6619 The Population Lab 05

Let us consider the problem of how to model the growth of a population of bacteria, people, or wolves. Let $P(t)$ denote the number of individuals in the population at time t. We constructed our first model by saying that the rate at which the number of individuals in the population increases is proportional to the number of individuals in the population. This gives the differential equation

$$\frac{dP}{dt} = kP \text{ with solution } P(t) = Ce^{kt},$$

where k is sometimes called the growth rate (but is not). The number k is different for different populations and must be found using some observed data about the population. Suppose $k = 0.04$, the initial population is 1000, and time is measured in years. A growth rate of $k = 0.04$ says that during the first year the population will grow from 1000 to $(1000 + 40)$ approximately.

This model for the growth of a population has a serious deficiency. This model says that the population grows and grows without bound. Suppose the population we are trying to model is a population of fish in a lake. Clearly the population cannot continue to grow and grow without ever reaching a maximum value. We need a new model that says that the number of fish cannot get larger than a certain number. There is only enough food and oxygen in the lake to support a certain number of fish. After this number is reached some of the fish die. We need a model that says that when the population reaches a certain level, then the environment can no longer support any more individuals. This number is called the carrying capacity, and we will denote it by N. In this case, it seems logical to assume that the rate at which the population grows is also proportional to $N - P$. This causes us to say that the rate of population growth is given by the logistic equation

$$\frac{dP}{dt} = kP(N - P).$$

Note that when $P(t) \equiv N$, then $\frac{dP}{dt} = 0$. This says that when the population reaches the carrying capacity, then the rate at which the population changes is zero. The population does not increase anymore. This makes this model seem more reasonable than our first model which allowed the population to grow without bound.

We can also add an outside influence to the growth of our population. For our population of fish in a lake we can assume that fishermen catch so many fish per year. Suppose time is measured in years, then let C denote the number of fish taken from the lake per year. We subtract this rate of loss from the rate of growth, that is, we subtract C fish per year from the growth rate. For any population we can think that the population is being harvested at a certain rate C. This means that the population is disappearing at the rate of C individuals per unit time and that this rate is not connected to natural growth. We write the differential equation for a population subject to harvesting as

$$\frac{dP}{dt} = kP(N - P) - C.$$

This is the logistic model for population $P(t)$, where t represents time. The number N is the carrying capacity and C is the harvesting rate. The actual numbers we use are small to make the solutions easier. You may like to think of N and P as being in units of one thousand or one million individuals, that is, $N = 50$ really means 50 thousand individuals. This lab will consist of solving this logistic equation in several cases. You also need to look at your solutions and draw some conclusions about the population growth.

Suppose that Lake Sunny has been stocked with a new kind of fish called " Wonderful Fish". It is a large lake. The carrying capacity is 80 thousand Wonderful Fish and the growth rate of Wonderful Fish is $k = 0.04$. We want to solve the following differential equation in several cases and explain the solution in terms of what will happen to the number of fish in the lake.

$$\frac{dP}{dt} = kP(N - P) - C.$$

1. a)Solve this differential equation when $k = 0.04$, $N = 50$, and $C = 0$. Solve this solutions for $P(t)$.

(b) Solve the initial value problem $P'(t) = 0.04P(50 - P)$, $P(0) = 25$. For this solution find $\lim_{t\to+\infty} P(t)$. When $P = 25$, what is the grwoth rate $\frac{dP}{dt}$?

(c) Solve the initial value problem $P'(t) = (0.04)P(50-P)$, $P(0) = 50$. What is the growth rate $\dfrac{dP}{dt}$ when $P = 50$?

(d) Solve the initial value problem $P'(t) = (0.04)P(50 - P)$, $P(0) = 80$. For this solution find $\lim_{t\to+\infty} P(t)$. When $P = 80$, what is the growth rate $\frac{dP}{dt}$?

2.a) Solve the differential equation when $k = 0.04$, $N = 50$, and $C = 21$. Solve this solution for $P(t)$.

(b) Solve the initial value problem $P'(t) = (0.04)P(50 - P) - 21$, $P(0) = 10$. For what value of t, with $t > 0$, is $P(t) = 0$?

(c) Solve the initial value problem $P'(t) = (0.04)P(50 - P) - 21$, $P(0) = 20$. Find $\lim_{t\to+\infty} P(t)$. You may use L'Hospital's rule to find the limit. What is the value of $\frac{dP}{dt}$, the growth rate, when $P = 20$?

(d) Solve the initial value problem $P'(t) = (0.04)P(50 - P) - 21$, $P(0) = 55$. When $P(0) = 55$, what is $\lim_{t\to+\infty} P(t)$? What is the value of $\frac{dP}{dt}$, the growth rate, when $P = 55$?

3.a) Solve the differential equation when $k = 0.04$, $N = 50$, and $C = 25$. Solve this solution for $P(t)$.

(b) Solve the initial value problem $P'(t) = (0.04)P(50 - P) - 25$, $P(0) = 25$. What is the value of $\frac{dP}{dt}$, the growth rate, when $P = 25$?

(c) Solve the initial value problem $P'(t) = (0.04)P(50 - P) - 25$, $P(0) = 50$. What is the value of $\frac{dp}{dt}$, the growth rate, when $P = 50$? What $P(0) = 50$, what is $\lim_{t \to +\infty} P(t)$?

4.a) Solve the differential equation in the case $k = 0.04$, $N = 50$, and $C = 29$. Solve this solution for $P(t)$.

(b) Solve the initial value problen $P'(t) = (0.04)P(50 - P) - 29$, $P(0) = 25$.

(c) Solve the initial value problem $P'(t) = (0.04)P(50 - P) - 29$, $P(0) = 35$. For what value of t, $t > 0$, is $P(t) = 0$?

5. Use Euler's method to obtain a solution for the initial value problem

$$\frac{dP}{dt} = (0.04)P(50 - P) - 21(1 + \sin 2t)$$

(a) $P(0) = 30$ (b) $P(0) = 60$

Find the solution on the interval $0 \leq t \leq 6$ in the two cases. You should use $n = 50$ or larger in the Euler method. The Euler method can be done on Excel. It can also be done using a calculator.

We are looking for a professional quality report. This includes careful explanations and detailed graphs and diagrams. You should explain each method you use, but you do not have to explain a method every time you use it. You should include at least one paragraph of introduction. Another paragraph of summary at the end would also be nice.

When you hand in your project it should have a cover page. The cover page should have the course (Math 1372), section number, title (Spring semester project population lab) and should list group members. If some member of your group does not do any work then leave their name off the report.

Also make it clear which question you are answering. Write your answers in a sentence. Do not just write: answer 4. Write something like: "when $P = 40$, the growth rate is $\frac{dP}{dt} = 4$".