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KILLING TENSORS AND NONORTHOGONAL VARIABLE SEPARATION FOR HAMILTON-JACOBI EQUATIONS*

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Abstract. Every separable coordinate system for the Hamilton–Jacobi equation on a Riemannian manifold V_n corresponds to a family of $n-1$ Killing tensors in involution, but the converse is false. For general n we find a practical characterization of those involutive families of Killing tensors that correspond to variable separation, orthogonal or not.

1. Introduction. We study the separation of variables problem for the Hamilton–Jacobi equation

$$g^{ij}\partial_{x^i}W\partial_{x^j}W=E, \qquad g^{ij}=g^{ji}, \qquad 1\leqq i,j\leqq n \tag{1.1}$$

($n\geqq 2$) and the relation between variable separation and second order Killing tensors on the (local) manifold V_n with metric tensor $\{g_{ij}\}$ in the local coordinates $\{x^i\}$. (We allow all coordinates and tensors to be complex and adopt the tensor notation in Eisenhart's book [1].)

In this paper we treat the general separation problem for (1.1), with emphasis on nonorthogonal separable coordinates. An analogous study for the more restricted orthogonal separation problem was presented in [2], and we assume familiarity with the basic definitions and results of that paper. Since every (multiplicative) separable system for the Helmholtz equation

$$\frac{1}{\sqrt{g}}\partial_{x^i}(\sqrt{g}g^{ij}\partial_{x^j}\Psi)=E\Psi, \qquad g=\det(g_{ij}) \tag{1.2}$$

is an (additively) separable system for (1.1), our treatment has direct applicability to the Helmholtz equation and the important families of special functions that arise as the separable solutions of this equation. (See [2] for a discussion of the relationship between these two equations together with additional references. The passage from (1.1) to (1.2) is closely analogous to the passage from classical mechanics to quantum mechanics.)

It is easily verified that to every separable coordinate system for (1.1), orthogonal or not, there corresponds a family of $n-1$ Killing tensors in involution. (The precise correspondence can be found in § 2.) However, not every such involutive family is associated with variable separation. In this paper we provide a solution to one of the fundamental problems in the theory of variable separation. We develop a decision procedure to determine precisely which families of Killing tensors are associated with separation, and for Killing tensors so associated we show how to construct the separable coordinates. Our procedure involves the determination of the eigenvalues and eigenforms of the Killing tensors, and is easy to implement for $n=3$, though less so for $n\geqq 4$.

It is important for many reasons to be able to compute separable coordinates directly from Killing tensors. Indeed, for flat spaces and spaces of constant curvature all second order Killing tensors can be expressed as second order polynomials in the Killing vectors, so for such spaces the possible involutive families of Killing tensors can be constructed explicitly through the use of Lie algebra techniques and then tested for

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variable separation. Furthermore, in the Lie theory treatment of special functions which arise through separation of variables in the Helmholtz equation [3] it is the symmetry operators, not the separable coordinates, that are fundamental.

Nonorthogonal separable coordinates, though considered from the earliest days in the classical literature (see, for example [4]), have received relatively little attention in comparison with orthogonal coordinates. However, nonorthogonal separable coordinates are of very frequent occurrence for the equations of mathematical physics, in particular for the real Klein–Gordon, wave, heat and time-dependent Schrödinger equations and their Hamilton–Jacobi counterparts. The special definition of nonorthogonal separation given in § 2 is due to the authors [5], [6] and clearly exhibits the nature of the separation. (Levi-Civita's classical definition in its original form [4] is, though intuitively appealing, very inconvenient for a detailed analysis of separable coordinate types.) Independently, Benenti [7] has arrived at our same classification of coordinates, which he calls "normal separable coordinates". He proves, roughly speaking, that all separable coordinates in the sense of Levi-Civita are equivalent to normal separable coordinates. (See [7], [8] for a more detailed discussion of the classical literature.)

In § 2 we discuss our definition of variable separation for the Hamilton–Jacobi equation in some detail, and show how to construct the involutive family of Killing tensors associated with a given separable system. In § 3 we show how to check if a given coordinate system $\{x^j\}$ permits variable separation in (1.1). Our results extend the well-known test for Stäckel form in the special case of orthogonal coordinates [1]. In § 4 we present our principal result: necessary and sufficient conditions that a given involutive family of Killing tensors determines a separable coordinate system. Our Theorem 4 is much stronger than earlier such results which have appeared in the literature [1], [8], because we have explicitly proved, rather than assumed, that the basis of differential forms which appears naturally in this problem is normalizable. (Hainzl [9] has studied variable separation for linear partial differential equations of arbitrary order through use of the Stäckel method and has obtained interesting partial analogues of our Theorems 2 and 3. However, when specialized to the Helmholtz equation his definition of separability omits the possibility of type 2 and nonorthogonal ignorable coordinates.) In § 5 we present a nontrivial example of the application of our Theorem 4 to three-dimensional Minkowski space.

2. Nonorthogonal separation. Our definition of separation of variables for the H–J equation (1.1) is identical with that presented in [5], [6], [10] and is based on a division of the separable coordinates into three classes: ignorable, essential of type 1 and essential of type 2. Let $\{x^1, \cdots, x^n\}$ be a coordinate system on the manifold with metric (g^{ij}) such that the n_1 coordinates x^a, $1 \leqq a \leqq n_1$, are essential of type 1, the n_2 coordinates x^r, $n_1+1 \leqq r \leqq n_1+n_2$, are essential of type 2, and the n_3 coordinates x^α, $n_1+n_2+1 \leqq \alpha \leqq n_1+n_2+n_3=n$, are ignorable. (In the following, indices a, b, c range from 1 to n_1, indices r, s, t range from n_1+1 to n_1+n_2, indices α, β, γ range from n_1+n_2+1 to n, and indices i, j, k range from 1 to n.) This means that the metric (g^{ij}), expressed in terms of coordinates $\{x^k\}$, is independent of the x^α, and that the separation equations take the form

$$W_a^2 + \sum_{\alpha,\beta=n_1+n_2+1}^{n} A_a^{\alpha,\beta}(x^a) W_\alpha W_\beta = \Phi_a(x^a; \lambda_1, \cdots, \lambda_{n_1+n_2}), \tag{2.1}$$

$$\sum_{\alpha=n_1+n_2+1}^{n} 2B_r^\alpha(x^r) W_r W_\alpha + \sum_{\alpha,\beta=n_1+n_2+1}^{n} C_r^{\alpha,\beta}(x^r) W_\alpha W_\beta = \Phi_r(x^r; \lambda_1, \cdots, \lambda_{n_1+n_2}), \qquad n_1+1 \leqq r \leqq n_1+n_2, \tag{2.2}$$

$$W_\alpha=\lambda_\alpha,\qquad n_1+n_2+1\leqq\alpha\leqq n. \tag{2.3}$$

Here $A_a^{\alpha,\beta}(=A_a^{\beta,\alpha})$, B_r^α, $C_r^{\alpha,\beta}(=C_r^{\beta,\alpha})$ and Φ_i are defined and analytic in a neighborhood $N\times S\subseteq C^{n_1+n_2}\times C^{n_1+n_2}$, where N is a neighborhood of $(x_0^1,\cdots,x_0^{n_1+n_2})$ and S is a neighborhood of $(0,\cdots,0)$ in the Euclidean space with coordinates $\lambda_1,\cdots,\lambda_{n_1+n_2}$. The parameters λ_α are arbitrary. Furthermore, the complex parameters $\lambda_1,\cdots,\lambda_{n_1+n_2}$ are independent; i.e. the Jacobian

$$\varphi(x^i,\lambda_1,\cdots,\lambda_{n_1+n_2})=\det\left(\frac{\partial\Phi_a}{\partial\lambda_i},\frac{\partial\Phi_r}{\partial\lambda_i}\right) \tag{2.4}$$

is nonzero in $N\times S$.

We say that the coordinates $\{x^i\}$ are *separable* for the H–J equation if there exist analytic functions A, B, C, Φ above and functions $U_a(x^i)$, $V_r(x^i)$, analytic in N, such that the H–J equation

$$\sum g^{ij}\partial_i W\partial_j W=E \tag{2.5}$$

can be written in the form

$$\sum_a U_a(x^i)\Phi_a+\sum_r V_r(x^i)\Phi_r=E \tag{2.6}$$

(identically in the parameters $\lambda_1=E, \lambda_2,\cdots,\lambda_n$), where $W=\sum_{j=1}^n W^{(j)}(x^j)$, $W_i=\partial_i W=\partial_i W^{(i)}$.

Comparison of (2.5) and (2.6) determines the functions U_a, V_r uniquely. Furthermore, differentiating (2.6) with respect to λ_l, we have

$$\sum_a U_a\frac{\partial\Phi_a}{\partial\lambda_l}+\sum_r V_r\frac{\partial\Phi_r}{\partial\lambda_l}=\delta_{1l},$$

and this leads to the usual Stäckel form

$$U_a(x^i)=\frac{\varphi^{a1}}{\varphi},\qquad V_r(x^i)=\frac{\varphi^{r1}}{\varphi}, \tag{2.7}$$

where φ^{lm} is the (lm)-cofactor of the matrix (2.4). The nonzero components of the contravariant matric tensor are thus

$$\begin{gathered}g^{ab}=\left(\frac{\varphi^{a1}}{\varphi}\right)\delta^{ab},\qquad g^{r\alpha}=g^{\alpha r}=\left(\frac{\varphi^{r1}}{\varphi}\right)B_r^\alpha(x^r),\\ \tfrac12 g^{\alpha\beta}=\sum_a A_a^{\alpha,\beta}(x^a)\frac{\varphi^{a1}}{\varphi}+\sum_r C_r^{\alpha,\beta}(x^r)\frac{\varphi^{r1}}{\varphi},\qquad \alpha\neq\beta,\\ g^{\alpha\alpha}=\sum_a A_a^{\alpha,\alpha}\frac{\varphi^{a1}}{\varphi}+\sum_r C_r^{\alpha,\alpha}\frac{\varphi^{r1}}{\varphi}.\end{gathered} \tag{2.8}$$

The generality of the functions Φ_l is illusory, due to the restrictive conditions (2.7) which require that the functions φ^{l1}/φ are independent of $\lambda_1,\cdots,\lambda_{n_1+n_2}$. Indeed, setting $\theta_{lm}(x^l)=\partial\Phi_l(x^l,\mathbf{0})/\partial\lambda_m$, $1\leqq l$, $m\leqq n_1+n_2$ and $\theta(x^i)=\varphi(x^i,\mathbf{0})$, where $\mathbf{0}=(0,\cdots,0)\in S$, we have

$$U_a=\frac{\theta^{a1}}{\theta}\not\equiv 0,\qquad V_r=\frac{\theta^{r1}}{\theta}\not\equiv 0. \tag{2.9}$$

Furthermore, since $\theta\neq 0$ in N there exist functions $G_l(\mathbf{x},\boldsymbol{\lambda})$, analytic in $N+S$, such that

$$\Phi_p(x^p,\boldsymbol{\lambda})=\sum_{m=1}^{n_1+n_2}G_m(\mathbf{x},\boldsymbol{\lambda})\theta_{pm}(x^p),\qquad 1\leqq p\leqq n_1+n_2. \tag{2.10}$$

Substituting (2.9) and (2.10) in (2.6) we find $G_1(\mathbf{x},\boldsymbol{\lambda})\equiv E=\lambda_1$. Furthermore, $\partial_{\lambda_m}G_l(\mathbf{x},\mathbf{0})=\delta_{ml}$ and, from the fact that the minors θ^{a1}, θ^{r1} are nonzero in a neighborhood of $\mathbf{x}_0$, $\partial_{x_j}G_l(\mathbf{x},\boldsymbol{\lambda})\equiv 0$ for $j=1,\cdots,n_1+n_2$, $l=2,\cdots,n_1+n_2$. Thus, $G_l(\mathbf{x},\boldsymbol{\lambda})\equiv G_l(\boldsymbol{\lambda})$ and, in terms of the new parameters $E_l=G_l(\boldsymbol{\lambda})$, $l=1,\cdots,n_1+n_2$, $E_\alpha=\lambda_\alpha$, $\alpha=n_1+n_2+1,\cdots,n$, the functions Φ_p assume the standard form

$$\Phi_p(x^p,\boldsymbol{\lambda})\equiv\Phi_p(x^p,\mathbf{E})=\sum_{l=1}^{n_1+n_2}E_l\theta_{pl}(x^p). \tag{2.11}$$

The separation equations (2.1)–(2.3) become

$$W_a^2+\sum_{\alpha,\beta=n_1+n_2+1}^{n}A_a^{\alpha,\beta}(x^a)E_\alpha E_\beta=\sum_{l=1}^{n_1+n_2}E_l\theta_{al}(x^a), \tag{2.12}$$

$$\sum_{\alpha=n_1+n_2+1}^{n}2B_r^\alpha(x^r)E_\alpha W_r+\sum_{\alpha,\beta=n_1+n_2+1}^{n}C_r^{\alpha,\beta}(x^r)E_\alpha E_\beta=\sum_{l=1}^{n_1+n_2}E_l\theta_{rl}(x^r), \tag{2.13}$$

$$W_\alpha=E_\alpha. \tag{2.14}$$

These expressions are the master equations for separation of variables in the Hamilton–Jacobi equation (2.5).

Remarks.

1) Since the metric tensor (g^{ij}) is nonsingular, $n_3\geqq n_2$.

2) From (2.11) we have

$$\sum_{l=1}^{n_1+n_2}\frac{\theta^{lm}}{\theta}\Phi_l=E_m,\qquad m=1,\cdots,n_1+n_2.$$

Thus,

$$\begin{aligned}&A_m(\mathbf{x},\mathbf{p})=E_m, &&m=1,\cdots,n_1+n_2,\\ &L_\alpha(\mathbf{x},\mathbf{p})=E_\alpha, &&\alpha=n_1+n_2+1,\cdots,n,\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}&A_m(\mathbf{x},\mathbf{p})=\sum_{i,j=1}^{n}a_{(m)}^{i,j}p_ip_j,\qquad L_\alpha(\mathbf{x},\mathbf{p})=p_\alpha,\\ &p_i=\partial_{x^i}W\end{aligned} \tag{2.16}$$

and the nonzero terms of the symmetric quadratic form $(a_{(m)}^{ij})$ are given by

$$\begin{aligned}a_{(m)}^{ab}&=\left(\frac{\theta^{am}}{\theta}\right)\delta^{ab},\qquad a_{(m)}^{r\alpha}=\left(\frac{\theta^{rm}}{\theta}\right)B_r^\alpha,\\ \tfrac{1}{2}a_{(m)}^{\alpha\beta}&=\sum_c A_c^{\alpha,\beta}\frac{\theta^{cm}}{\theta}\sum_r C_r^{\alpha,\beta}\frac{\theta^{rm}}{\theta},\qquad \alpha\neq\beta,\\ a_{(m)}^{\alpha\alpha}&=\sum_c A_c^{\alpha,\alpha}\frac{\theta^{cm}}{\theta}+\sum_r C_r^{\alpha,\alpha}\frac{\theta^{rm}}{\theta}.\end{aligned} \tag{2.17}$$

(Note that $A_1=E_1$ is the original Hamilton–Jacobi equation.)

3) By definition, the quadratic form $H=\sum_{l=1}^{n_1+n_2}H_l^{-2}p_l^2$ is in *Stäckel form* if $H_l^2=\theta/\theta^{l1}$, where

$$\Theta=(\theta_{lm}(x^m))$$

is a Stäckel matrix, $\theta=\det\Theta$ and θ^{l1} is the $(l,1)$ minor of Θ. It is well known [1] that necessary and sufficient conditions that H be in Stäckel form are

$$\partial^2_{x^j x^k} \ln H_i^2 - \partial_{x^j} \ln H_i^2 \; \partial_{x^k} \ln H_i^2 + \partial_{x^j} \ln H_i^2 \; \partial_{x^k} \ln H_j^2 + \partial_{x^k} \ln H_i^2 \; \partial_{x^j} \ln H_k^2 = 0, \qquad j \neq k \tag{2.18}$$

4) If H is in Stäckel form as in 3), the expressions $\theta^{lm}/\theta \equiv \rho_l^{(m)}\theta^{l1}/\theta = \rho_l^{(m)} H_l^{-2}$ are characterized by the equations

$$\partial_{x^k}\rho_l = (\rho_k - \rho_l)\,\partial_{x^k}(\ln H_l^{-2}), \qquad k \neq l, \qquad \partial_{x^l}\rho_l = 0; \tag{2.19}$$

see [1]. In particular, (2.18) constitute the integrability conditions for the system (2.19), and this system admits an (n_1+n_2)-dimensional space of vector-valued solutions $(\rho_1, \cdots, \rho_{n_1+n_2})$. To any basis of solutions $(\rho_j^{(m)})$ with $\rho_j^{(1)} \equiv 1$ there corresponds a Stäckel matrix Θ with $\theta^{lm}/\theta = \rho_l^{(m)} H_l^{-2}$.

5) To understand the significance of the quadratic forms A_m and linear forms L_α (2.16), we use the natural symplectic structure on the cotangent bundle $\tilde{V}_n$ of the Riemannian manifold V_n. Corresponding to local coordinates $\{x^i\}$ on V_n we have coordinates $\{x^i, p_j\}$ on $\tilde{V}_n$. If $\{\hat{x}^k(x^j)\}$ is another local coordinate system on V_n then it corresponds to $\{\hat{x}^k, \hat{p}_k\}$ where $\hat{p}_k = p_l \partial x^l/\partial \hat{x}^k$. The *Poisson bracket* of two functions $F(x^i, p_j)$, $G(x^i, p_j)$ on $\hat{V}_n$ is the function

$$[F, G] = \partial_{x^i}F\,\partial_{p_i}G - \partial_{p_i}F\,\partial_{x^i}G. \tag{2.20}$$

(We are employing the summation convention for variables that range from 1 to n.)

It is straightforward, though tedious, to verify the relations

$$[A_l, A_m] = 0, \qquad [L_\alpha, A_l] = 0, \qquad [L_\alpha, L_\beta] = 0. \tag{2.21}$$

(For $n \leqq 4$ these relations were already noted in [5] and [6]. We will give an explicit proof for general n in § 3.) Thus, the A_m for $m \geqq 2$ are second order Killing tensors and the L_α are Killing vectors (first order Killing tensors) for the manifold V_n. Moreover, the family of $n-1$ Killing tensors $\{A_m (m \geqq 2), L_\alpha\}$ is in involution.

The relations (2.21) associating separable coordinates on V_n with an involutive family of Killing tensors are not difficult to prove. Much more difficult is the characterization of exactly those involutive families of Killing tensors that define variable separation and the development of a constructive procedure to determine the coordinates from a knowledge of the Killing tensors. For orthogonal separable coordinate systems this problem was given an elegant solution in [2]. For the more general case in which the coordinates may not be orthogonal, we provide a (less elegant) solution in the following two sections.

3. Generalized Stäckel form. Here, we are given a Riemannian manifold V_n and the contravariant metric tensor g^{ij} on V_n, expressed in terms of the local coordinates $x^1, \cdots, x^n$. We wish to determine necessary and sufficient conditions on the g^{ij} in order that the Hamilton–Jacobi equation (1.1) permit separation in these local coordinates.

If $g^{ij} = H_i^{-2}\delta^{ij}$, i.e., if the coordinates $\{x^k\}$ are orthogonal, then the necessary and sufficient condition for separation is that $H = g^{ij}p_ip_j$ be in Stäckel form [1, App. 13]. In other words, the relations (2.18) must be satisfied.

For nonorthogonal coordinates the conditions are somewhat more complicated. To derive these conditions we need some preliminary lemmas related to Stäckel form. Let $ds^2 = h_i^2(dx^i)^2 = g_{ij}\,dy^i\,dy^j$ be a metric that is in Stäckel form with respect to the local coordinates $y^1, \cdots, y^N$; i.e., there exists an $N\times N$ Stäckel matrix Θ such that $h_i^2=\theta/\theta^{i1}$, where $\theta=\det\Theta$ and θ^{i1} is the $(i1)$ minor of Θ. A scalar valued function $f(\mathbf{y})$ is a *Stäckel multiplier* (*for* ds^2) if the metric $d\hat{s}^2=f\,ds^2=fh_i^2(dy^i)^2$ is also in Stäckel form.

LEMMA 1. *f is a Stäckel multiplier for ds^2 if and only if it satisfies the relations*

$$\partial_{y^jy^k}f+\partial_{y^j}f\partial_{y^k}\ln h_j^2+\partial_{y^k}f\partial_{y^j}\ln h_k^2=0,\qquad j\neq k. \tag{3.1}$$

Proof. These relations follow directly from the fact that (2.18) must hold for $H_i^2=h_i^2$ and also for $H_i^2=fh_i^2$ if f is a Stäckel multiplier. □

LEMMA 2. *f is a Stäckel multiplier for ds^2 if and only if there exist local analytic functions $\varphi_l=\varphi_l(y^l)$ such that*

$$f(\mathbf{y})=\sum_{l=1}^{N}\varphi_l(y^l)h_l^{-2}. \tag{3.2}$$

Proof. Suppose f is a Stäckel multiplier for ds^2. Then there exists a Stäckel matrix $\hat{\Theta}$ such that $fh_l^2=\hat{\theta}/\hat{\theta}^{l1}$. But $h_l^2=\theta/\theta^{l1}$, so $f\theta/\hat{\theta}=\theta^{l1}/\hat{\theta}^{l1}$, a function independent of y^l. Since the preceding relation holds for all l we have $f\theta/\hat{\theta}=\theta^{l1}/\hat{\theta}^{l1}=\mathcal{K}\in C$ and, without loss of generality, we can renormalize $\hat{\Theta}$ so that $\mathcal{K}=1$. Thus,

$$f=\frac{\hat{\theta}}{\theta}=\sum_l \hat{\theta}_{l1}\frac{\hat{\theta}^{l1}}{\theta}=\sum_l \hat{\theta}_{l1}\frac{\theta^{l1}}{\theta}$$

and we obtain (3.2) with $\varphi_l=\hat{\theta}_{l1}$.

Conversely, if f can be expressed in the form (3.2), where $h_l^2=\theta/\theta^{l1}$ is in Stäckel form, then it follows directly from (2.18) with $H_l^2=h_l^2$ that relations (3.1) are satisfied. Hence, f is a Stäckel multiplier. □

Note that (3.2) is the general solution of (3.1).

Let (g^{ij}) be a given contravariant metric in the coordinates $x^1, \cdots x^n$. We wish to determine if these coordinates permit separation for the Hamilton–Jacobi equation. It is convenient to reorder the coordinates in a standard form. Let n_3 be the number of ignorable variables x^α (recall that x^α is ignorable if $\partial_{x^\alpha}g^{ij}=0$ for all i, j). Of the remaining $n-n_3$ variables, suppose n_2 variables x^r have the property $g^{rr}=0$ and the remaining n_1 variables x^a satisfy $g^{aa}\neq 0$. We relabel the variables so that $1\leqq a\leqq n_1$, $n_1+1\leqq r\leqq n_1+n_2$, and $n_1+n_2+1\leqq \alpha\leqq n_1+n_2+n_3=n$.

THEOREM 1. *Suppose (g^{ij}) is in standard form with respect to the variables $\{x^i\}$. The Hamilton–Jacobi equation* (1.1) *is separable for this system if and only if*:

1) *The contravariant metric assumes the form*

$$(g^{ij})=\begin{array}{cc} \begin{array}{ccc} \quad n_1 & \quad\quad n_2 & \quad\quad n_3 \end{array} & \\ \left[\begin{array}{c|c|c} \delta^{ab}H_a^{-2} & 0 & 0 \\ \hline 0 & 0 & H_r^{-2}B_r^{\alpha} \\ \hline 0 & H_r^{-2}B_r^{\alpha} & g^{\alpha\beta} \end{array}\right] & \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \end{array}$$

where $B_r^\alpha=B_r^\alpha(x^r)$.

2) *The metric*

$$d\tilde{s}^2=\sum_{a=1}^{n_1}H_a^2(dx^a)^2+\sum_{r=n_1+1}^{n_1+n_2}H_r^2(dx^r)^2$$

is in Stäckel form; i.e., relations (2.18) *hold for $1\leqq i, j, k\leqq n_1+n_2$.*

3) *Each $g^{\alpha\beta}(\mathbf{x})$ is a Stäckel multiplier for the metric $d\tilde{s}^2$.*

Proof. The theorem follows immediately from expressions (2.8) and Lemmas 1 and 2.

Note that Theorem 1 reduces the problem of determining whether the Hamilton–Jacobi equation is separable in given coordinates to the verification of two systems of partial differential equations. If the coordinates are orthogonal, then $n_2=0$ and the separation requirement is simply that the metric be in Stäckel form.

Let $A=a^{ij}(\mathbf{x})p_ip_j$, $B=b^{ij}(\mathbf{x})p_ip_j$ be symmetric quadratic functions on $\tilde{V}_n$. It follows from (2.20) that these functions are in involution with respect to the Poisson bracket if and only if

$$a^{[i,j}\partial_j b^{\boxed{k,l]}}=b^{[i,j}\partial_j a^{\boxed{k,l]}},\qquad 1\leqq i,k,l\leqq n, \tag{3.3}$$

where

$$a^{[i,j}\partial_j b^{\boxed{k,l]}}=a^{ij}\partial_j b^{kl}+a^{lj}\partial_j b^{ik}+a^{kj}\partial_j b^{li}.$$

A scalar-valued function $\rho(\mathbf{x})$ is a *root* of the form $a^{ij}(x)$ if

$$\det(a^{ij}(\mathbf{x})-\rho(\mathbf{x})g^{ij}(\mathbf{x}))=0 \tag{3.4}$$

in a coordinate neighborhood, where (g^{ij}) is the metric on V_n. A form $\psi=\lambda_j(\mathbf{x})\,dx^j$ such that

$$(a^{ij}-\rho g^{ij})\lambda_j=0,\qquad \psi\neq 0$$

in the same coordinate neighborhood is an *eigenform* corresponding to the root ρ.

THEOREM 2. *Let (g^{ij}) be the contravariant metric tensor on V_n in the coordinates $\{x^i\}$. If the Hamilton–Jacobi equation is separable in these coordinates, then there exists a Q-dimensional vector space $\mathcal{A}$ of second order Killing tensors on V_n such that*

(3.5)

(1) $[A,B]=0$ *for each* $A,B\in\mathcal{A}$.

(2) *For each of the n_1 essential coordinates of type 2, x^a, the form dx^a is a simultaneous eigenform for every $A\in\mathcal{A}$, with root $\overset{A}{\rho}_a$.*

(3) *For each of the n_2 essential coordinates of type 1, x^r, the form dx^r is a simultaneous eigenform for every $A\in\mathcal{A}$, with root $\overset{A}{\rho}_r$. The root $\overset{A}{\rho}_r$ has multiplicity 2 but corresponds to only one eigenform.*

(4) $\partial_i(a^{\alpha\beta})-\overset{A}{\rho}_i\,\partial_i g^{\alpha\beta}=0$, $i=1,\cdots,n_1+n_2$ *for all $A\in\mathcal{A}$, and all n_3 ignorable variables $\alpha,\beta=n_1+n_2+1,\cdots,n$.*

(5) $[A,L_\alpha]=0$ *for each $A\in\mathcal{A}$ and $L_\alpha=p_\alpha$, $\alpha=n_1+n_2+1,\cdots,n$.*

(6) $Q=n+n_3(n_3-1)/2$.

This theorem is easily obtained from the proof of the following deeper result. Let $\{x^i\}$ be a coordinate system on V_n with coordinates divided into three classes, containing n_1, n_2, and n_3 variables respectively $(n=n_1+n_2+n_3)$. (We will call them essential variables of type 1, essential variables of type 2 and ignorable variables, respectively, even though at this point they have nothing to do with separation.) Let $H=g^{ij}p_ip_j$.

THEOREM 3. *Suppose there exists a Q-dimensional vector space $\mathcal{A}$ of second order Killing tensors on V_n such that $H\in\mathcal{A}$ and conditions* (1)–(6) *in* (3.5), *are satisfied. Furthermore, suppose $g^{ab}=0$ if $1\leqq a<b\leqq n_1$ and $g^{ar}=g^{a\alpha}=g^{rs}=0$ for $1\leqq a\leqq n_1$, $n_1+1\leqq r$, $s\leqq n_1+n_2$, $n_1+n_2+1\leqq\alpha\leqq n$. Then the Hamilton–Jacobi equation* (1.1) *is separable in the coordinates $\{x^i\}$. The Killing tensors A_m, $m=1,\cdots,n_1+n_2$,* (2.16), *and $L_\alpha L_\beta=p_\alpha p_\beta$, $n_1+n_2+1\leqq\alpha\leqq\beta\leqq n$, form a basis for $\mathcal{A}$.*

Proof. From conditions (2), (3) and our assumptions on the vanishing of certain matrix elements of (g^{ij}), we see that the matrix corresponding to any $A\in\mathcal{A}$ takes the form

$$(a^{ij})=\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c}\begin{array}{ccc} n_1 & n_2 & n_3 \end{array}\\ \left[\begin{array}{c|c|c} \delta^{ab}\rho_a H_a^{-2} & 0 & 0 \\ \hline 0 & 0 & \rho_a g^{a\alpha} \\ \hline 0 & \rho_a g^{a\alpha} & a^{\alpha\beta} \end{array}\right]\end{array}\begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \end{array} \tag{3.6}$$

If $(\rho_i^A)=(\rho_i^B)$ for $A, B\in\mathscr{A}$, it follows from (3.6) and condition (4) that $A-B$ is a linear combination of the $n_3(n_3+1)/2$ Killing tensors $L_\alpha L_\beta = p_\alpha p_\beta$, $\alpha\geqq\beta$. It follows that for each $\mathbf{x}$ the set of (n_1+n_2)-tuples $\{(\rho_i^A(\mathbf{x})), A\in\mathscr{A}\}$ spans $C^{n_1+n_2}$.

The relation $[H, A]=0$ is equivalent to

$$g^{[i,j}\partial_j a^{\boxed{k,l]}} = a^{[i,j}\partial_j g^{\boxed{k,l]}}. \tag{3.7}$$

Setting $(i, k, l)=(a, b, c)$ in (3.7) and utilizing (3.6) we obtain

$$\partial_a\rho_b=(\rho_a-\rho_b)\partial_a(\ln H_b^{-2}), \qquad \partial_a\rho_a=0. \tag{3.8}$$

Setting $(i, k, l)=(a, r, \alpha)$ in (3.7) we find

$$\partial_a\rho_r=(\rho_a-\rho_r)\partial_a \ln g^{r\alpha} \quad \text{if } g^{r\alpha}\not\equiv 0. \tag{3.9}$$

For $(i, j, k)=(r, \alpha, \beta)$ we obtain

$$g^{\beta s}g^{\alpha r}\partial_s\rho_r+g^{\beta r}g^{\alpha s}\partial_s\rho_r=(\rho_s-\rho_r)g^{\beta s}\partial_s g^{\alpha r}+(\rho_s-\rho_r)g^{\alpha s}\partial_s g^{\beta r} \qquad (\text{sum on } s). \tag{3.10}$$

The case $(i, j, k)=(a, a, \alpha)$ leads to

$$\partial_r\rho_a=(\rho_r-\rho_a)\partial_r \ln H_a^{-2}. \tag{3.11}$$

The cases $(i, j, k)=(a, \alpha, \beta)$, (α, β, γ) are satisfied as a consequence of condition (4), and all remaining cases are satisfied identically.

Multiplying both sides of (3.10) by $g_{R\alpha}g_{S\beta}$, $(n_1+1\leqq R, S\leqq n_1+n_2)$, and summing on α and β we find

$$\delta^r_R\partial_S\rho_R+\delta^r_S\partial_R\rho_S=(\rho_S-\rho_r)g_{R\alpha}\partial_S g^{\alpha r}+(\rho_R-\rho_r)g_{S\beta}\partial_R g^{\beta r}. \tag{3.12}$$

Setting $R=r$, $S=s$ in (3.12), solving for $\partial_s\rho_R$, substituting this result in (3.10) and equating coefficients of ρ_s, $s\neq r$, we find after some manipulation

$$\partial_r(\ln g^{\gamma s})=\partial_r(\ln g^{\alpha s})=\sum_\beta g_{s\beta}\partial_r(g^{\beta s}), \qquad r\neq s \tag{3.13}$$

for all α, γ such that $g^{\gamma s}$, $g^{\alpha s}\not\equiv 0$.

Since (g^{ij}) is nonsingular, for each s, $n_1+1\leqq s\leqq n_1+n_2$, there is at least one $\alpha=\alpha(s)$ such that $g^{\alpha s}\not\equiv 0$. We define $H_s^{-2}=g^{\alpha(s)s}$. It follows from (3.9) and (3.13) that there exist functions $B_r^\gamma(x^r)$ such that

$$g^{\gamma r}=g^{r\gamma}=H_r^{-2}(\mathbf{x})B_r^\gamma(x^r), \quad n_1+1\leqq r\leqq n_1+n_2, \quad n_1+n_2+1\leqq\gamma\leqq n.$$

Thus, expressions (3.8), (3.9), (3.11) and (3.12) reduce to

$$\partial_i\rho_j=(\rho_i-\rho_j)\partial_j(\ln H_j^{-2}), \qquad 1\leqq i, j\leqq n_1+n_2. \tag{3.14}$$

The integrability conditions for the system (3.14) are precisely (2.18); i.e., the metric $d\hat{s}^2=\sum_{i=1}^{n_1+n_2}H_i^2(dx^i)^2$ must be in Stäckel form. Similarly, the integrability requirements $\partial_i\partial_j a^{\alpha\beta}=\partial_j\partial_i a^{\alpha\beta}$ for condition (4) are (through use of (3.14)) simply that each $g^{\alpha\beta}$ be a Stäckel multiplier for the metric $d\hat{s}^2$. Thus the contravariant metric (g^{ij}) takes the form (2.8); hence the Hamilton–Jacobi equation separates in the coordinates $\mathbf{x}$. The stated relation between the $A \in \mathscr{A}$ and the quadratic forms A_m of § 2 is provided by (2.17) and (2.19). In particular, expressions (2.17) for the $a^{\alpha\beta}$ satisfy conditions (4) and are determined by these conditions to within additive constants. □

The role of condition (4) needs clarification. It is not difficult to construct examples of Killing tensors that satisfy conditions (2), (3) and (5) but violate condition (4). However, we have

COROLLARY 1. *Let (g^{ij}) be the metric for V_n in the separable coordinates $\{x^i\}$, the coordinates ordered as in Theorems 2 and 3, and let $\mathscr{A}$ be the space of second order Killing tensors described in Theorem 2. Suppose C is a second order Killing tensor satisfying conditions (2), (3) and (5) of Theorem 2 and such that $[C, A]=0$ for all $A \in \mathscr{A}$. Then $C \in \mathscr{A}$; i.e., C satisfies condition (4).*

Proof. Let (ρ_i^C) be the roots of C. Then there exists a $B \in \mathscr{A}$ such that $(\rho_i^C) \equiv (\rho_i^B)$. Thus, the Killing tensor $F = C - B$ has roots $\rho_i^F \equiv 0$ and takes the form $F = f^{\alpha\beta} p_\alpha p_\beta$. The condition $[F, A]=0$ for all $A \in \mathscr{A}$ becomes

$$\rho_a^A g^{\alpha r} \partial_r (f^{\beta\gamma}) + \rho_r^A g^{\gamma r} \partial_r (f^{\alpha\beta}) + \rho_r^A g^{\beta r} \partial_r (f^{\gamma\alpha}) = 0, \qquad \partial_a f^{\alpha\beta} = 0. \tag{3.15}$$

The coefficient of ρ_r^A in (3.15) must vanish, so we have

$$g^{\alpha r} \partial_r f^{\beta\gamma} + g^{\gamma r} \partial_r f^{\alpha\beta} + g^{\beta r} \partial_r f^{\gamma\alpha} = 0 \qquad \text{(no sum on } r\text{)}. \tag{3.16}$$

(Recall that for fixed r there is at least one γ such that $g^{\gamma r} \not\equiv 0$.)

Suppose $g^{\alpha r} \not\equiv 0$. Setting $(\alpha, \beta, \gamma) = (\alpha, \alpha, \alpha)$ in (3.16) we find $g^{\alpha r} \partial_r f^{\alpha\alpha} = 0$, so that $\partial_r f^{\alpha\alpha} \equiv 0$. On the other hand, if $g^{\alpha r} \equiv 0$ but $g^{\gamma r} \not\equiv 0$, then setting (α, β, γ) in (3.16) $g^{\gamma r} \partial_r f^{\alpha\alpha} = 0$. Thus in all cases $\partial_r f^{\alpha\alpha} \equiv 0$.

If $g^{\alpha r} \not\equiv 0$, then setting $(\alpha, \beta, \gamma) = (\alpha, \beta, \alpha)$ in (3.16) we find $g^{\alpha r} \partial_r f^{\alpha\beta} = 0$, so $\partial_r f^{\alpha\beta} \equiv 0$. However, if $g^{\alpha r} \equiv 0$ but $g^{\gamma r} \not\equiv 0$, then, since $\partial_r f^{\beta\gamma} = \partial_r f^{\gamma\alpha} \equiv 0$, (3.16) becomes $g^{\gamma r} \partial_r f^{\alpha\beta} = 0$. Thus in all cases $\partial_r f^{\alpha\beta} \equiv 0$.

We have shown that $f^{\alpha\beta}$ is a constant, hence that $F = f^{\alpha\beta} p_\alpha p_\beta = f^{\alpha\beta} L_\alpha L_\beta \in \mathscr{A}$. □

Remark. It is sufficient to require that condition (4) of Theorem 2 be valid for $i = n_1 + 1, \cdots, n_1 + n_2$, since the requirement $[H, A] = 0$ for $(i, j, k) = (a, \alpha, \beta)$ yields this condition for $i = 1, \cdots, n_1$.

4. The main result. We come now to the fundamental question: given an involutive family of $n-1$ Killing tensors, how do we determine if this family corresponds to a separable coordinate system for the Hamilton–Jacobi equation?

Let $\{x^j\}$ be a local coordinate system on the Riemannian manifold V_n and let $\theta_{(j)} = \lambda_{i(j)}\, dx^i$, $1 \leqq j \leqq n$, be a local basis of one-forms on V_n. The dual basis of vector fields is $X^{(h)} = \Lambda^{i(h)} \partial_{x^i}$, $1 \leqq h \leqq n$, where $\Lambda^{i(h)} \lambda_{i(j)} = \delta_{(j)}^{(h)}$. We say that the forms $\{\theta_{(j)}\}$ are *normalizable* if there exist local analytic functions $g_{(j)}$, y^j such that $\theta_{(j)} = g_{(j)}\, dy^j$, (no sum). (Equivalently, $X^{(h)} = g_{(h)}^{-1} \partial_{y^h}$.) It is classical that the forms are normalizable if and only if the coefficient of $X^{(l)}$ is zero in the expansion of $[X^{(h)}, X^{(k)}]$ in terms of the $\{X^{(j)}\}$ basis whenever $h, k \neq l$; see [1, § 35].

LEMMA 3. *The one-forms $\{\theta_{(j)}\}$ are normalizable if and only if*

$$(\partial_{x^j} \lambda_{i(l)} - \partial_{x^i} \lambda_{j(l)}) \Lambda^{i(h)} \Lambda^{j(k)} = 0, \qquad h, k \neq l. \tag{4.1}$$

This condition can also be expressed in terms of the inner products

$$G_{(h,l)} = \lambda^i_{(h)} \lambda_{i(l)}. \tag{4.2}$$

We have $\lambda_{j(l)} = G_{(h,l)} \Lambda_j^{(h)}$ or $\Lambda^{j(h)} = \lambda^j_{(l)} G^{(l,h)}$ where $G^{(h,l)} G_{(l,j)} = \delta_{(j)}^{(h)}$. Thus condition (4.1) can be written in the form

$$G^{(h,h')} G^{(k,k')} (\gamma_{(lh'k')} - \gamma_{(lk'h')}) = 0, \qquad h, k \neq l, \tag{4.3}$$

where

$$\gamma_{(lhk)}=\lambda_{i(l),j}\lambda^{i}_{(h)}\lambda^{j}_{(k)} \tag{4.4}$$

and $\lambda_{i(l),j}$ is the jth covariant derivative of $\lambda_{i(l)}$ [1]. Let $H=g^{ij}p_ip_j$.

THEOREM 4. *Suppose there exists a Q-dimensional vector space $\mathcal{A}$ of second order Killing tensors on V_n such that $H\in\mathcal{A}$ and:*

(1) *$[A, B]=0$ for each $A, B\in\mathcal{A}$.*

(2) *There is a basis of one forms $\theta_{(h)}=\lambda_{i(h)}\,dx^i$, $1\leqq h\leqq n$ such that*

(a) *the n_1 forms $\theta_{(a)}$, $1\leqq a\leqq n_1$ are simultaneous eigenforms for every $A\in\mathcal{A}$ with root ρ^A_a:*

$$(a^{ij}-\rho^A_a g^{ij})\lambda_{j(a)}=0,$$

(b) *the n_2-forms $\theta_{(r)}$, $n_1+1\leqq r\leqq n_1+n_2$, are simultaneous eigenforms for every $A\in\mathcal{A}$ with root ρ^A_r:*

$$(a^{ij}-\rho^A_r g^{ij})\lambda_{j(r)}=0. \tag{4.5}$$

The root ρ^A_r has multiplicity 2 but corresponds to only one eigenform.

(3) *$X^{(h)}(\lambda_{i(\alpha)}a^{ij}\lambda_{j(\beta)})=\rho^A_h X^{(h)}(\lambda_{i(\alpha)}g^{ij}\lambda_{j(\beta)})$, $h=1,\cdots,n_1+n_2$ for all $A\in\mathcal{A}$ and all $\alpha,\beta=n_1+n_2+1,\cdots,n$.*

(4) *$[L_\alpha, L_\beta]=0$ where $L_\alpha=\Lambda^{i(\alpha)}p_i$.*

(5) *$[A, L_\alpha]=0$ for each $A\in\mathcal{A}$.*

(6) *$Q=\frac{1}{2}(2n+n_3^2-n_3)$, where $n_3=n-n_1-n_2$.*

(7) *$G_{(ab)}=0$ if $1\leqq a<b\leqq n_1$, and $G_{(ar)}=G_{(a\alpha)}=G_{(rs)}=0$ for $1\leqq a\leqq n_1$, $n_1+1\leqq r, s\leqq n_1+n_2$, $n_1+n_2+1\leqq\alpha\leqq n$.*

Then there exist local coordinates $\{y^i\}$ for V_n such that $\theta_{(j)}=f^{(j)}(\mathbf{y})\,dy^j$ for suitably chosen functions $f^{(j)}$, and the Hamilton–Jacobi equation is separable in these coordinates. Conversely, to every separable coordinate system $\{y^i\}$ for the Hamilton–Jacobi equation there corresponds a family $\mathcal{A}$ of second order Killing tensors on V_n with properties (1)–(7).

Proof. It is enough to show that conditions (1)–(7) imply that the one-forms $\theta_{(j)}$ are normalizable; the remainder of the proof follows immediately from Theorems 2 and 3.

From conditions (4) and (5) it follows that there exists a coordinate system $\{x^i\}$ on V_n such that $\theta_{(\alpha)}=dx^\alpha+\sum_{h=1}^{n_1+n_2}\lambda_{(\alpha)h}(x^a, x^r)\,dx^h$ and $X^{(\alpha)}=\partial_{x^\alpha}$. Clearly, conditions (4.1) hold for $h=\alpha$ and any values of k, l.

Some other conditions (4.1) follow directly from [2, proof of Theorem 5]. It follows from that proof that conditions (1), (2) and (7) imply $\gamma_{(lhk)}=\gamma_{(lkh)}=0$ for pairwise distinct numbers l, h, k such that $1\leqq l, h, k\leqq n_1+n_2$. Thus, (4.1) holds for $l=1,\cdots,n_1+n_2$ and $1\leqq h, k\leqq n_1$, $h, k\neq l$.

The remainder of the proof is essentially a systematic exploitation of condition (1) for $A, B\in\mathcal{A}$. Writing this condition in the form (3.3), multiplying by $\lambda_{(m_1)i}\lambda_{(m_2)j}\lambda_{(m_3)k}$ and summing for $i, j, k=1,\cdots,n$, we obtain an identity $E^{A,B}_{m_1,m_2,m_3}$. This identity can be simplified through use of conditions (2) and (3). In particular, condition (2) leads to

$$\partial_u a^{jv}=-a^{jw}\Lambda^{v(z)}\partial_u\lambda_{w(z)}+\partial_u(\rho^{(z)}g^{jw})\Lambda^{v(z)}\lambda_{w(z)}+\rho^{(z)}g^{jw}\Lambda^{v(z)}\partial_u\lambda_{w(z)}, \tag{4.6}$$

where in this and the following expressions u, v, w, z range from 1 to n_1+n_2 and $\partial_i=\partial_{x^i}$. Condition (3) leads to

$$\partial_u(\lambda_{i(\alpha)}a^{ij}\lambda_{j(\beta)})=\lambda_{u(z)}\Lambda^{v(z)}\rho^{(z)}\partial_v(\lambda_{i(\alpha)}g^{ij}\lambda_{j(\beta)}). \tag{4.7}$$

Furthermore, convariant differentiation of (4.2) leads to the relation

$$\gamma_{(hlk)}+\gamma_{(lhk)}=\lambda^{j}_{(k)}G_{(hl),j}. \tag{4.8}$$

Through use of these relations we can express the identities $E^{A,B}_{m_1,m_2,m_3}$ in terms of $\lambda_{i(u)}$, $\Lambda^{i(u)}$, $\rho^{(u)}$ and g^{ij} alone.

Let A, $B\in\mathscr{A}$ have roots $\{\rho_i\}$, $\{\mu_i\}$, respectively. Equating coefficients of $\rho_r\mu_s$ in $E^{A,B}_{c,\alpha,\beta}$ we find

$$G^{(rh')}G^{(s,k')}(\gamma_{(ch'k')}-\gamma_{(ck'h')})=0; \tag{4.9}$$

i.e., (4.1) holds for $(h, k, l)=(r, s, c)$. Equating coefficients of $\rho_a\mu_r$ in $E^{A,B}_{a,b,\alpha}$, $a\neq b$, we find similarly that (4.1) holds for $(h, k, l)=(r, a, c)$. Thus the forms $\theta_{(c)}$ are normalizable.

Equating coefficients of $\rho_r\mu_s$ in $E^{A,B}_{t,\alpha,\beta}$, r, s, t pairwise distinct, we verify that (4.1) holds for $(h, k, l)=(r, s, t)$. Finally, equating coefficients of ρ_r in $E^{A,H}_{a,t,\alpha}$, $r\neq t$, we verify that (4.1) holds for $(h, k, l)=(r, a, t)$. This shows that the forms $\theta_{(t)}$ are normalizable.

We see at this point that, by renormalization of $\theta_{(a)}$, $\theta_{(r)}$ if necessary, we can find local coordinates $\{y^i\}$ such that

$$\begin{gathered}\theta_{(a)}=dy^a,\quad a=1,\cdots,n_1,\qquad \theta_{(r)}=dy^r,\quad r=n_1+1,\cdots,n_1+n_2,\\ \theta_{(\alpha)}=dy^\alpha+\sum_{h=1}^{n_1+n_2}\lambda_{h(\alpha)}\,dy^h.\end{gathered} \tag{4.10}$$

Replacing $\theta_{(\alpha)}$ by $\hat\theta_{(\alpha)}=\theta_{(\alpha)}-\sum_{r=n_1+1}^{n_1+n_2}\lambda_{r(\alpha)}\theta_{(r)}$, we see that the new forms $\theta_{(\alpha)}$ (dropping the hat) satisfy conditions (1)–(7), since $G_{(ar)}=0$, and further that $\lambda_{r(\alpha)}=0$ for the new forms. Equating coefficients of ρ_b in $E^{A,H}_{b,c,\alpha}$, $b\neq c$, we find $\partial_b\lambda_{c(\alpha)}=\partial_c\lambda_{b(\alpha)}$, and equating coefficients of $\rho_c\mu_r$ in $E^{A,B}_{c,\alpha,\beta}$ we obtain $\partial_r\lambda_{(\beta)c}=0$. Thus $\theta_{(\alpha)}=dy^\alpha+df^\alpha$, where $df^\alpha=\sum_{c=1}^{n_1}\lambda_{c(\alpha)}\,dy^c$. Setting

$$z^h=y^h,\quad h=1,\cdots,n_1+n_2,\qquad z^\alpha=y^\alpha+f^\alpha, \tag{4.11}$$

we have $\theta_{(j)}=dz^j$, $X^{(j)}=\partial_{z^j}$, $j=1,\cdots,n$ and our one-forms are normalizable. □

Remark. It is sufficient to require that condition (3) of Theorem 4 be valid for $h_{n1}+1,\cdots,n_1+n_2$ since the identity $E^{A,H}_{a,\alpha,\beta}$ yields this condition for $h=1,\cdots,n_1$. Thus condition (3) is unnecessary when $n_2=0$.

5. An example. To show how Theorem 4 can be employed in practice we treat a single example in some detail. The real Hamilton–Jacobi equation

$$W_t^2-W_x^2-W_y^2=E \tag{5.1}$$

admits the pseudo-Euclidean algebra $e(2, 1)$ as its symmetry algebra of Killing vectors. A basis for the symmetry algebra is

$$\begin{gathered}K_1=xp_t+tp_x,\quad K_2=yp_t+tp_y,\quad L_3=yp_x-xp_y\\ P_0=p_t,\quad P_1=p_x,\quad P_2=p_y.\end{gathered} \tag{5.2}$$

As is well known (e.g., [2]), the space of second order Killing tensors for the pseudo-Riemannian manifold with (5.1) as its associated equation is spanned by products of Killing vectors (5.2). Thus, it is easy to display the second order Killing tensors for this manifold.

Recall that two separable coordinate systems for a Hamilton–Jacobi equation are considered as equivalent if the defining symmetry operators for the two systems are equivalent under the adjoint action of the local Lie symmetry group of the equation [5], [6], [10]. Thus, if we are looking for all separable coordinate systems with one ignorable variable we can limit our search to those cases where the Killing vector X_α corresponding to this variable is an explicitly chosen representative of one of the conjugacy classes of one-dimensional subalgebras of $e(2, 1)$. We consider the particularly interesting case where $L_\alpha = P_0 + P_2$. (As shown in [5], all nonorthogonal separable coordinates for (5.1) correspond to this case. Moreover, it is easily shown that any coordinate system with $P_0 + P_2$ as a generator for an ignorable coordinate must necessarily be nonorthogonal [10].) For such a system the Killing tensor A must commute with L_α. Thus A can be chosen from the real vector space of homogeneous second order polynomials in the symmetries (5.2). Furthermore, we can identify two Killing tensors that lie on the same orbit under the adjoint action of the normalizer for $P_0 + P_2$. The normalizer has basis $\{K_2, L_3 - K_1, P_0 + P_2, P_1, P_0 - P_2\}$. (See [11] for a more detailed discussion of this problem.) One family of orbit representatives is

$$(5.3)\quad (L_3 - K_1)^2 + 4P_1^2 + a(L_3 - K_1)(P_0 - P_2) + b(P_0 - P_2)^2 + c(P_0 + P_2)^2 + dP_1(P_0 - P_2).$$

(That is, two such representatives lie on the same orbit if and only if they are identical. We could, of course, easily compute all possible families of orbit representatives and apply the following considerations to each such family.) Group theory can take us no further than this point. We still have to determine which, if any, of the Killing tensors (5.3) actually correspond to separable coordinates.

In the following it is convenient to choose new coordinates $\{x, \tau, w\}$ such that $\tau = \frac{1}{2}(y + t)$, $w = \frac{1}{2}(y - t)$, so $p_\tau = p_y + p_t$, $p_w = p_y - p_t$. In terms of these coordinates,

$$(5.4)\qquad A - \rho H = \begin{bmatrix} 4w^2 + 4 + \rho & -2xw & -aw - \dfrac{d}{2} \\ -2xw & x^2 + c & \dfrac{ax + \rho}{2} \\ -aw - \dfrac{d}{2} & \dfrac{ax + \rho}{2} & b \end{bmatrix}.$$

Since ρ_2 is a double root, we must have $f'(\rho_2) = 0$. Also, $f(\rho) = \frac{1}{4}(\rho - \rho_2)^2(\rho - \rho_1)$. It is orthogonal, we must have $n_1 = n_2 = 1$ for any separable coordinates. Thus A must have a single root ρ_1 and a distinct root ρ_2 of multiplicity 2 which has only one eigenform. The characteristic equation $f(\rho) = \det(A - \rho H) = 0$ reads

$$(5.5)\qquad \begin{aligned} &\frac{\rho^3}{4} + \rho^2\left(w^2 + 1 + \frac{ax}{2}\right) - \rho\left(bx^2 - \frac{a^2}{4}x^2 - 2ax + cb - axw^2 + dxw\right) \\ &\quad + \left(dacw - c(4b - a^2)w^2 + \left(-4b + a^2 + \frac{d^2}{4}\right)x^2 + \frac{d^2c}{4} - 4cb\right) = 0. \end{aligned}$$

Since ρ_2 is a double root, we must have $f'(\rho_2) = 0$. Also, $f(\rho) = \frac{1}{4}(\rho - \rho_2)^2(\rho - \rho_1)$. It is straightforward, though tedious, to verify that these conditions on ρ_1, ρ_2 are inconsistent unless $a = b = c = d = 0$, in which case

$$(5.6)\qquad \rho_1 = -4(w^2 + 1), \qquad \rho_2 = 0.$$

Thus,

$$(5.7)\qquad A = \begin{bmatrix} 4w^2 + 4 & -2xw & 0 \\ -2xw & x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\theta_{(1)}=(2w^2+2)\,dx-2xw\,dw$, $\theta_{(2)}=dw$. To satisfy conditions (5) and (7) of Theorem 4 we must require $\theta_{(3)}=xw(w^2+1)^{-1}\,dx+d\tau+f\,dw$. We choose f such that $\theta_{(3)}$ is a perfect differential and obtain

$$(5.8)\qquad (\lambda_{j(k)})=\begin{bmatrix} 2w^2+2 & 0 & \dfrac{xw}{(w^2+1)} \\ 0 & 0 & 1 \\ -2xw & 1 & \dfrac{x^2(1-w^2)}{2(1+w^2)^2} \end{bmatrix},\qquad (\Lambda^{(l)j})=\begin{bmatrix} \dfrac{1}{2w^2+2} & \dfrac{-xw}{2(w^2+1)^2} & 0 \\ \dfrac{xw}{w^2+1} & \dfrac{-x^2}{2(1+w^2)} & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Condition (3) can be verified directly.

Finally, $Q=3$ and $\mathscr{A}$ has the basis $\{A, H, X^{(3)}\}$.

We conclude that among the operators (5.3) only $A=(L_3-K_1)^2+4P_1^2$ corresponds to a separable coordinate system. Furthermore, in this case it is now straightforward to derive the separable coordinates. They are $\{x^1, x^2, x^3\}$, where

$$(5.9)\qquad x=x^1[1+(x^2)^2]^{1/2},\qquad \tau=\frac{[x^3-(x^1)^2x^2]}{2},\qquad w=x^2.$$

Indeed,

$$(5.10)\qquad X^{(1)}=\tfrac{1}{2}(1+w^2)^{-3/2}\partial_1,\qquad X^{(2)}=\partial_2,\qquad X^{(3)}=2\partial_1.$$

REFERENCES

[1] L. P. EISENHART, *Riemannian Geometry*, Princeton University Press, Princeton, NJ, (2nd printing), 1949.

[2] E. G. KALNINS AND W. MILLER, Jr., *Killing tensors and variable separation for Hamilton–Jacobi and Helmoholtz equations*, this Journal, 11 (1980), pp. 1011–1026.

[3] W. MILLER, Jr., *Symmetry and Separation of Variables*, Addison-Wesley, Reading, MA, 1977.

[4] T. LEVI-CIVITA, *Sulla integrazione della equazione di Hamilton–Jacobi per separazione di variabili*, Math. Ann., 59 (1904), pp. 383–397.

[5] E. G. KALNINS AND W. MILLER, Jr., *Separable coordinates for three-dimensional complex Riemannian spaces*, J. Differential Geometry, 14 (1979), pp. 221–236.

[6] C. P. BOYER, E. G. KALNINS AND W. MILLER, Jr., *Separable coordinates for four-dimensional Riemannian spaces*, Commun. Math. Phys., 59 (1978), pp. 255–302.

[7] S. BENENTI, *Separability structures on Riemannian manifolds*, Proceedings of Conference on Differential Geometrical Methods in Mathematical Physics, Salamanca 1979, Lecture Notes in Mathematics, Springer-Verlag, to appear.

[8] S. BENENTI AND M. FRANCAVIGLIA, *The theory of separability of the Hamilton–Jacobi equation and its application to general relativity*, in General Relativity and Gravitation, Vol. 1, A. Held, ed., Plenum, New York, 1980.

[9] J. HAINZL, *Separation of variables and commuting operators*, Math. Meth. Appl. Sci., 1 (1979), pp. 468–479.

[10] E. G. KALNINS AND W. MILLER, Jr., *Nonorthogonal separable coordinate systems for the flat 4-space Helmholtz equation*, J. Phys. A. (GB), 12 (1979), pp. 1129–1147.

[11] W. MILLER, Jr., J. PATERA AND P. WINTERNITZ, *Subgroups of Lie groups and separation of variables*, J. Math. Phys. to appear.