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Math 5467. Midterm Exam II

April 15, 2002

(There are a total of 100 points on this exam)

Problem 1 *(15 points) The half band filter*

$$P(\omega) = 1 + \sum_{n \text{ odd}} \mathbf{p}(n) e^{-in\omega}$$

satisfies $P(\pi) = 0$. *From these two facts alone, deduce that* $P(0) = 2$.

ANSWER: $P(0) = 1 + \sum_{n \text{ odd}} \mathbf{p}(n)$. *Now* $P(\pi) = 1 + \sum_{n \text{ odd}} \mathbf{p}(n)(-1)^n = 1 - \sum_{n \text{ odd}} \mathbf{p}(n)$, *so* $\sum_{n \text{ odd}} \mathbf{p}(n) = 1$ *and* $P(0) = 1 + 1 = 2$.

Problem 2 *Let* V_0 *be the space of all square integrable signals* $f(t)$ *that are peicewise linear and continuous with possible corners occuring at only the integers* $0, \pm 1, \pm 2, \cdots$. *Similarly, for every integer* j *let* V_j *be the space of all square integrable signals* $f(t)$ *that are peicewise linear and continuous with possible corners occuring at only the dyadic points* $k/2^j$ *for* k *an integer. Let* $\phi(t)$ *be the hat function*

$$\phi(t) = \begin{cases} t+1, & -1 \le t \le 0, \\ 1-t, & 0 < t \le 1, \\ 0, & |t| > 1. \end{cases}$$

In class we saw that $f(t) = \sum_k f(k)\phi(t-k)$ *for every* $f \in V_0$, *so the hat function and its integer translates form a (nonorthogonal) basis for* V_0. *This defines the linear spline multiresolution analysis, though it is not orthonormal.*

- *2a (5 points) Verify that the spaces V_j are nested, i.e., $V_j \subset V_{j+1}$.*

  ANSWER: $f \in V_j \longleftrightarrow f$ square int., piecewise linear and continuous with breaks at points $k/2^j$. $g \in V_{j+1} \longleftrightarrow g$ square int., piecewise linear and continuous with breaks at points $\ell/2^{j+1}$. But then $f \in V_j$ satisfies the conditions for V_{j+1} where $k/2^j = 2k/2^{j+1} = \ell/2^{j+1}$, ℓ even, and for ℓ odd there is in fact no break (but that is OK).

  However, if $g \in V_{j+1}$ and there is a break for some ℓ odd, so that $g'(\ell/2^{j+1} - 0) \neq g'(\ell/2^{j+1} + 0)$ then $g \notin V_j$.

- *2b (5 points) Verify that $f(t) \in V_j \Longleftrightarrow f(2^{-j}t) \in V_0$.*

  ANSWER: $f(t) \in V_j \longleftrightarrow f$ square int., piecewise linear and continuous with breaks at points $k/2^j \longleftrightarrow f(2^{-j}$ square int., piecewise linear and continuous with breaks at integer points $k \Longleftrightarrow f(2^{-j}t) \in V_0$.

- *2c (10 points) What is the scaling function basis for V_1?*

  ANSWER: $\phi_{1k}(t) = \phi(2t - k)$ where k runs over the integers. (There is no particular need to normalize the basis, since it isn't ON.) In other words, $\phi_{1k}(t)$ is the continuous piecewise linear function that is zero at all half-integer points, except that $\phi_{1k}(k/2) = 1$. For a proof, note that $f(t) = \sum_k f(k)\phi(t-k)$ for any $f(t) \in V_0$. But $g(t) \in V_1 \longleftrightarrow h(t) = g(t/2) \in V_0$. Hence If $g(t) \in V_1$ we can expand $h(t)$ uniquely as

  $$h(t) = g(t/2) = \sum_k h(k)\phi(t-k).$$

  Hence, replacing t by $2t$, $g(t) = \sum_k g(k/2)\phi(2t-k)$.

- *2d (10 points) Compute the coefficients $\mathbf{c}(k)$ in the dilation equation $\phi(t) = \sqrt{2}\sum_k \mathbf{c}(k)\phi(2t-k)$, where $\phi(t)$ is the hat function.*

  ANSWER: Set $g(t) = \phi(t)$ in part 2b. We have $\phi(k/2) = 0$ except that $\phi(-1/2) = \phi(1/2) = 1/2$ and $\phi(2/2) = 1$. Thus

  $$\phi(t) = \frac{1}{2}\phi(2t+1) + \phi(2t) + \frac{1}{2}\phi(2t-1),$$

  and $\mathbf{c}(1) = \mathbf{c}(-1) = 1/2\sqrt{2}$, $\mathbf{c}(0) = 1/\sqrt{2}$.

Problem 3 *Dubechies D_4 is the 4-tap filter with z-transform $C(z)$ such that*

$$C(z) = \frac{1}{4\sqrt{2}}\left((1+\sqrt{3}) + (3+\sqrt{3})z^{-1} + (3-\sqrt{3})z^{-2} + (1-\sqrt{3})z^{-3}\right).$$

It satisfies the properties $C(1) = \sqrt{2}$, $C(-1) = 0$ (low pass) and $|C(z)|^2 + |C(-z)|^2 = 2$ (double-shift orthogonality).

1. *(10 points) Among all 4-tap filters satisfying these equations, D_4 is the unique filter with a certain property. What is that property?*

   ANSWER: $C'(-1) = 0$.

2. *(10 points) Verify that D_4 has the property.*

   ANSWER:

   $$C'(z) = \frac{1}{4\sqrt{2}}\left(-(3+\sqrt{3})z^{-2} - 2(3-\sqrt{3})z^{-3} - 3(1-\sqrt{3})z^{-4}\right),$$

   so

   $$C'(-1) = \frac{1}{4\sqrt{2}}\left(-(3+\sqrt{3}) + 2(3-\sqrt{3}) - 3(1-\sqrt{3})\right) = 0.$$

Problem 4 *(20 points) Let $\phi(t)$ be a continuous scaling function satisfying the dilation equation*

$$\phi(t) = \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell)\phi(2t-\ell)$$

where the filter coefficients are normalized by

$$\sum_{k=0}^{N} \mathbf{c}(k) = \sqrt{2}$$

and $\phi(t)$ is normalized by $\int \phi(t)dt = 1$. In class and in the notes, we proved that $\sum_k \phi(t+k) = 1$ for all t, so in particular $\sum_k \phi(k) = 1$. Show that

$$\sum_k \phi(\frac{k}{2}) = 2.$$

ANSWER:

$$\sum_k \phi(\frac{k}{2}) = \sum_k \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell)\phi(2\frac{k}{2}-\ell) = \sqrt{2}\sum_k \sum_{\ell=0}^{N} \mathbf{c}(\ell)\phi(k-\ell)$$

$$= \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell)\sum_k \phi(k-\ell) = \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell)\sum_{k'} \phi(k') = \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell) = 2.$$

Problem 5 *(15 points) Find all sets $\mathbf{c}(k)$ with exactly 3 nonzero coefficients that determine a scaling function $\phi(t)$ with dilation equation*

$$\phi(t) = \sqrt{2}\sum_{k=0}^{N} \mathbf{c}(k)\phi(2t-k),$$

and whose integer translates form an ON set. Jusify your answer.

ANSWER: Orthogonality of $\{\phi(t-k)\} \longrightarrow$ double-shift orthogonality for the filter coefficients $\{\mathbf{c}(k)\}$. However, double-shift orthogonality can be satisfied only for FIR filters with N odd. In this case $N+1=3$, so $N=2$ is even. hence no such sets can exist.

COMMENT: Even if one allows interspersed zero filter coefficients, it is still impossible to have exactly 3 nonzero coefficients. Indeed suppose the nonzero coefficients were $\mathbf{c}(0), \mathbf{c}(a), \mathbf{c}(N)$, with all other $\mathbf{c}(j) = 0$. We need to verify double-shift orthogonality. One row vector will look like

$$\mathbf{r} = \cdots, \mathbf{c}(0), \cdots, \mathbf{c}(a), \cdots \mathbf{c}(N), \cdots$$

where only 3 terms are nonzero. Let $\mathbf{S}^{2k}\mathbf{r}$ be the vector that is obtained by right-shifting $\mathbf{r}$ $2k$ places (i.e., by right double-shifting $\mathbf{r}$ k times). For double-shift orthogonality we must have $(\mathbf{r}, \mathbf{S}^{2k}\mathbf{r}) = 0$ for all $k \neq 0$. Now if $\mathbf{c}(0)\mathbf{c}(a)\mathbf{c}(N) \neq 0$ and we have double-shift orthogonality then N must be odd. For if $N = 2k_0$ were even we would have $(\mathbf{r}, \mathbf{S}^{2k_0}\mathbf{r}) = 0 = \mathbf{c}(0)\overline{\mathbf{c}(N)}$, which is impossible. Similarly a must be odd, for if $a = 2k_1$ were even we would have $(\mathbf{r}, \mathbf{S}^{2k_1}\mathbf{r}) = 0 = \mathbf{c}(0)\overline{\mathbf{c}(a)}$, which is impossible. (There could be no overlap between $\mathbf{c}(N)$ and $\mathbf{c}(a)$ because $N-a$ would be odd.) Thus both N and a must be odd. But then we have $N - a = 2k_2$ even so that $(\mathbf{r}, \mathbf{S}^{2k_2}\mathbf{r}) = 0 = \mathbf{c}(a)\overline{\mathbf{c}(N)}$, which is impossible. Q.E.D.