(z)^{-1}$ is single-valued and holomorphic on $M - \Sigma Q_\alpha$.

Next we examine $w_1(z)\, w_2(z)^{-1}$ in a neighborhood of a singular point Q. Here

$$w_1(z) = (I + z P_1^{(1)} + z^2 P_2^{(1)} + \cdots)\, z^{R_1} K_1$$

$$w_2(z) = (I + z P_1^{(2)} + z^2 P_2^{(2)} + \cdots)\, z^{R_2} K_2$$

where R_1 and R_2 are each similar to

$$\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \Lambda$$

Say $S_1 R_1 S_1^{-1} = \Lambda$

$$w_1(z) = (I + z P_1^{(1)} + \cdots)\, S_1^{-1} S_1 z^{R_1} S_1^{-1} S_1 K_1$$

or

$$w_1(z) = (I + z P_1^{(1)} + \cdots)\, S_1^{-1} z^{\Lambda} \hat{K}_1$$

where $z^\Lambda = \operatorname{diag}(z^{\lambda_1}, z^{\lambda_2}, \ldots, z^{\lambda_n})$.

Also

$$w_2(z) = (I + z P_1^{(2)} + \cdots)\, S_2^{-1} z^{\Lambda} \hat{K}_2 .$$

Thus

$$w_1(z)\, w_2(z)^{-1} = (I + z P_1^{(1)} + \cdots) S_1^{-1} z^{\Lambda} \hat{K}_1 \hat{K}_2^{-1} z^{-\Lambda} S_2 (I - z P_1^{(2)} + \cdots).$$

Now we show $\hat{K}_1 \hat{K}_2^{-1}$ commutes with z^Λ. Then $w_1(z)\, w_2(z)^{-1}$ is holomorphic at Q.

Then $w_1(z)\, w_2^{-1}(z) = I$

so $w_1(z) = w_2(z)$ and $\mathcal{S}_1)$ is $\mathcal{S}_2)$.
Now show $\hat{K}_1 \hat{K}_2^{-1}$ commutes with z^Λ.

At the start of a small loop around Q we have

$$w_1(z_0) = w_2(z_0) C_Q .$$

After encircling we must have

$$w_1(z) \to (I + zP_1^{(1)} + \cdots) S_1^{-1} z^\Lambda e^{2\pi i\Lambda} \hat{K}_1$$

$$w_2(z) \to (I + zP_1^{(2)} + \cdots) S_2^{-1} z^\Lambda e^{2\pi i\Lambda} \hat{K}_2 .$$

Around Q we have the monodromy matrix

$\hat{K}_1^{-1} e^{2\pi i\Lambda} \hat{K}_1$ for $\mathcal{S}_1)$ and

$\hat{K}_2^{-1} e^{2\pi i\Lambda} \hat{K}_2$ for $\mathcal{S}_2)$.

But these are the same so

$$\hat{K}_1^{-1} e^{2\pi i\Lambda} \hat{K}_1 = \hat{K}_2^{-1} e^{2\pi i\Lambda} \hat{K}_2 .$$

Thus

$$(\hat{K}_1 \hat{K}_2^{-1})^{-1} e^{2\pi i\Lambda} \hat{K}_1 \hat{K}_2^{-1} = e^{2\pi i\Lambda} .$$

Now $e^{2\pi i\Lambda}$ is diagonal, with distinct diagonal elements, and hence $\hat{K}_2 \hat{K}_1^{-1}$ is diagonal, and so commutes with z^Λ.

Q. E. D.

Let M be a Riemann surface and D a closed discrete (possibly empty) set of points in M. Consider a base point $P \in M - D$ and a homomorphism

$$\Psi : \pi_1(M-D) \to GL(n, \mathbb{C}).$$

We shall construct a differential system $\frac{dw}{dz} = A(z)\, w$, with $A(z)$ a $n \times n$ holomorphic matrix (differential) on $M - D$, (or if D is empty and M compact $A(z)$ is meromorphic) having the prescribed singularities (possibly essential) at points of D and the prescribed monodromy, in the sense that the fundamental solution matrix which reduces to I at P yields the given representation of the monodromy group in $GL(n, \mathbb{C})$.

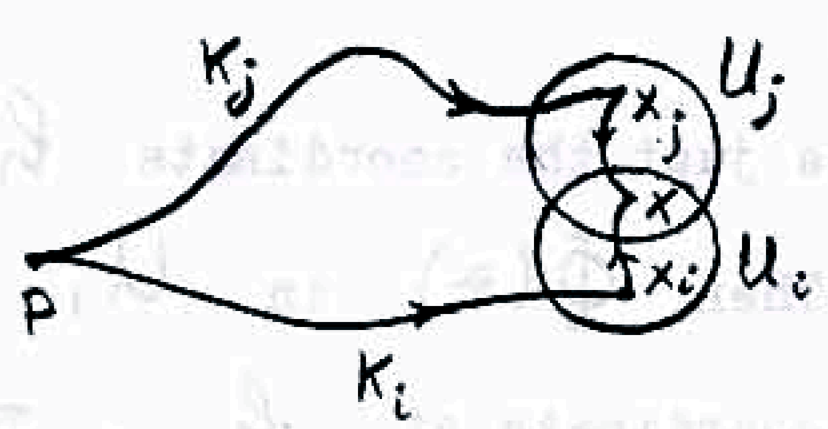
<u>Theorem 21.</u> Let M be a Riemann surface and D a discrete, (closed) (possibly empty) point set such that $M - D$ is not compact. Then there exists a differential system, holomorphic on $M - D$, and with singularities only at D, with a prescribed monodromy group,

$$\Psi : \pi_1(M-D) \to GL(n, \mathbb{C}).$$

<u>Proof.</u>

1. Construction of principal $GL(n, \mathbb{C})$ bundle over $M - D$.

Choose a covering of $M - D$ by local coordinate discs $\{U_\alpha\}$ with simply-connected intersections. On the overlap of U_i and U_j we define the transition functions $g_{ij} \in GL(n, \mathbb{C})$ as follows.



Select a path K_i in $M - D$ from P to a reference point $x_i \in U_i$ and a similar path K_j from P to $x_j \in U_j$.

For a point $x \in U_i \cap U_j$ choose a path ℓ_i from x_i to x in U_i, and a path ℓ_j from x_j to x in U_j. Define

$$g_{ij}(x) = \Psi[K_i \ell_i \ell_j^{-1} K_j^{-1}] \in GL(n, \mathbb{C})$$

Note that $g_{ij}(x)$ is a constant matrix on $U_i \cap U_j$ and is independent of auxiliary paths ℓ_i and ℓ_j. Over U_i and U_j we have the local products $U_i \times GL(n, \mathbb{C})$ and $U_j \times GL(n, \mathbb{C})$ with the identification $(x, g_{ij}M)_i \longleftrightarrow (x, M)_j$. This defines the required holomorphic fiber bundle $\mathcal{P}$ with base $M-D$ and fiber and group $GL(n, \mathbb{C})$.

2. A holomorphic cross-section of $h: M-D \to \mathcal{P}$ enables us to construct a fundamental solution matrix (a holomorphic function from the universal covering space of $M-D$ to $GL(n,\mathbb{C})$), which defines the required differential system.

For consider such a cross-section h. Consider the matrix function $\Phi(z)$ which is prescribed as the coordinate of h in a chosen local coordinate system U_0 around P. We can assume that the cross-section h has been multiplied by a constant matrix on the right so that $\Phi(P)=I$. Now continue $\Phi(z)$ analytically along paths in $M-D$.

We next show that $\Phi(z)$ has the required monodromy. Let $\mathcal{C}(t)$, $0\le t\le 1$, be a closed path in $M-D$, based at P. There are a finite number of the coordinate systems $\{U_\alpha\}$ which cover the compact set $\mathcal{C}$. Let $0<t_1<t_2<\cdots<t_k=1$ yield points on $\mathcal{C}$ in the coordinate patches $U_0, U_1, U_2, \ldots, U_k = U_0$ where each U_α intersects its neighbors in the ordered string.

Now in U_0 around P, $\Phi(z)$ is just the coordinate h_0 (in the local product coordinates) of h. Then $\Phi(z)$ in U_1 is just $\Phi(z) = g_{01} h_1$, where h_1 is the coordinate of h. Then $\Phi(z)$ continued around $\mathcal{C}$ is just $\Phi(z) = g_{01} g_{12} \cdots g_{k-1,k} h_0$ in the system U_0. Thus the monodromy matrix for $\mathcal{C}$ is just

$$\mathcal{J} = g_{01} g_{12} \cdots g_{k-1,k} .$$

But

$$g_{01} = \Psi(K_0 \ell_0 \ell_1^{-1} K_1^{-1}), \qquad g_{12} = \Psi(K_1 \hat{\ell}_1 \ell_2^{-1} K_2^{-1}),\ etc.$$

Thus $\mathcal{J} = \Psi(\mathcal{C})$ since the composite path

$$K_0 \ell_0 \ell_1^{-1} K_1^{-1} K_1 \hat{\ell}_1 \ell_2^{-1} K_2^{-1} K_2 \hat{\ell}_2 \ell_3^{-1} K_3^{-1} \cdots K_{k-1} \hat{\ell}_{k-1} \ell_k^{-1} K_k$$

is homotopic to $\mathcal{C}$ in $M-D$.

3. We now prove the existence of a holomorphic section of $M-D$ into $\mathcal{P}$.

H. Grauert has proved: a holomorphic fiber bundle over a holomorphically complete base space is holomorphically trivial (a product bundle) if and only if it is topologically trivial. A non-compact Riemann surface $M-D$ is holomorphically complete and, in this case, a simple proof of this proposition has been given by H. Röhrl. Thus we must only show that $\mathcal{P}$ is a topological product of $M-D$ and $GL(n, \mathbb{C})$. Since $\mathcal{P}$ is a principal bundle we need only prove that there is a continuous cross-section of $M-D$ into $\mathcal{P}$. We do this by obstruction theory.

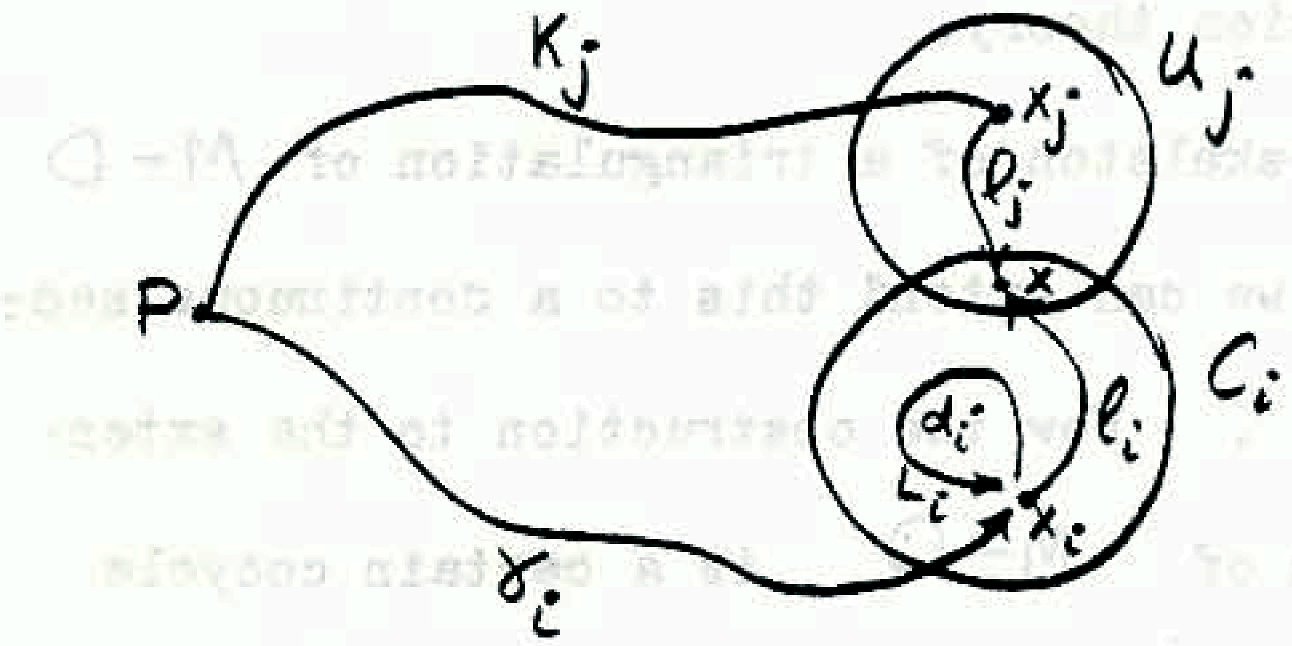
Consider a section from the 0-skeleton of a triangulation of $M-D$ into $\mathcal{P}$. Since $\mathcal{P}$ is connected we can extend this to a continuous section over the 1-skeleton of $M-D$. Now the obstruction to the extension of the section to the 2-skeleton of $M-D$ is a certain cocycle in $H^2(M-D)$, the (infinite cochain) cohomology group with coefficients in $\pi_1(GL(n, \mathbb{C}))$. But the second cohomology group of $M-D$ with integer coefficients is zero. By the universal coefficient theorem the obstruction is zero and the continuous cross-section of $M-D$ into $\mathcal{P}$ exists. Q. E. D.

Theorem 22. Let M be a non-compact Riemann surface and D a (closed), discrete (possibly empty) set of points of M. Let $\Psi: \pi_1(M-D) \to GL(n, \mathbb{C})$ be a prescribed homomorphism for the base point $P \in M-D$. Then there exists a Fuchsian differential system $\frac{dw}{dz} = A(z)w$ on M, with the prescribed singularities at D and the prescribed monodromy group.

Proof.

1. Construct the principal holomorphic fiber bundle $\mathcal{P}$ over $M-D$ just as in the previous theorem. We now extend $\mathcal{P}$ to $\hat{\mathcal{P}}$, a holomorphic principle fiber bundle over M. Use a covering of $M-D$

$\{U_\alpha\}$ as above and in addition a collection of coordinate discs $\{C_\alpha\}$ centered at the points of D and such that each intersection of members of the covering is simply connected. For each $U_i \cap U_j$ we define $g_{ij} \in GL(n, \mathbb{C})$ using chosen paths from P in $M - D$ as above. Require that $C_i \cap C_j$ is empty for each pair of distinct points d_i and d_j in D. Now define the transition functions for each $U_j \cap C_i$ as follows.



Choose a path K_j from P to a reference point $x_j \in U_j$ in $M - D$ and a path γ_i from P to a reference point $x_i \in C_i$ in $M - D$. Also choose ℓ_i from x_i to x in C_i and ℓ_j from x_j to x in U_j.

Further choose a closed path L_i, based at x_i, lying in $C_i - d_i$ and generating the fundamental group of $C_i - d_i$.

Call t the local coordinate in C_i with $t = 0$ at d_i. Then, choose L_i so that,

$$L_i(\log t)_{x_i} - (\log t)_{x_i} = 2\pi\sqrt{-1}$$

for a chosen branch of $\log t$ at x_i. Define the matrix function

$$f_i(x) = \exp\left\{-\frac{1}{2\pi\sqrt{-1}} \log \Psi[\gamma_i L_i \gamma_i^{-1}] \log t(x)\right\},$$

and note that f_i is multiple-valued in C_i. Define, for

$$x \in U_j \cap C_i \quad , \quad g_{ij}(x) = f_i(x_i)\Psi[\gamma_i \ell_i \ell_j^{-1} K_j^{-1}]$$

where ℓ_i is used to compute $f_i(x_i)$.

Observe that $g_{ij}(x) \in GL(n, \mathbb{C})$ is independent of the choice of ℓ_j and of ℓ_i, since we can only replace ℓ_i by a curve homotopic in $C_i - d_i$ to $L_i^n \ell_i$, for an integer n.

Thus the holomorphic principle bundle $\hat{\wp}$ has been constructed over the base space M.

2. A holomorphic cross-section (which exists as above)

$$h: M \to \hat{\wp}$$

yields a holomorphic non-singular fundamental solution matrix $\Phi(z)$ on $M-D$, just as above. Also $\Phi(z)$ displays the correct monodromy.

3. We now show that $\Phi(z)$ has a Fuchsian (regular singularity at most) singularity at each point of D. Now in C_i we have $\Phi(z)$ is the coordinate of the section h_i multiplied by some constant matrices and by some $f_i(x_i)$. Thus $\Phi(z)$ is a holomorphic non-singular matrix h_i multiplied by $t^A B$ where B is a non-singular constant matrix and $A = -\frac{1}{2\pi\sqrt{-1}} \log[\gamma_i L_i \gamma_i^{-1}]$. Q. E. D.

<u>Remark.</u> If M is non-compact and D is non-empty, the exponents of the Fuchsian equation $\Phi(z)$ can be prescribed arbitrarily at the points of D.

For it is only necessary to define the matrix function $f_i(x_i)$, and the corresponding transition function $g_{ij}(x)$, appropriately in the theorem. However, even prescribing the exponents and the monodromy group of a Fuchsian differential system does not enforce uniqueness when M is non-compact, since there always exists non-constant holomorphic functions on M.

<u>Theorem 23.</u> Let M be a compact Riemann surface and D a discrete closed (possibly empty) set of points of M. Let

$$\Psi: \pi_1(M-D) \to GL(n, \mathbb{C})$$

be a homomorphism, for the base point $P \in M-D$. Then there exists a Fuchsian differential system $\frac{dw}{dz} = A(z)w$ on M, with singularities only at D (and at one additional point if D is empty), and with the prescribed monodromy group.

Proof.

Consider the holomorphic principal fiber bundle over M, just as in the above theorem. Select a point $Q \in M$ (take $Q \in D$ if D is non-empty), and then a holomorphic cross-section $\ell : M - Q \to \hat{\wp}$ exists, just as above, and yields a differential system with the required Fuchsian singularities at $M - Q - D$. Moreover this has the required monodromy if there exists a local coordinate system near Q in which the fundamental solution matrix is single-valued and meromorphic. H. Röhrl has proved that there exists such a holomorphic cross-section of $M - Q$ into $\hat{\wp}$, which is the restriction of a meromorphic cross-section of M into a vector bundle associated with $\hat{\wp}$ having fiber $\mathbb{C}^{n^2}$.

Problems

Consider the torus T^2 as the Riemann surface obtained by identifying points in the complex z-plane $\mathbb{C}$ under the translation group generated by $z \to z+1$ and $z \to z+i$. A meromorphic function on T^2 corresponds to a doubly periodic meromorphic function (an elliptic function) on $\mathbb{C}$ with periods 1 and i. Take the fundamental domain $D: \left\{ z = x+iy \,\middle|\, \begin{array}{l} -\frac{1}{2} \le x < \frac{1}{2} \\ -\frac{1}{2} \le y < \frac{1}{2} \end{array} \right\}$ and the local coordinates on T^2 which are the projections of $z_1 = z$, $z_2 = z + i/2$, $z_3 = z - i/2$, $z_4 = z + \frac{1}{2}$, $z_5 = z - \frac{1}{2}$, $z_6 = z + \frac{1}{2} + \frac{i}{2}$, $z_7 = z + \frac{1}{2} - \frac{i}{2}$, $z_8 = z - \frac{1}{2} + \frac{i}{2}$, $z_9 = z - \frac{1}{2} - \frac{i}{2}$ where z ranges over the interior of D. Using these local coordinates on T^2, a differential equation with meromorphic coefficients on T^2,

$\mathcal{D}$) $\quad w^{(n)} + a_{n-1}(z)w^{(n-1)} + \dots + a_0(z)w = 0$

corresponds to a differential equation on $\mathbb{C}$ with elliptic functions as coefficients.

1. Find all Fuchsian (linear homogeneous) differential equations on T^2 which have no singularities, that is, the coefficients are holomorphic everywhere.

2. Show that the Lamé equation

$$\frac{d^2w}{dz^2} - [a\wp(z) + b]w = 0, \quad \text{complex } a \neq 0,\ b,$$

where $\wp(z) = \frac{1}{z^2} + \sum_{K,K'}{}' \left[\frac{1}{(z-K-K'i)^2} - \frac{1}{(K+K'i)^2}\right]$ is the Weierstrass elliptic function, is Fuchsian with one singularity P on T^2. Prove that no solution of the Lamé equation has a branch point at P if and only if $a = n(n+1)$ for an integer $n = 1, 2, 3, \dots$. (Hint: $\wp(z) = \wp(-z)$ and so if $w_1(z)$ is a solution, so are $\frac{w_1(z)+w_1(-z)}{2}$ and $\frac{w_1(z)-w_1(-z)}{2}$.)

3. Prove that the Lamé equation $w'' - [n(n+1)\wp(z) + b]w = 0$, $n = 1, 2, 3, \dots$, has a commutative monodromy group but that this group cannot consist of just the one element I. (It is known that the fundamental group of $T^2 - P$ is generated by the three loops $z \to z+1$, $z \to z+i$, and the loop encircling P .)

BIBLIOGRAPHY

Several Complex Variables

1. S. Bochner and W. T. Martin, Several Complex Variables, Princeton 1948.
2. W. Osgood, Lehrbuch der Funktionentheorie II, Berlin 1924.
3. G. Springer, Riemann Surfaces, Boston 1958.

Elementary Texts

4. E. Poole, Theory of Linear Differential Equations, Oxford 1936.
5. E. Rainville, Intermediate Differential Equations, New York 1943.

Advanced Texts.

6. L. Bieberbach, Theorie der gewöhnlichen Differentialgleichungen, auf functionentheoretischer Grundlage dargestellt, Berlin 1953.
7. E. Coddington and N. Levinson, Theory of Differential Equations, New York 1955.
8. L. Fuchs, Gesammelte Werke I, pp. 159, 205.
9. V. Golubev, Lectures on the Analytic Theory of Differential Equations, Moscow 1950.
10. E. Ince, Ordinary Differential Equations, New York (Dover)
11. F. Klein, Über lineare Differentialgleichungen der zweiten Ordnung, Leipzig 1906.
12. ______, Über die hypergeometrische Funktion, Gottingen 1894.
13. L. Schlesinger, Vorlesungen uber lineare Differentialgleichungen, Berlin 1908.
14. ____________, Einführung in die Theorie der gewohnlichen Differentialgleichungen, Berlin 1922.
15. ____________, Handbuch der Theorie der linearen Differentialgleichungen, Berlin 1895 and 1898.

Articles

16. G. D. Birkhoff, Proc. Am. Acad. 49 (1913) p. 521.

17. H. Grauert, Math. Ann. 1958.

18. D. Hilbert, Gott. Nach. (1905) p. 307.

19. J. Plemelj, Monatsh. Math. Phys. 19 (1908) p. 211.

20. B. Riemann, Gesammelte Werke (2nd edition) p. 379.

21. H. Röhrl, Math. Ann. 1957.