Math 5467. Midterm Exam II (take-home problem) [Solution]

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Let $\phi(t)$ be a continuous scaling function satisfying the dilation equation

$$\phi(t) = \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell)\phi(2t-\ell)$$

where the filter coefficients are normalized by

$$\sum_{k=0}^{N} \mathbf{c}(k) = \sqrt{2}$$

and $\phi(t)$ is normalized by $\int \phi(t)dt = 1$. In class and in the notes, we show that $\sum_k \phi(t+k) = 1$ for all t, so in particular $\sum_k \phi(k) = 1$. From these facts alone, show that

$$\sum_k \phi(\frac{k}{2^j}) = 2^j \qquad j = 0, 1, 2, \cdots.$$

Hint: Try induction on j.

Solution: Proof by induction on j. The result is true for $j = 0$. We assume that it is true for $j = J$:

$$\sum_k \phi(\frac{k}{2^J}) = 2^J.$$

Then

$$\sum_k \phi(\frac{k}{2^{J+1}}) = \sum_k \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell)\phi(\frac{k}{2^J} - \ell) = \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell)\sum_k \phi(\frac{k}{2^J} - \ell)$$

(setting $k' = k - 2^J\ell$)

$$= \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell)\sum_{k'} \phi(\frac{k'}{2^J}) = \sqrt{2}\sum_{\ell=0}^{N} \mathbf{c}(\ell)2^J = 2^{J+1}.$$

This verifies the induction step.