MODELS OF q-ALGEBRA REPRESENTATIONS: q-INTEGRAL TRANSFORMS AND "ADDITION THEOREMS"

E.G. KALNINS† AND WILLARD MILLER, JR.‡

ABSTRACT. In his classic book on group representations and special functions Vilenkin studied the matrix elements of irreducible representations of the Euclidean and oscillator Lie algebras with respect to countable bases of eigenfunctions of the Cartan subalgebras, and he computed the summation identities for Bessel functions and Laguerre polynomials associated with the addition theorems for these matrix elements. He also studied matrix elements of the pseudo-Euclidean and pseudo-oscillator algebras with respect to the continuum bases of generalized eigenfunctions of the Cartan subalgebras of these Lie algebras and this resulted in realizations of the addition theorems for the matrix elements as integral transform identities for Bessel functions and for confluent hypergeometric functions. Here we work out q-analogs of these results in which the usual exponential function mapping from the Lie algebra to the Lie group is replaced by the q-exponential mappings E_q and e_q. This study of representations of the Euclidean quantum algebra and the q-oscillator algebra (not a quantum algebra) leads to summation, integral transform and q-integral transform identities for q-analogs of the Bessel and confluent hypergeometric functions, extending the results of Vilenkin for the $q = 1$ case.

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1. Introduction. This paper is part of a series on the study of function space models of irreducible representations of q-algebras [1-4]. These algebras and models are motivated by recurrence relations satisfied by q-hypergeometric functions [5-11] and our treatment is an alternative to the theory of quantum groups [12-23]. In our earlier papers we considered irreducible representations of q-analogs of the three-dimensional Euclidean Lie algebra and the four-dimensional oscillator algebra (not a quantum algebra). We replaced the usual exponential function mapping from the Lie algebra to the Lie group by the q-exponential mappings E_q and e_q. In place of the usual matrix elements on the group (arising from an irreducible representation) we found several different types of matrix elements expressible in terms of q-hypergeometric series. These q-matrix elements do not satisfy group homomorphism properties, so they do not lead to addition theorems in the usual sense, but to various q-analogs of addition theorems. All of the matrix elements

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†Department of Mathematics and Statistics, University of Waikato, Hamilton, New Zealand
‡School of Mathematics and Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, Minnesota 55455. Work supported in part by the National Science Foundation under grant DMS 91–100324

are determined with respect to countable bases of eigenfunctions of the "Cartan subalgebra" of the q-algebra.

In his classic book [24] Vilenkin studied the matrix elements of irreducible representations of the Euclidean and oscillator Lie algebras with respect to these same countable bases, and he computed the identities for Bessel functions and Laguerre polynomials associated with the addition theorems for these matrix elements. However, he also studied matrix elements of the pseudo-Euclidean and pseudo-oscillator algebras with respect to the continuum bases of generalized eigenfunctions of the Cartan subalgebras of these Lie algebras. (See also [25, 26] in these regards.) These studies resulted in realizations of the addition theorems for the matrix elements as integral transform identities for Bessel functions and for confluent hypergeometric functions. Here we work out q-analogs of these results.

In §2 we introduce a family of four-parameter q-matrix elements for the unitary irreducible representations of the Euclidean Lie algebra with respect to the standard countable eigenbasis and work out an associated "addition theorem" for these matrix elements". (These functions were introduced earlier in [27] as generating functions for q-Bessel functions but their role as matrix elements obeying an "addition theorem" was not pursued. Simultaneously with the issuance of a first preprint of our results Koelink [28] issued a preprint in which he proved this same addition theorem and reinterpreted it to yield a q-analogue of an integral of Weber and Sonine and of the Fourier-Bessel transform.) Then, in analogy with Vilenkin's work for true group representations, we introduce a q-analog of matrix elements of the pseudo-Euclidean group with respect to a continuum basis of generalized eigenfunctions. This study involves use of the Mellin transform and leads to integral transform identities for q-Bessel functions, interpreted as "addition theorems" that in the limit as $q \to 1$ go to identities derived by Vilenkin. In §3 we introduce a different q-analog of the pseudo-Euclidean group and apply the same procedures. This time it is the complex Fourier series that is relevant and the "addition theorems" lead to q-integral transform identities for q-Bessel functions.

In §3 and §4 we apply the same ideas to q-analogs of the oscillator and pseudo-oscillator algebras (these are not quantum groups) and obtain discrete, integral transform and q-integral transform identities for q-analogs of the confluent hypergeometric functions, extending the results of Vilenkin for the $q = 1$ case.

The notation used for q-series and q-integrals in this paper follows that of Gasper and Rahman [29].

2. Matrix elements of $m(2)$ representations. The three dimensional Lie algebra $m(2)$ is determined by its generators H, E_+, E_- which obey the commutation relations

$$[H, E_+] = E_+, \quad [H, E_-] = -E_-, \qquad (2.1)$$
$$[E_+, E_-] = 0.$$

We consider irreducible representations $(\omega$ of $m(2)$, characterized by the positive number ω. The spectrum of H corresponding to (ω) is the set $Z = \{m : m \text{an integer}\}$ and the complex representation space has basis vectors f_m, $m \in Z$, such that

$$E_\pm f_m = \omega f_{m\pm 1}, \; H f_m = m f_m, \; E_+ E_- f_m = \omega^2 f_m, \qquad (2.2)$$

where $C \equiv E_+E_-$ is an invariant operator. A simple realization of (ω) is given by the operators

$$H = m_0 + z\frac{d}{dz},\ E_+ = \omega z,\ E_- = \frac{\omega}{z} \tag{2.3}$$

acting on the space of all linear combinations of the functions z^n, z a complex variable, $n \in Z$, with basis vectors $f_m(z) = z^m$.

We can introduce an inner product such that $< f_n, f_{n'} >= \delta_{nn'}$, $n, n' \in Z$. On the dense subspace $\mathcal{K}$ of all finite linear combinations of the basis vectors we have

$$< E_+f, f' > = < f, E_-f' >, \quad < Hf, f' > = < f, Hf' >, \tag{2.4}$$

for all $f, f' \in \mathcal{K}$, so $H = H^*$ and $E_+^* = E_-$. In terms of the operators (2.3) we can obtain a realization of (ω) and its Hilbert space structure by setting $z = e^{i\theta}$:

$$\begin{aligned} &H = -i\frac{d}{d\theta},\ E_+ = \omega e^{i\theta},\ E_- = \omega e^{-i\theta}, \\ &f_n(z) = e^{in\theta},\ < f, f' >= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{f'(e^{i\theta})}\, d\theta. \end{aligned} \tag{2.5}$$

Matrix elements $T_{m'm}$ of the complex motion group in the representation (ω) are typically defined by the expansions

$$e^{\beta E_+}e^{\alpha E_-}e^{\tau H}f_m = \sum_{m'=-\infty}^{\infty} T_{m'm}(\alpha, \beta, \tau)f_{m'}, \tag{2.6}$$

[2, 10, 30]. The group multiplication property of the operators on the left-hand side of (2.6) leads to addition theorems for the matrix elements. For convenience in the computations to follow we shall limit ourselves to the case where $\tau = 0$.

With the q-analogs of the exponential function

$$\begin{aligned} e_q(x) &= \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty},\ |x| < 1, \\ E_q(x) &= \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}x^k}{(q;q)_k} = (-x;q)_\infty, \end{aligned} \tag{2.7}$$

we employ the model (2.3) to define the following q-analogs of matrix elements of (ω), [2]:

$$\begin{aligned} &a)\ e_q(\beta E_+)e_q(\alpha E_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e,e)}(\alpha,\beta)f_{n'},\ |\omega\alpha|, |\omega\beta| < 1 \\ &b)\ e_q(\beta E_+)E_q(\alpha E_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e,E)}(\alpha,\beta)f_{n'},\ |\omega\beta| < 1 \\ &c)\ E_q(\beta E_+)e_q(\alpha E_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(E,e)}(\alpha,\beta)f_{n'},\ |\omega\alpha| < 1 \\ &d)\ E_q(\beta E_+)E_q(\alpha E_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(E,E)}(\alpha,\beta)f_{n'}, \\ &e)\ e_q(\alpha E_+)E_q(\beta E_+)e_q(\gamma E_-)E_q(\delta E_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}(\alpha,\beta,\gamma,\delta)f_{n'}, \\ &\qquad |\omega\gamma| < |z| < 1/|\omega\alpha|. \end{aligned} \tag{2.8}$$

Here, $0 < q < 1$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. (All of these matrix elements, except (2.8e), were studied in [2].) Since $E_+^* = E_-$ we have

$$\begin{aligned} T^{(e,e)}_{n'n}(\alpha,\beta) &= < e_q(\beta E_+)e_q(\alpha E_-)f_n,\ f_{n'} >=< f_n,\ e_q(\overline{\alpha}E_+)e_q(\overline{\beta}E_-)f_{n'} > \\ &= \overline{T^{(e,e)}_{nn'}(\overline{\beta},\overline{\alpha})} = T^{(e,e)}_{nn'}(\beta,\alpha), \end{aligned} \tag{2.9}$$

$$T^{(e,E)}_{n'n}(\alpha,\beta) = T^{(E,e)}_{nn'}(\beta,\alpha), \quad T^{(E,E)}_{n'n}(\alpha,\beta) = T^{(E,E)}_{nn'}(\beta,\alpha),$$

$$T_{n'n}(\alpha,\beta,\gamma,\delta) = T_{nn'}(\gamma,\delta,\alpha,\beta). \tag{2.10}$$

Furthermore, since $e_q(x)E_q(-x) = 1$, we have the identities

$$\sum_{\ell=-\infty}^{\infty} T^{(e,e)}_{n'\ell}(\alpha,\beta)T^{(E,E)}_{\ell n}(-\alpha,-\beta) = \delta_{n'n},\ |\omega\alpha|, |\omega\beta| < 1 \tag{2.11}$$

$$\sum_{\ell=-\infty}^{\infty} T^{(e,E)}_{n'\ell}(\alpha,\beta)T^{(E,e)}_{\ell n}(-\alpha,-\beta) = \delta_{n'n},\ |\omega\alpha|, |\omega\beta| < 1, \tag{2.12}$$

and, of primary interest here:

$$\begin{aligned} T_{n'n}(\alpha,\beta,\gamma,\delta) &= \sum_{\ell=-\infty}^{\infty} T^{(e,e)}_{n'\ell}(\gamma,\alpha)T^{(E,E)}_{\ell n}(\delta,\beta), \quad |\gamma\omega|, |\alpha\omega| < 1, \\ &= \sum_{\ell=-\infty}^{\infty} T^{(e,E)}_{n'\ell}(\delta,\alpha)T^{(E,e)}_{\ell n}(\gamma,\beta), \quad |\gamma\omega|, |\alpha\omega| < 1, \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\sum_{\ell=-\infty}^{\infty} T_{n'\ell}(\alpha,\beta,\gamma,\delta)T_{\ell n}(\alpha',-\alpha,\gamma',-\gamma) = T_{n'n}(\alpha',\beta,\gamma',\delta), \\ &\qquad\qquad |\alpha\gamma\omega^2|, |\alpha'\gamma'\omega^2|, |\alpha'\gamma\omega^2|, |\alpha\gamma'\omega^2| < 1. \end{aligned} \tag{2.14}$$

(Note that our operator derivations of these formulas and of many formulas to follow lead automatically to formal power series identities in the 'group parameters'. These identities must then be examined case by case to determine when the series are convergent as analytic functions of the group parameters.) Using the model (2.3) to treat (2.8) as generating functions for the matrix elements and computing the coefficients of $z^{n'}$ in the resulting expressions we obtain the explicit results:

$$\begin{aligned} T^{(e,e)}_{n'n}(\alpha,\beta) &= \frac{(q^{n-n'+1};q)_\infty(\alpha\omega)^{n-n'}}{(q;q)_\infty} {}_2\phi_1\left(\begin{matrix} 0, & 0 \\ q^{n-n'+1} & \end{matrix}; q, \alpha\beta\omega^2\right) \\ &= \frac{(q^{n-n'+1};q)_\infty(\alpha\omega)^{n-n'}}{(q;q)_\infty(\alpha\beta\omega^2;q)_\infty} {}_0\phi_1\left(\begin{matrix} - \\ q^{n-n'+1} \end{matrix}; q, \alpha\beta\omega^2 q^{n-n'+1}\right), \end{aligned} \tag{2.15}$$

(2.16)
$$T^{(e,E)}_{n'n}(\alpha,\beta)=\frac{(q^{n-n'+1};q)_\infty(\alpha\omega)^{n-n'}}{(q;q)_\infty}q^{(n-n')(n-n'-1)/2}{}_1\phi_1\left(\begin{matrix}0\\ q^{n-n'+1}\end{matrix};q,-\alpha\beta\omega^2q^{n-n'}\right)$$
$$=\frac{(q^{n'-n+1};q)_\infty(\beta\omega)^{n'-n}}{(q;q)_\infty}{}_1\phi_1\left(\begin{matrix}0\\ q^{n'-n+1}\end{matrix};q,-\alpha\beta\omega^2\right),$$

$$T^{(E,e)}_{n'n}(\alpha,\beta)=T^{(e,E)}_{nn'}(\beta,\alpha), \tag{2.17}$$

(2.18)
$$T^{(E,E)}_{n'n}(\alpha,\beta)=\frac{(q^{n-n'+1};q)_\infty(\alpha\omega)^{n-n'}}{(q;q)_\infty}q^{(n-n')(n-n'-1)/2}{}_0\phi_1\left(\begin{matrix}-\\ q^{n-n'+1}\end{matrix};q,\alpha\beta\omega^2q^{n-n'}\right).$$

If $\alpha\beta\neq 0$ we can express these results in terms of the Jackson q-Bessel functions [28, page 25]

$$J^{(1)}_\nu(z;q)=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(\frac{z}{2})^\nu{}_2\phi_1\left(\begin{matrix}0, & 0\\ q^{\nu+1}\end{matrix};q,-\frac{z^2}{4}\right),$$

$$J^{(2)}_\nu(z;q)=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(\frac{z}{2})^\nu{}_0\phi_1\left(\begin{matrix}-\\ q^{\nu+1}\end{matrix};q,-\frac{z^2q^{\nu+1}}{4}\right), \tag{2.19}$$

$$J^{(2)}_\nu(z;q)=(-z^2/4;q)_\infty J^{(1)}_\nu(z;q),$$

and the Hahn-Exton q-Bessel function [27]

$$J_\nu(z;q)=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}z^\nu{}_1\phi_1\left(\begin{matrix}0\\ q^{\nu+1}\end{matrix};q,qz^2\right). \tag{2.20}$$

Indeed, setting $\alpha=ire^{i\psi}$, $\beta=ire^{-i\psi}$, we see that in terms of the new complex coordinates $[r,e^{i\psi}]$ we have

$$\begin{aligned}T^{(e,e)}_{n'n}(\alpha,\beta)\equiv T^{(e,e)}_{n'n}[r,e^{i\psi}]&=\frac{e^{i(\frac{\pi}{2}+\psi)(n-n')}}{(-r^2\omega^2;q)_\infty}J^{(2)}_{n-n'}(2r\omega;q)\\ T^{(e,E)}_{n'n}[r,e^{i\psi}]&=e^{i(\frac{\pi}{2}-\psi)(n'-n)}q^{(n'-n)/2}J_{n'-n}(r\omega q^{-\frac12};q)\\ T^{(E,e)}_{n'n}[r,e^{i\psi}]&=e^{i(\frac{\pi}{2}+\psi)(n-n')}q^{(n-n')/2}J_{n-n'}(r\omega q^{-\frac12};q)\\ T^{(E,E)}_{n'n}[r,e^{i\psi}]&=e^{i(\frac{\pi}{2}+\psi)(n-n')}q^{(n-n')^2/2}J^{(2)}_{n-n'}(2r\omega q^{-\frac12};q).\end{aligned} \tag{2.21}$$

(Note that $J_{-n}(z;q)=(-1)^nq^{n/2}J_n(zq^{n/2};q)$, $J^{(2)}_{-n}(z;q)=(-1)^nJ^{(2)}_n(z;q)$ for integer n.)

For the matrix elements (2.8e) we obtain (through the use of the q-binomial theorem on the factors involving E_+ and, separately, on the factors involving E_-)

(2.22)
$$T_{n'n}(\alpha,\beta,\gamma,\delta)=\frac{(\gamma\omega)^{n-n'}(-\delta/\gamma,q)_{n-n'}}{(q;q)_{n-n'}}{}_2\phi_1\left(\begin{matrix}-\beta/\alpha, & -\delta q^{n-n'}/\gamma\\ q^{n-n'+1}\end{matrix};q,\alpha\gamma\omega^2\right)$$

$$= \frac{(\alpha\omega)^{n'-n}(-\beta/\alpha,q)_{n'-n}}{(q;q)_{n'-n}} {}_2\phi_1\left(\begin{matrix}-\beta q^{n'-n}/\alpha, & -\delta/\gamma \\ q^{n'-n+1} & \end{matrix}; q, \alpha\gamma\omega^2\right), |\alpha\gamma\omega^2| < 1.$$

Alternatively, we could obtain this result by writing the matrix element as a contour integral

$$T_{n'n}(\alpha,\beta,\gamma,\delta) = \frac{1}{2\pi i}\oint \frac{(-\beta\omega z;q)_\infty(-\delta\omega/z;q)_\infty}{(\alpha\omega z;q)_\infty(\gamma\omega/z;q)_\infty} z^{n-n'-1}\, dz, \tag{2.23}$$

where the contour is the unit circle centered at the origin of the complex z-plane, and evaluating the integral by residues. Setting $\alpha = (1-q)\alpha', \cdots, \delta = (1-q)\delta'$ we see that in the limit as $q \to 1$,

$$T_{n'n}(\alpha,\beta,\gamma,\delta) \to \frac{[\omega(\gamma'+\delta')]^{n-n'}}{\Gamma(n-n'+1)} {}_0F_1\left(\begin{matrix}- \\ n-n'+1\end{matrix}; (\alpha+\beta)(\gamma+\delta)\omega^2\right), \tag{2.24}$$

expressible in terms of ordinary Bessel functions, [24].

The second of equations (2.13) was already derived by Koornwinder and Swarttouw [27, Proposition 4.1] where it was interpreted as a q-analog of Graf's addition formula for Bessel functions:

$$\left(\frac{y-x/s}{y-xs}\right)^{n/2} J_n\left(\sqrt{(y-x/s)(y-xs)}\right) = \sum_{k=-\infty}^{\infty} s^k J_{n+k}(y)J_k(x).$$

The first equation (2.13) and (2.14) have similar interpretations.

The "addition theorem" (2.14) reads, [28],

(2.25)
$$\begin{aligned}
&\frac{(\gamma'\omega)^{n-n'}(-\delta/\gamma';q)_{n-n'}}{(q;q)_{n-n'}} {}_2\phi_1\left(\begin{matrix}-\beta/\alpha', & -\delta q^{n-n'}/\gamma' \\ q^{n-n'+1} & \end{matrix}; q, \alpha'\gamma'\omega^2\right) = \\
&\sum_{\ell=-\infty}^{\infty} \frac{(\gamma\omega)^{\ell-n'}(-\delta/\gamma;q)_{\ell-n'}}{(q;q)_{\ell-n'}} {}_2\phi_1\left(\begin{matrix}-\beta/\alpha, & -\delta q^{\ell-n'}/\gamma \\ q^{\ell-n'+1} & \end{matrix}; q, \alpha\gamma\omega^2\right) \times \\
&\quad \frac{(\alpha'\omega)^{\ell-n}(\alpha/\alpha';q)_{\ell-n}}{(q;q)_{\ell-n}} {}_2\phi_1\left(\begin{matrix}\alpha q^{\ell-n}/\alpha', & \gamma/\gamma' \\ q^{\ell-n+1} & \end{matrix}; q, \alpha'\gamma'\omega^2\right), \\
&\quad |\alpha\gamma\omega^2|, |\alpha'\gamma'\omega^2|, |\alpha'\gamma\omega^2|, |\alpha\gamma'\omega^2| < 1.
\end{aligned}$$

Next we introduce a model of a q-analog of the pseudo-Euclidean group. The model consists of a Hilbert space of complex valued functions $f(x) \equiv \hat{f}(\theta)$ where $x = e^\theta$ and θ is a real variable, such that $||f||^2 < \infty$ and the inner product is

$$< f, g >= \int_0^\infty f(x)\overline{g}(x)\, \frac{dx}{x} = \int_{-\infty}^\infty \hat{f}(\theta)\overline{\hat{g}}(\theta)\, d\theta, \tag{2.26}$$

and $||f||^2 =< f, f >$. The formal action of the q-algebra is

$$E_+ = \omega x = \omega e^\theta \quad E_- = \frac{\omega}{x} = \omega e^{-\theta}, \quad H = x\frac{d}{dx} = \frac{d}{d\theta}.$$

The action of the "pseudo-Euclidean group" is given by the formal operator $\mathbf{T}(\alpha,\beta,\gamma,\delta)$ where

(2.27)
$$\begin{aligned}\mathbf{T}(\alpha,\beta,\gamma,\delta)\hat{f}(\theta) &= e_q(-\alpha E_+)E_q(\beta E_+)e_q(-\gamma E_-)E_q(\delta E_-)\hat{f}(\theta)\\ &= \frac{(-\beta\omega e^{\theta},-\delta\omega e^{-\theta};q)_\infty}{(-\alpha\omega e^{\theta},-\gamma\omega e^{-\theta};q)_\infty}\hat{f}(\theta).\end{aligned}$$

We require that neither α or γ is negative, so that the denominator in (2.27) never vanishes. Then for various values of the parameters $\alpha,\beta,\gamma,\delta$ the operator $\mathbf{T}$ corresponds to multiplication by a bounded function of θ. In particular, the multiplier is bounded under the following circumstances:

$$\begin{cases} 1) & |\beta/\alpha|<1, |\delta/\gamma|<1\\ 2) & |\beta/\alpha|<1, \delta=\gamma=0\\ 3) & |\delta/\gamma|<1, \beta=\alpha=0\\ 4) & \alpha=\beta=\gamma=\delta=0.\end{cases}$$

For these estimates we make use of the identities
(2.28)
$$(Aq^{-n};q)_\infty=(-A)^n q^{-\frac{n(n+1)}{2}}(\frac{q}{A};q)_n(A;q)_\infty,\quad \frac{(Aq^{-n};q)_\infty}{(Bq^{-n};q)_\infty}=\left(\frac{A}{B}\right)\frac{(\frac{q}{A};q)_n(A;q)_\infty}{(\frac{q}{B};q)_n(B;q)_\infty}.$$

Following [24, Chapter 5], we will compute the matrix elements of the operator $\mathbf{T}$ with respect to a continuum basis in which H is diagonalized. We first restrict our attention to the subspace $\mathcal{G}$ of the Hilbert space where $\mathcal{G}$ consists of those functions $\hat{f}$ that are C^∞ with compact support. Then as shown in [24, 26], the complex Fourier or Mellin transform

$$F(\lambda)=\int_{-\infty}^{\infty}\hat{f}(\theta)e^{\lambda\theta}\,d\theta=\int_0^\infty f(x)x^{\lambda-1}\,dx \tag{2.29}$$

has the properties that 1) $F(\lambda)$ is an entire (analytic) function of λ, 2) $|F(\lambda)|< Ce^{k|\mathrm{Re}\ \lambda|}$ for some positive constants C,k and 3) F decreases rapidly on every straight line parallel to the imaginary axis in the complex λ-plane, i.e., $\lim_{t\to\infty}|t|^n|F(c+it)|=0$ for $n=0,1,2,\cdots$.. (We denote the space of transforms of functions in $\mathcal{G}$ by $\hat{\mathcal{G}}$.) Furthermore we have the inversion formula

$$\hat{f}(\theta)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}F(\mu)e^{-\mu\theta}\,d\mu, \tag{2.30}$$

for any real number a.

Now, the induced action of the operator $q^{\phi H}$ on the transformed functions F becomes diagonal:

$$q^{\phi H}F(\lambda)\equiv\int_{-\infty}^{\infty}\left[q^{\phi H}\hat{f}(\theta)\right]e^{\lambda\theta}\,d\theta=\int_{-\infty}^{\infty}f(\theta+\phi)e^{\lambda\theta}\,d\theta=e^{-\lambda\phi}F(\lambda).$$

Furthermore, the induced action of the operator $\mathbf{T}$ on $\hat{\mathcal{G}}$ is given by

$$\mathbf{T}(\alpha,\beta,\gamma,\delta)F(\lambda)=\int_{-\infty}^{\infty}\frac{(-\beta\omega e^{\theta},-\delta\omega e^{-\theta};q)_\infty}{(-\alpha\omega e^{\theta},-\gamma\omega e^{-\theta};q)_\infty}\hat{f}(\theta)e^{\lambda\theta}\ d\theta,$$

or

$$\mathbf{T}(\alpha,\beta,\gamma,\delta)F(\lambda)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(-\beta\omega e^{\theta},-\delta\omega e^{-\theta};q)_\infty}{(-\alpha\omega e^{\theta},-\gamma\omega e^{-\theta};q)_\infty}d\theta\int_{a-i\infty}^{a+i\infty}F(\mu)e^{(\lambda-\mu)\theta}\ d\mu.$$

If $|\beta q^{a-\lambda}/\alpha|<1, |\delta q^{\lambda-a}/\gamma|<1$, then the iterated integral is absolutely convergent and we can interchange the order of integration to obtain

$$\mathbf{T}(\alpha,\beta,\gamma,\delta)F(\lambda)=\int_{a-i\infty}^{a+i\infty}K(\lambda,\mu;\alpha,\beta,\gamma,\delta)F(\mu)\ d\mu, \tag{2.31}$$

where

$$\begin{aligned}K(\lambda,\mu;\alpha,\beta,\gamma,\delta)&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(-\beta\omega e^{\theta},-\delta\omega e^{-\theta};q)_\infty}{(-\alpha\omega e^{\theta},-\gamma\omega e^{-\theta};q)_\infty}e^{(\lambda-\mu)\theta}\ d\theta\\ &=\frac{1}{2\pi i}\int_{0}^{\infty}\frac{(-\beta\omega x,-\delta\omega/x;q)_\infty}{(-\alpha\omega x,-\gamma\omega/x;q)_\infty}x^{\lambda-\mu-1}dx.\end{aligned}$$

To compute the kernel function K we evaluate the contour integral

$$I_{N,M}=\frac{1}{2\pi i}\oint_{C_{N,M}}\frac{(-\beta\omega z,-\delta\omega/z;q)_\infty}{(-\alpha\omega z,-\gamma\omega/z;q)_\infty}z^{\lambda-\mu-1}dz \tag{2.32}$$

along the closed contour $C_{N,M}$ on the Riemann surface of the integrand, where N, M are positive integers and the contour is made up of the curves

$$C_{N,M}\quad\begin{cases}1)\quad z=t, & \gamma\omega q^{M+1/2}\le t\le\frac{1}{\alpha\omega}q^{-N-1/2}\\ 2)\quad z=\frac{1}{\alpha\omega}q^{-N-1/2}e^{it}, & 0\le t\le 2\pi\\ 3)\quad z=e^{2\pi i}t, & \frac{1}{\alpha\omega}q^{-N-1/2}\ge t\ge\gamma\omega q^{M+1/2}\\ 4)\quad z=\gamma\omega q^{M+1/2}e^{it}, & 2\pi\ge t\ge 0.\end{cases}$$

In the limit as $N\to\infty, M\to\infty$ the integrals on the large and on the small circle go to zero if $|\beta q^{-\lambda+\mu}/\alpha|<1, |\delta q^{\lambda-\mu}/\gamma|<1$. Then, evaluating (2.32) by residues and (temporarily) assuming that $|\alpha\gamma\omega^2|<1$ to make use of Heine's transformation [29, page 9], we obtain

$$K(\lambda,\mu;\alpha,\beta,\gamma,\delta)=\frac{1}{2\pi i}\int_{0}^{\infty}\frac{(-\beta\omega x,-\delta\omega/x;q)_\infty}{(-\alpha\omega x,-\gamma\omega/x;q)_\infty}x^{\lambda-\mu-1}dx=$$

(2.33)

$$\frac{1}{2i\sin\pi(\mu-\lambda)}\left[\frac{(\frac{\beta}{\alpha},q^{\mu-\lambda+1};q)_\infty}{(q,\frac{\beta}{\alpha}q^{\mu-\lambda};q)_\infty}(\frac{1}{\alpha\omega})^{\lambda-\mu}{}_2\phi_1\left(\begin{matrix}\frac{\delta}{\gamma}, & \frac{\beta}{\alpha}q^{\mu-\lambda}\\ q^{\mu-\lambda+1} & \end{matrix};q,\alpha\gamma\omega^2\right)\right.$$
$$\left.+\frac{(\frac{\delta}{\gamma},q^{\lambda-\mu+1};q)_\infty}{(q,\frac{\delta}{\gamma}q^{\lambda-\mu};q)_\infty}(\gamma\omega)^{\lambda-\mu}{}_2\phi_1\left(\begin{matrix}\frac{\beta}{\alpha}, & \frac{\delta}{\gamma}q^{\lambda-\mu}\\ q^{\lambda-\mu+1} & \end{matrix};q,\alpha\gamma\omega^2\right)\right].$$

(The apparent singularities at $\alpha\gamma\omega^2 = q^{-n}$, $n = 0, 1, \cdots$, are removable.) The following special cases of (2.33) are of interest:

$$K(\lambda,\mu;\alpha,\alpha,\gamma,\delta) = \frac{1}{2i\sin\pi(\mu-\lambda)} \frac{(\frac{\delta}{\gamma}, q^{\lambda-\mu+1}; q)_\infty}{(q, \frac{\delta}{\gamma}q^{\lambda-\mu}; q)_\infty} (\gamma\omega)^{\lambda-\mu}, \quad |q^{-\lambda+\mu}| < 1, |\delta q^{\lambda-\mu}/\gamma| < 1,$$

(2.34)

$$K(\lambda,\mu;\alpha,\beta,\gamma,\gamma) = \frac{1}{2i\sin\pi(\mu-\lambda)} \frac{(\frac{\beta}{\alpha}, q^{\mu-\lambda+1}; q)_\infty}{(q, \frac{\beta}{\alpha}q^{\mu-\lambda}; q)_\infty} (\frac{1}{\alpha\omega})^{\lambda-\mu}, \quad |\beta q^{-\lambda+\mu}/\alpha| < 1, |q^{\lambda-\mu}| < 1,$$

$$K(\lambda,\mu;\alpha,0,\gamma,0) = \frac{1}{2i\sin\pi(\mu-\lambda)(\alpha\gamma\omega^2;q)_\infty} \left[\frac{(\frac{1}{\alpha\omega})^{\lambda-\mu}}{(q,;q)_\infty} {}_1\phi_1 \left(\begin{matrix} \alpha\gamma\omega^2 \\ 0 \end{matrix} ; q, q^{\mu-\lambda+1} \right) \right.$$
$$\left. + \frac{(\gamma\omega)^{\lambda-\mu}}{(q;q)_\infty} {}_1\phi_1 \left(\begin{matrix} \alpha\gamma\omega^2 \\ 0 \end{matrix} ; q, q^{\mu-\lambda+1} \right) \right].$$

Setting $\alpha = (1-q)\alpha', \cdots, \delta = (1-q)\delta'$ in (2.33) we see that in the limit as $q \to 1-$,

$$K(\lambda,\mu;\alpha,\beta,\gamma,\delta) \to \frac{1}{2i\sin\pi(\mu-\lambda)} \left[\frac{[\omega(\alpha'-\beta')]^{\mu-\lambda}}{\Gamma(\mu-\lambda+1)} {}_0F_1 \left(\begin{matrix} - \\ \mu-\lambda+1 \end{matrix} ; (\alpha'-\beta')(\gamma'-\delta')\omega^2 \right) \right.$$

(2.35)
$$\left. + \frac{[\omega(\gamma'-\delta')]^{\lambda-\mu}}{\Gamma(\lambda-\mu+1)} {}_0F_1 \left(\begin{matrix} - \\ \lambda-\mu+1 \end{matrix} ; (\alpha'-\beta')(\gamma'-\delta')\omega^2 \right) \right].$$

From the expression (2.27) we have the "addition formula"

$$\mathbf{T}(\alpha,\beta,\gamma,\delta)\mathbf{T}(\alpha',\alpha,\gamma',\gamma) = \mathbf{T}(\alpha',\beta,\gamma',\delta)$$

which, for the kernel functions, takes the form

(2.36)
$$\int_{b-i\infty}^{b+i\infty} K(\lambda,\mu;\alpha',\beta,\gamma',\delta)F(\mu)\,d\mu =$$
$$\int_{b-i\infty}^{b+i\infty} K(\lambda,\nu;\alpha,\beta,\gamma,\delta)\,d\nu \int_{a-i\infty}^{a+i\infty} K(\nu,\mu;\alpha',\alpha,\gamma',\gamma)F(\mu)\,d\mu,$$

see [24, page 268]. Then, if the integrals in (2.36) are absolutely convergent we have the functional relations

(2.37)
$$K(\lambda,\mu;\alpha',\beta,\gamma',\delta) = \int_{a-i\infty}^{a+i\infty} K(\lambda,\nu;\alpha,\beta,\gamma,\delta)K(\nu,\mu;\alpha',\alpha,\gamma',\gamma)\,d\nu,$$

$$|\alpha q^\mu/\alpha'|, |\delta q^\lambda/\gamma| < |q^a| < |\alpha q^\lambda/\beta|, |\gamma' q^\mu/\gamma|.$$

Two special cases of (2.37) are of particular interest. If $\alpha = \beta, \gamma' = \gamma$ we have

$$K(\lambda,\mu;\alpha',\beta,\gamma,\delta) = \frac{1}{4}\int_{a-i\infty}^{a+i\infty} \frac{(\gamma\omega)^{\lambda-\nu}(\frac{1}{\alpha'\omega})^{\nu-\mu}}{\sin\pi(\nu-\lambda)\sin\pi(\nu-\mu)} \frac{(\frac{\delta}{\gamma}, q^{\lambda-\nu+1}, \frac{\beta}{\alpha'}, q^{\mu-\nu+1}; q)_\infty}{(q, \frac{\delta}{\gamma}q^{\lambda-\nu}, q, \frac{\beta}{\alpha'}q^{\mu-\nu}; q)_\infty}\,d\nu,$$

$$|\delta q^\lambda/\gamma|, |\beta q^\mu/\alpha'| < q^a < |q^\lambda|, |q^\mu|,$$

and if just $\gamma = \gamma'$ we have

$$K(\lambda,\mu;\alpha',\beta,\gamma,\delta) = \int_{a-i\infty}^{a+i\infty} K(\lambda,\nu;\alpha,\beta,\gamma,\delta) \frac{(\frac{1}{\alpha'\omega})^{\nu-\mu}}{2i\sin\pi(\mu-\nu)} \frac{(\frac{\alpha}{\alpha'}, q^{\mu-\nu+1}; q)_\infty}{(q, \frac{\alpha}{\alpha'}q^{\mu-\nu}; q)_\infty}\,d\nu.$$

3. A discrete model of $m(2)$ representations. In this section we study a model of an alternate q-analog of the pseudo-Euclidean group. Here the generators of our algebra are q^H, E_+, E_- which obey the relations

$$q^H E_+ = qE_+q^H, \quad qq^H E_- = E_-q^H, \quad [E_+, E_-] = 0. \tag{3.1}$$

We consider the following class of irreducible representations (ω) for this algebra, characterized by the positive number ω. The Hilbert space consists of complex functions $f(x)$ with domain $x = q^n$, $n = 0, \pm 1, \pm 2, \cdots$ and such that $(f, f) < \infty$, where the inner product is

$$(f, g) = \sum_{n=-\infty}^{\infty} f(q^n)\overline{g}(q^n). \tag{3.2}$$

The action of the algebra on this Hilbert space is given by the operators

$$E_+ = \omega x, \quad E_- = \frac{\omega}{x}, \quad q^H f(x) = f(qx). \tag{3.3}$$

To define these operators rigorously we can restrict their action to, say, the dense subspace $\mathfrak{L}$ of all functions in the Hilbert space that are nonzero at only a finite number of points. Then it is easy to show that $E_+^* = E_+$, $E_-^* = E_-$, $(q^H)^* = (q^H)^{-1}$. We define the (inverse) Fourier transform $\mathfrak{F}$ of $f \in \mathfrak{L}$ by

$$\mathfrak{F}(\lambda) \equiv \mathfrak{F}\,[z] = (f, x^{\overline{\lambda}}) = \sum_{n=-\infty}^{\infty} f(q^n)q^{n\lambda}, \quad \lambda \in \mathbb{C}, \tag{3.4}$$

where $z = q^\lambda$. Then the induced action of the algebra on the transform space $\hat{\mathfrak{L}}$ is

$$q^H\mathfrak{F}\,[z] = z^{-1}\mathfrak{F}\,[z], \quad E_\pm\mathfrak{F}\,[z] = \omega\mathfrak{F}\,[q^{\pm 1}z],$$

so the operator q^H is diagonalized in the transform space. Clearly, every $\mathfrak{F} \in \hat{\mathfrak{L}}$ is analytic for all $z \neq 0$. We can recover f from its transform $\mathfrak{F}$ via the formula

$$f(q^m) = \frac{1}{2\pi i}\oint \mathfrak{F}\,[z]z^{-m-1}\,dz \tag{3.5}$$

where the integration path is a simple closed curve around the origin in the z-plane.

Now the action of the "pseudo-Euclidean group" is given by the operator $\mathbf{T}(\alpha, \beta, \gamma, \delta)$ where

$$\begin{aligned}\mathbf{T}(\alpha, \beta, \gamma, \delta)f(x) &= e_q(-\alpha E_+)E_q(\beta E_+)e_q(-\gamma E_-)E_q(\delta E_-)f(x) \\ &= \frac{(-\beta\omega x, -\delta\omega/x; q)_\infty}{(-\alpha\omega x, -\gamma\omega/x; q)_\infty}f(x), \quad f \in \mathfrak{L}.\end{aligned} \tag{3.6}$$

The induced action of $\mathbf{T}$ on $\hat{\mathfrak{L}}$ is given by

$$\mathbf{T}(\alpha, \beta, \gamma, \delta)\mathfrak{F}\,[z] = \oint K(z/w; \alpha, \beta, \gamma, \delta)\mathfrak{F}\,[w]\,\frac{dw}{w}, \tag{3.7}$$

where

$$
\begin{aligned}
K(z/w;\alpha,\beta,\gamma,\delta) &= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\frac{(-\beta\omega q^n,-\delta\omega q^{-n};q)_\infty}{(-\alpha\omega q^n,-\gamma\omega q^{-n};q)_\infty}\left(\frac{z}{w}\right)^n \\
&= \frac{1}{2\pi i(1-q)}\int_0^\infty \frac{(-\beta\omega x,-\delta\omega/x;q)_\infty}{(-\alpha\omega x,-\gamma\omega/x;q)_\infty}x^{\lambda-\mu-1}\,d_qx, \quad z=q^\lambda, w=q^\mu \\
&= \frac{1}{2\pi i}\frac{(-\beta\omega,-\delta\omega;q)_\infty}{(-\alpha\omega,-\gamma\omega;q)_\infty}{}_2\psi_2\left(\begin{matrix}-\alpha\omega, & -q/\delta\omega \\ -\beta\omega, & -q/\gamma\omega\end{matrix};q,\frac{\delta z}{\gamma w}\right), \qquad (3.8)
\end{aligned}
$$

and we require $|\frac{\delta}{\gamma}|<|\frac{w}{z}|<|\frac{\alpha}{\beta}|$. Here

$$
{}_2\psi_2\left(\begin{matrix}a_1, & a_2 \\ b_1, & b_2\end{matrix};q,z\right) = \sum_{n=-\infty}^{\infty}\frac{(a_1,a_2;q)_n}{(b_1,b_2;q)_n}z^n, \quad |\frac{b_1b_2}{a_1a_2}|<|z|<1.
$$

Using Ramanujan's ${}_1\psi_1$ sum, [29, page 239] we have the following special cases of (3.8):

$$
\begin{aligned}
K(z/w;\alpha,\alpha,\gamma,\delta) &= \frac{1}{2\pi i}\frac{(q,\delta/\gamma,-qz/\gamma\omega w,-\omega w/z;q)_\infty}{(-\gamma\omega,-q/\gamma\omega,\delta z/\gamma w,w/z;q)_\infty}, \quad |\frac{\delta}{\gamma}|<|\frac{w}{z}|<1 \\
K(z/w;\alpha,\beta,\gamma,\gamma) &= \frac{1}{2\pi i}\frac{(q,\beta/\alpha,-\alpha\omega z/w,qw/\alpha\omega z;q)_\infty}{(-\alpha\omega,-q/\alpha\omega,z/w,\beta w/\alpha z;q)_\infty}, \quad |\frac{\beta}{\alpha}|<|\frac{z}{w}|<1.
\end{aligned}
$$

Now the formula

$$
\mathbf{T}(\alpha,\beta,\gamma,\delta)\mathbf{T}(\alpha',\alpha,\gamma',\gamma) = \mathbf{T}(\alpha',\beta,\gamma',\delta)
$$

leads to the functional relation

$$
K(z/w;\alpha',\beta,\gamma',\delta) = \oint K(z/y;\alpha,\beta,\gamma,\delta)K(y/w;\alpha',\alpha,\gamma',\gamma)\,\frac{dy}{y}, \qquad (3.9)
$$

where, choosing the integration path as the unit circle $|y|=1$, we have the requirements $|\beta/\alpha|<|z|<|\gamma/\delta|$, $|\gamma/\gamma'|<|w|<|\alpha'/\alpha|$.

Two special cases are of particular interest. If $\alpha=\beta,\gamma'=\gamma$ we have

$$
\begin{aligned}
&(-\beta\omega,-\delta\omega;q)_\infty\ {}_2\psi_2\left(\begin{matrix}-\alpha'\omega, & -q/\delta\omega \\ -\beta\omega, & -q/\gamma\omega\end{matrix};q,\frac{\delta z}{\gamma w}\right) \\
&\quad = \frac{1}{2\pi i}\oint\frac{(q,\delta/\gamma,-qz/\gamma\omega y,-\omega y/z,q,\beta/\alpha',-\alpha'\omega y/w,qw/\alpha'\omega y;q)_\infty}{(-q/\gamma\omega,\delta z/\gamma y,y/z,\omega,-q/\alpha'\omega,y/w,\beta w/\alpha' y;q)_\infty}\,\frac{dy}{y}
\end{aligned}
$$

$$
1<|w|<|\alpha'/\alpha|, \quad 1<|z|<|\gamma/\delta|,
$$

and if just $\gamma=\gamma'$ we have

$$
\begin{aligned}
&{}_2\psi_2\left(\begin{matrix}-\alpha'\omega, & -q/\delta\omega \\ -\beta\omega, & -q/\gamma\omega\end{matrix};q,\frac{\delta z}{\gamma w}\right) \\
&\quad = \oint {}_2\psi_2\left(\begin{matrix}-\alpha\omega, & -q/\delta\omega \\ -\beta\omega, & -q/\gamma\omega\end{matrix};q,\frac{\delta z}{\gamma y}\right)\frac{(q,\alpha/\alpha',-\alpha'\omega y/w,qw/\alpha'\omega y;q)_\infty}{(-\alpha\omega,-q/\alpha'\omega,y/w,\alpha w/\alpha' y;q)_\infty}\,\frac{dy}{y},
\end{aligned}
$$

$$
1<|w|<|\alpha'/\alpha|, \quad |\beta/\alpha|<|z|<|\gamma/\delta|.
$$

4. Matrix elements of oscillator algebra representations. In [1] a q-analog of the oscillator algebra was introduced. This is the associative algebra generated by the four elements H, E_+, E_-, $\mathcal{E}$ that obey the commutation relations

$$\begin{aligned}&[H,E_+]=E_+, \quad [H,E_-]=-E_-,\\ &[E_+,E_-]=-q^{-H}\mathcal{E}, \quad [\mathcal{E},E_\pm]=[\mathcal{E},H]=0.\end{aligned} \tag{4.1}$$

It admits a class of algebraically irreducible representations $\uparrow_{\ell,\lambda}$ where ℓ,λ are real numbers and $\ell>0$. These are defined on a Hilbert space $\mathcal{H}$ with orthogonal basis $\{f_n : n=0,1,\cdots\}$ where

$$\begin{aligned}E_+f_n&=\ell q^{-(n+1)/2}f_{n+1}\\ E_-f_n&=\ell q^{-n/2}\frac{1-q^n}{1-q}f_{n-1}\\ Hf_n&=(\lambda+n)f_n \quad \mathcal{E}f_n=\ell^2q^{\lambda-1}f_n.\end{aligned} \tag{4.2}$$

Furthermore, the formal adjoints satisfy $(E_+)^*=E_-$, $H^*=H$, $\mathcal{E}^*=\mathcal{E}$. The elements $\mathcal{C}=qq^{-H}\mathcal{E}+(q-1)E_+E_-$ and $\mathcal{E}$ lie in the center of this algebra, and corresponding to the irreducible representation $\uparrow_{\ell,\lambda}$ we have $\mathcal{C}=\ell^2I$, $\mathcal{E}=\ell^2q^{\lambda-1}I$ where I is the identity operator on $\mathcal{H}$.

A convenient model of $\uparrow_{\ell,\lambda}$, [1], is determined by the orthonormal basis functions

$$e_n=q^{n(n+1)/4}\sqrt{\frac{(1-q)^n}{(q;q)_n}}z^n, \quad n=0,1,\cdots, \tag{4.3}$$

(so that $f_n(z)=q^{n(n+1)/4}z^n$) and the operators

$$\begin{aligned}&E_+=\ell zI, \quad E_-=-\frac{\ell}{(1-q)z}(1-T_z^{-1})\\ &H=\lambda+z\frac{d}{dz}, \quad \mathcal{E}=\ell^2q^{\lambda-1}I.\end{aligned} \tag{4.4}$$

The inner product is

$$(f,g) \ = \iint_{-\infty}^{\infty} f(z)\overline{g(z)}\rho(z,\overline{z})\ dxdy$$

where $z=x+iy$ and

$$\rho(z,\overline{z})=\frac{1-q}{(-(1-q)z\overline{z};q)_\infty\pi\ln q^{-1}}.$$

The model Hilbert space $\mathcal{H}(z)$ consists of all functions

$$f'(z)=\sum_{n=0}^{\infty}c_nz^n \quad \text{such that} \quad \sum_{n=0}^{\infty}\frac{|c_n|^2q^{-n(n+1)/2}}{(1-q)^n}<\infty.$$

This is a space of entire functions; it has the kernel function

$$S(\overline{z'},z)=\sum_{n=0}^{\infty}e'_n(\overline{z'})e'_n(z)=(-(1-q)q\overline{z'}z;q)_\infty. \tag{4.5}$$

Although the parameter λ is essential in the computation of tensor products of pairs of irreducible representations $\uparrow_{\ell,\lambda}$, [1,4], it disappears from the final expressions for the matrix elements studied in this paper. Thus we henceforth set $\lambda = 0$. Using the relations (2.3) we have the following q-analogs of the matrix elements of $\uparrow_{\ell,0}$:

$$(4.6)\qquad \begin{aligned}
(e+,e-):&\quad e_q(\beta E_+)e_q(\alpha E_-)f_n = \sum_{n'} T^{(e+,e-)}_{n'n}(\alpha,\beta)f_{n'},\\
(e+,E-):&\quad e_q(\beta E_+)E_q(\alpha E_-)f_n = \sum_{n'} T^{(e+,E-)}_{n'n}(\alpha,\beta)f_{n'},\\
(e-,E+):&\quad e_q(\beta E_-)E_q(\alpha E_+)f_n = \sum_{n'} T^{(e-,E+)}_{n'n}(\alpha,\beta)f_{n'},\\
(E+,e-):&\quad E_q(\beta E_+)e_q(\alpha E_-)f_n = \sum_{n'} T^{(E+,e-)}_{n'n}(\alpha,\beta)f_{n'},\\
(E-,e+):&\quad E_q(\beta E_-)e_q(\alpha E_+)f_n = \sum_{n'} T^{(E-,e+)}_{n'n}(\alpha,\beta)f_{n'},\\
(E+,E-):&\quad E_q(\beta E_+)E_q(\alpha E_-)f_n = \sum_{n'} T^{(E+,E-)}_{n'n}(\alpha,\beta)f_{n'},\\
(E-,E+):&\quad E_q(\beta E_-)E_q(\alpha E_+)f_n = \sum_{n'} T^{(E-,E+)}_{n'n}(\alpha,\beta)f_{n'}.
\end{aligned}$$

$$(e+,E-,e-,E+):\quad e_q(\delta E_+)E_q(\gamma E_-)e_q(\beta E_-)E_q(\alpha E_+)f_n = \sum_{n'} T_{n'n}(\alpha,\beta,\gamma,\delta)f_{n'}.$$

(The series for the matrix elements $T^{(e-,e+)}_{n'n}(\alpha,\beta)$ does not converge.) All of these sets of matrix elements were studied in [4], with the exception of (e+,E-,e-,E+) which will be the focus of attention here.

Since $E_+^* = E_-$ the following relationships hold:

$$(4.7)\qquad \begin{gathered}
T^{(e+,e-)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(e+,e-)}_{nn'}(\beta,\alpha),\\
T^{(e+,E-)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(E+,e-)}_{nn'}(\beta,\alpha),\quad T^{(e-,E+)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(E-,e+)}_{nn'}(\beta,\alpha),\\
T^{(E+,E-)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(E+,E-)}_{nn'}(\beta,\alpha),\quad T^{(E-,E+)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(E-,E+)}_{nn'}(\beta,\alpha).
\end{gathered}$$

Here

$$A_{n'n} = \frac{(q;q)_{n'}}{(q;q)_n}(1-q)^{n-n'}.$$

Since $e_q(z)E_q(-z) = 1$, we have the identities

$$\begin{aligned}
a)&\ \sum_h T^{(e+,e-)}_{n'h}(\alpha,\beta)T^{(E-,E+)}_{hn}(-\beta,-\alpha) = \delta_{n'n}\\
b)&\ \sum_h T^{(E-,e+)}_{n'h}(\alpha,\beta)T^{(E+,e-)}_{hn}(-\beta,-\alpha) = \delta_{n'n}.
\end{aligned}$$

Using the model (4.4) to compute the matrix elements (which are model independent) we obtain the explicit results

$$
\begin{aligned}
&T^{(e+,e-)}_{n'n}(\alpha,\beta)\\
&=\frac{(q^{n'-n+1};q)_\infty(\beta\ell)^{n'-n}}{(q;q)_\infty}q^{(n-n')(n+n'+1)/4}{}_2\phi_1\left(\begin{matrix}q^{-n}, & 0\\ q^{n'-n+1}\end{matrix};q,\frac{-\alpha\beta\ell^2}{1-q}\right)\\
&=\frac{(q^{n-n'+1};q)_\infty(q^{n'+1};q)_\infty(\alpha\ell)^{n-n'}}{(q;q)_\infty(q^{n+1};q)_\infty(1-q)^{n-n'}}q^{(n'-n)(n'+n+1)/4}{}_2\phi_1\left(\begin{matrix}q^{-n'}, & 0\\ q^{n-n'+1}\end{matrix};q,\frac{-\alpha\beta\ell^2}{1-q}\right)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(E+,e-)}_{n'n}(\alpha,\beta)\\
&=\frac{(q^{n'+1};q)_\infty(q^{n-n'+1};q)_\infty(\alpha\ell)^{n-n'}}{(q;q)_\infty(q^{n+1};q)_\infty(1-q)^{n-n'}}q^{(n'-n)(n'+n+1)/4}{}_1\phi_1\left(\begin{matrix}q^{-n'}\\ q^{n-n'+1}\end{matrix};q,\frac{\alpha\beta\ell^2}{1-q}\right)\\
&=\frac{(q^{n'-n+1};q)_\infty(\beta\ell)^{n'-n}}{(q;q)_\infty}q^{(n'-n)(n'-3n-3)/4}{}_1\phi_1\left(\begin{matrix}q^{-n}\\ q^{n'-n+1}\end{matrix};q,\frac{\alpha\beta\ell^2q^{n'-n}}{1-q}\right)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(E+,E-)}_{n'n}(\alpha,\beta)\\
&=\frac{(q^{n'-n+1};q)_\infty(\beta\ell)^{n'-n}}{(q;q)_\infty}q^{(n'-n)(n'-3n-3)/4}{}_1\phi_2\left(\begin{matrix}q^{-n}\\ q^{n'-n+1}, & 0\end{matrix};q,\frac{-\alpha\beta\ell^2q^{n'-n}}{1-q}\right)\\
&=\frac{(q^{n-n'+1};q)_\infty(q^{n'+1};q)_\infty(\alpha\ell)^{n-n'}}{(q;q)_\infty(q^{n+1};q)_\infty(1-q)^{n-n'}}q^{(n-n')(n-3n'-3)/4}{}_1\phi_2\left(\begin{matrix}q^{-n'}\\ q^{n-n'+1}, & 0\end{matrix};q,\frac{-\alpha\beta\ell^2q^{n-n'}}{1-q}\right)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(E-,E+)}_{n'n}(\alpha,\beta)\\
&=\frac{(\frac{-\alpha\beta\ell^2}{(1-q)q};q)_\infty(q^{n'-n+1};q)_\infty(\alpha\ell)^{n'-n}}{(q;q)_\infty}q^{(n'-n)(n'-3n-3)/4}{}_2\phi_1\left(\begin{matrix}q^{-n}, & 0\\ q^{n'-n+1}\end{matrix};q,\frac{-\alpha\beta\ell^2}{(1-q)q}\right)\\
&=\frac{(\frac{-\alpha\beta\ell^2}{(1-q)q};q)_\infty(q^{n-n'+1};q)_\infty(q^{n'+1};q)_\infty(\beta\ell)^{n-n'}}{(q;q)_\infty(q^{n+1};q)_\infty(1-q)^{n-n'}}q^{(n-n')(n-3n'-3)/4}\\
&\qquad {}_2\phi_1\left(\begin{matrix}q^{-n'}, & 0\\ q^{n-n'+1}\end{matrix};q,\frac{-\alpha\beta\ell^2}{(1-q)q}\right).
\end{aligned}
$$

The matrix elements $T^{(e+,e-)}, T^{(E+,e-)}, T^{(E+,E-)}$ are polynomials in α and β and the matrix elements $T^{(E-,E+)}$ are entire analytic functions of these variables.

In [4] it is shown that all of the remaining matrix elements (4.6), except the last, can be expressed in terms of these four. Indeed, we have the operator identities

$$
e_q\left(\frac{\alpha\beta\ell^2q^{-H-1}}{1-q}\right)e_q(\beta E_+)E_q(\alpha E_-)=E_q(\alpha E_-)e_q(\beta E_+), \tag{4.8}
$$

$$
E_q(\beta E_+)e_q(\alpha E_-)e_q\left(\frac{\alpha\beta\ell^2q^{-H-1}}{1-q}\right)=e_q(\alpha E_-)E_q(\beta E_+) \tag{4.9}
$$

which imply

$$T^{(E-,e+)}_{n'n}(\beta,\alpha) = e_q\left(\frac{\alpha\beta\ell^2 q^{-n'-1}}{1-q}\right) T^{(e+,E-)}_{n'n}(\alpha,\beta), \tag{4.10}$$
$$T^{(e-,E+)}_{n'n}(\beta,\alpha) = e_q\left(\frac{\alpha\beta\ell^2 q^{-n-1}}{1-q}\right) T^{(E+,e-)}_{n'n}(\alpha,\beta).$$

Thus the matrix elements $T^{(E-,e+)}_{n'n}$ are well defined for $|\alpha\beta\ell^2 q^{-n'-1}/(1-q)| < 1$ and the matrix elements $T^{(e-,E+)}_{n'n}$ are well defined for $|\alpha\beta\ell^2 q^{-n-1}/(1-q)| < 1$.

From identities (4.8), (4.9) we can express the $(e+,E-,e-,E+)$ operator in the alternate forms

$$\begin{aligned}\mathbf{T}(\alpha,\beta,\gamma,\delta) &= e_q(\delta E_+)E_q(\gamma E_-)e_q(\beta E_-)E_q(\alpha E_+)\\ &= e_q(\delta E_+)E_q(\alpha E_+)E_q(\frac{\alpha\gamma\ell^2 q^{-H-1}}{1-q})E_q(\gamma E_-)e_q(\beta E_-)e_q(\frac{\alpha\beta\ell^2 q^{-H-1}}{1-q}).\end{aligned} \tag{4.11}$$

An explicit computation of the matrix elements yields

$$\begin{aligned} T_{n'n}(\alpha,\beta,\gamma,\delta) &= \frac{(-\frac{\alpha\gamma\ell^2}{1-q}q^{-n-1};q)_\infty(-\frac{\alpha}{\delta};q)_{n'-n}}{(\frac{\alpha\beta\ell^2}{1-q}q^{-n-1};q)_\infty(q;q)_{n'-n}}(\ell\delta)^{n'-n}q^{(n+n'+1)(n-n')/4}\\ &\quad\times {}_3\phi_2\left(\begin{matrix}-\frac{\alpha}{\delta}q^{n'-n}, & -\frac{\gamma}{\beta}, & q^{-n}\\ q^{n'-n+1}, & -\frac{\alpha\gamma\ell^2}{1-q}q^{-n-1} & \end{matrix}; q, -\frac{\beta\delta\ell^2}{1-q}\right)\\ &= \frac{(-\frac{\alpha\gamma\ell^2}{1-q}q^{-n'-1};q)_\infty(-\frac{\gamma}{\beta};q)_{n-n'}(q;q)_n}{(\frac{\alpha\beta\ell^2}{1-q}q^{-n-1};q)_\infty(q;q)_{n-n'}(q;q)_{n'}}\left(\frac{\ell\beta}{1-q}\right)^{n-n'}q^{(n+n'+1)(n'-n)/4}\\ &\quad\times {}_3\phi_2\left(\begin{matrix}-\frac{\alpha}{\delta}, & -\frac{\gamma}{\beta}q^{n-n'}, & q^{-n'}\\ q^{n-n'+1}, & -\frac{\alpha\gamma\ell^2}{1-q}q^{-n'-1} & \end{matrix}; q, -\frac{\beta\delta\ell^2}{1-q}\right),\\ &\quad |\alpha\beta\ell^2 q^{-n-1}/(1-q)| < 1.\end{aligned} \tag{4.12}$$

Indeed, from (4.11) and the fact that $E_+^* = E_-$ we have the identity

$$T_{n'n}(\alpha,\beta,\gamma,\delta) = (1-q)^{n'-n}\frac{(q;q)_n(\frac{\gamma\delta\ell^2}{1-q}q^{-n'-1};q)_\infty}{(q;q)'_n(\frac{\alpha\beta\ell^2}{1-q}q^{-n-1};q)_\infty}T_{nn'}(\gamma,\delta,\alpha,\beta).$$

Using Sears' ${}_3\phi_2$ transform [29, page 61] we have the alternate form

$$\begin{aligned} T_{n'n}(\alpha,\beta,\gamma,\delta) &= \frac{(-\frac{\alpha\gamma\ell^2}{1-q}q^{-1};q)_\infty(-\frac{\gamma}{\beta};q)_{n-n'}(q;q)_n}{(\frac{\alpha\beta\ell^2}{1-q}q^{n'-n-1};q)_\infty(q;q)_{n-n'}(q;q)_{n'}}\left(\frac{\ell\beta}{1-q}\right)^{n-n'}q^{(n+n'+1)(n'-n)/4}\times\\ &\quad {}_3\phi_2\left(\begin{matrix}-\frac{\delta}{\alpha}q^{n-n'+1}, & -\frac{\gamma}{\beta}q^{n-n'}, & q^{-n'}\\ q^{n-n'+1}, & \frac{1-q}{\alpha\beta\ell^2}q^{n-n'+2} & \end{matrix}; q, q\right).\end{aligned}$$

Setting $\alpha = (1-q)\alpha', \cdots, \delta = (1-q)\delta'$ in (4.12) we see that in the limit as $q \to 1-$,

$$
\begin{aligned}
T_{n'n}(\alpha,\beta,\gamma,\delta) \to & \frac{[\ell(\alpha'+\delta')]^{n'-n}}{(n'-n)!} e^{\ell^2\alpha'(\beta'+\gamma')} {}_1F_1\left(\begin{matrix} -n, \\ n'-n+1 \end{matrix}; -\ell^2(\alpha'+\delta')(\beta'+\gamma') \right), \\
& \text{if } n' \geq n, \\
\to & [\ell(\beta'+\gamma')]^{n-n'} \binom{n}{n'} e^{\ell^2\alpha'(\beta'+\gamma')} {}_1F_1\left(\begin{matrix} -n', \\ n-n'+1 \end{matrix}; -\ell^2(\alpha'+\delta')(\beta'+\gamma') \right), \\
& \text{if } n \geq n'.
\end{aligned}
$$

In order to derive identities for the matrix elements of the operators $\mathbf{T}(\alpha,\beta,\gamma,\delta)$, (4.11) we cannot just multiply two general operators of this sort, as was done in the previous sections. Indeed, the resulting formal power series diverge. However, just as in (2.13) we can write $\mathbf{T}$ as a composition of two two-factor operators and obtain the identities

$$
\begin{aligned}
T_{n'n}(\alpha,\beta,\gamma,\delta) &= \sum_{k=0}^{\infty} T_{n'k}^{(e+,E-)}(\gamma,\delta) T_{kn}^{(e-,E+)}(\alpha,\beta), \\
T_{n'n}(\alpha,\beta,\gamma,\delta) &= \sum_{k=0}^{\infty} T_{n'k}^{(e+,e-)}(\beta,\delta) T_{kn}^{(E-,E+)}(\alpha,\gamma) \\
& |\alpha\beta\ell^2/q^{n+1}(1-q)| < 1.
\end{aligned}
\tag{4.13}
$$

(Once the common factor $e_q(\alpha\beta\ell^2 q^{-n-1}/(1-q))$ is removed from both sides of the first equation (4.13), this formula holds for all values of the parameters.) Note that relations (2.10) are special cases of (4.13). These identities are q-analogs of an addition formula for the confluent hypergeometric functions [30, Chapter 4]:

$$
\frac{(n+n')!}{n!} e^{-\alpha\gamma} [-\gamma(\alpha+\delta)]^n {}_1F_1\left(\begin{matrix} -n' \\ n+1 \end{matrix}; (\beta+\gamma)(\alpha+\delta) \right) =
$$

$$
\sum_{j=0}^{\infty} \frac{(-\alpha\gamma)^{j-n'} j!}{\Gamma(j-n'+1)\Gamma(j-n-n'+1)} {}_1F_1\left(\begin{matrix} -n' \\ j-n'+1 \end{matrix}; \alpha\beta \right) {}_1F_1\left(\begin{matrix} -n-n' \\ j-n-n'+1 \end{matrix}; \gamma\delta \right).
$$

5. Continuum bases for oscillator representations. Next we introduce a model of a q-analog of the pseudo-oscillator group. The model consists of a Hilbert space of complex valued functions $f(x)$ where x is a positive real variable, such that $||f||^2 < \infty$ and the inner product is

$$
< f, g > = \int_0^{\infty} f(x)\overline{g}(x) \; \frac{dx}{x} \tag{5.1}
$$

and $||f||^2 = < f, f >$. The formal action of the q-algebra is

$$
\begin{aligned}
E_+ &= \ell x I, \quad E_- = -\frac{\ell}{(1-q)x}(1 - T_x^{-1}) \\
H &= x\frac{d}{dx}, \quad \mathcal{E} = \ell^2 q^{-1} I.
\end{aligned}
\tag{5.2}
$$

The action of the "pseudo-oscillator group" is given by the formal operator $\mathbf{T}(\alpha,\beta,\gamma,\delta)$ where

(5.3)
$$\begin{aligned}\mathbf{T}(\alpha,\beta,\gamma,\delta)f(x) &= e_q(\delta E_+)E_q(\gamma E_-)e_q(\beta E_-)E_q(\alpha E_+)f(x)\\ &= \frac{\left(-\alpha\ell x, \frac{\gamma\ell}{(1-q)x};q\right)_\infty}{\left(\delta\ell x, -\frac{\beta\ell}{(1-q)x};q\right)_\infty}\sum_{n=0}^{\infty}\left(\frac{\alpha\beta\ell^2}{q(1-q)}\right)^n\frac{\left(-\frac{q}{\alpha\ell x}, -\frac{\gamma}{\beta};q\right)_n}{\left(q, \frac{\gamma\ell}{(1-q)x};q\right)_n}T_x^{-n}f(x).\end{aligned}$$

(To derive this expression we have made use of the q-Gauss formula [29, page 10].) We require that neither δ or $-\beta$ is positive, so that (5.3) is well defined when acting on the subspace $\mathcal{G}$ of the Hilbert space, where $\mathcal{G}$ consists of those functions $f(x)$ that are C^∞ with compact support in $(0,\infty)$. Indeed, for each such f the summation in (5.3) is finite. Using the identity (2.28) we can show that if $f \in \mathcal{G}$ has support in a proper subset of the interval $(0,K]$ then the function $\mathbf{T}(\alpha,\beta,\gamma,\delta)f(x)$ vanishes for $x \geq K$ and if $|\gamma/\beta| < 1$ this function remains bounded as $x \to 0+$.

Following [24, Chapter 8], we will compute the matrix elements of the operator $\mathbf{T}$ with respect to a continuum basis in which H is diagonalized. We recall that the Mellin transform of $f \in \mathcal{G}$,

(5.4)
$$F(\lambda) = \int_0^\infty f(x)x^{\lambda-1}\,dx$$

has the properties that 1) $F(\lambda)$ is an entire (analytic) function of λ, 2) $|F(\lambda)| < Ce^{k|\mathrm{Re}\ \lambda|}$ for some positive constants C, k and 3) F decreases rapidly on every straight line parallel to the imaginary axis in the complex λ-plane. (We denote the space of transforms of functions in $\mathcal{G}$ by $\hat{\mathcal{G}}$.) Furthermore we have the inversion formula

$$f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(\mu)x^{-\mu}\,d\mu,$$

for any real number a.

Clearly, the induced action of the operator $q^{\phi H}$ on the transformed functions F is diagonal:

$$q^{\phi H}F(\lambda) = e^{-\lambda\phi}F(\lambda).$$

Furthermore, the induced action of the operator $\mathbf{T}$ on $\hat{\mathcal{G}}$ is given by (assuming that neither δ or $-\beta$ is positive and that $|\gamma/\beta| < 1$),

$$\begin{aligned}&\mathbf{T}(\alpha,\beta,\gamma,\delta)F(\lambda)\\ &= \int_0^\infty \frac{\left(-\alpha\ell x, \frac{\gamma\ell}{(1-q)x};q\right)_\infty}{\left(\delta\ell x, -\frac{\beta\ell}{(1-q)x};q\right)_\infty}\sum_{n=0}^{\infty}\left(\frac{\alpha\beta\ell^2}{q(1-q)}\right)^n\frac{\left(-\frac{q}{\alpha\ell x}, -\frac{\gamma}{\beta};q\right)_n}{\left(q, \frac{\gamma\ell}{(1-q)x};q\right)_n}f(q^{-n}x)x^{\lambda-1}\,dx\\ &\qquad \mathrm{Re}\ \lambda > 0,\end{aligned}$$

or

$$\mathbf{T}(\alpha,\beta,\gamma,\delta)F(\lambda)=\frac{1}{2\pi i}\int_0^\infty \frac{\left(-\alpha\ell x,\frac{\gamma\ell}{(1-q)x};q\right)_\infty}{\left(\delta\ell x,-\frac{\beta\ell}{(1-q)x};q\right)_\infty}\,dx$$

$$\times\sum_{n=0}^\infty\left(\frac{\alpha\beta\ell^2}{q(1-q)}\right)^n\frac{\left(-\frac{q}{\alpha\ell x},-\frac{\gamma}{\beta};q\right)_n}{\left(q,\frac{\gamma\ell}{(1-q)x};q\right)_n}\int_{a-i\infty}^{a+i\infty}F(\mu)q^{n\mu}x^{\lambda-\mu-1}\,d\mu$$

$$=\frac{1}{2\pi i}\int_0^\infty \frac{\left(-\alpha\ell x,\frac{\gamma\ell}{(1-q)x};q\right)_\infty}{\left(\delta\ell x,-\frac{\beta\ell}{(1-q)x};q\right)_\infty}\,dx\,\times$$

$$\int_{a-i\infty}^{a+i\infty}{}_2\phi_1\left(\begin{matrix}-\frac{q}{\alpha\ell x}, & -\frac{\gamma}{\beta}\\ \frac{\gamma\ell}{(1-q)x} & \end{matrix};q,\frac{\alpha\beta\ell^2q^{\mu-1}}{1-q}\right)F(\mu)x^{\lambda-\mu-1}\,d\mu,$$

if the contour is chosen so that $|\alpha\beta\ell^2q^{a-1}/(1-q)|<1$. Using Heine's and Jackson's transformations [29, page 241] we can write this result as

$$\mathbf{T}(\alpha,\beta,\gamma,\delta)F(\lambda)=\frac{1}{2\pi i}\int_0^\infty\frac{(-\alpha\ell x;q)_\infty}{\left(\delta\ell x,-\frac{\beta\ell}{(1-q)x};q\right)_\infty}\,dx\int_{a-i\infty}^{a+i\infty}\frac{\left(-\frac{\alpha\gamma\ell^2}{1-q}q^{\mu-1},-\frac{\beta\ell}{(1-q)x}q^\mu;q\right)_\infty}{\left(\frac{\alpha\beta\ell^2}{1-q}q^{\mu-1};q\right)_\infty}$$

$$\times\ {}_2\phi_2\left(\begin{matrix}q^\mu, & \frac{\alpha\beta\ell^2}{1-q}q^{\mu-1}\\ -\frac{\alpha\gamma\ell^2}{1-q}q^{\mu-1}, & -\frac{\beta\ell}{(1-q)x}q^\mu\end{matrix};q,\frac{\gamma\ell}{(1-q)x}\right)F(\mu)x^{\lambda-\mu-1}\,d\mu.$$

If, in addition, $|\gamma/\beta|<q^a<|\delta q^\lambda/\alpha|,|q^\lambda|<1$, then the iterated integral is absolutely convergent and we can interchange the order of integration to obtain

$$\mathbf{T}(\alpha,\beta,\gamma,\delta)F(\lambda)=\int_{a-i\infty}^{a+i\infty}K(\lambda,\mu;\alpha,\beta,\gamma,\delta)F(\mu)\,d\mu, \tag{5.5}$$

where

$$K(\lambda,\mu;\alpha,\beta,\gamma,\delta)=\frac{1}{2\pi i}\int_0^\infty\frac{\left(-\alpha\ell x,-\frac{\alpha\gamma\ell^2}{1-q}q^{\mu-1},-\frac{\beta\ell}{(1-q)x}q^\mu;q\right)_\infty}{\left(\delta\ell x,-\frac{\beta\ell}{(1-q)x},\frac{\alpha\beta\ell^2}{1-q}q^{\mu-1};q\right)_\infty}$$

$$\times\ {}_2\phi_2\left(\begin{matrix}q^\mu, & \frac{\alpha\beta\ell^2}{1-q}q^{\mu-1}\\ -\frac{\alpha\gamma\ell^2}{1-q}q^{\mu-1}, & -\frac{\beta\ell}{(1-q)x}q^\mu\end{matrix};q,\frac{\gamma\ell}{(1-q)x}\right)x^{\lambda-\mu-1}\,dx,$$

$$|\gamma/\beta|<|q^\mu|<|\delta q^\lambda/\alpha|,\ |q^\lambda|<1,\ |q^\mu|<|q(1-q)/\alpha\beta\ell^2|,\ \delta,-\beta\ \text{nonpositive}.$$

To compute the kernel function K we evaluate the contour integral

(5.6)

$$I_{N,M}=\frac{1}{2\pi i}\frac{(-\frac{\alpha\gamma\ell^2}{1-q}q^{\mu-1};q)_\infty}{(\frac{\alpha\beta\ell^2}{1-q}q^{\mu-1};q)_\infty}\oint_{C_{N,M}}\frac{\left(-\alpha\ell z,-\frac{\beta\ell}{(1-q)z}q^\mu;q\right)_\infty}{\left(\delta\ell z,-\frac{\beta\ell}{(1-q)z};q\right)_\infty}$$

$$\times\ {}_2\phi_2\left(\begin{matrix}q^\mu, & \frac{\alpha\beta\ell^2}{1-q}q^{\mu-1}\\ -\frac{\alpha\gamma\ell^2}{1-q}q^{\mu-1}, & -\frac{\beta\ell}{(1-q)z}q^\mu\end{matrix};q,\frac{\gamma\ell}{(1-q)z}\right)z^{\lambda-\mu-1}\,dz,$$

along the closed contour $C_{N,M}$ of (2.32). In the limit as $N \to \infty, M \to \infty$ the integrals on the large and on the small circle go to zero. Then, evaluating (5.6) by residues and using the formula

$$\sum_{k,n=0}^{\infty} \frac{(a;q)_k(b;q)_n(c;q)_{n+k}x^k y^n}{(d;q)_{n+k}(q;q)_k(q;q)n} = \frac{(c,ax,by;q)_\infty}{(d,x,y;q)_\infty} {}_3\phi_2\left(\begin{matrix} \frac{d}{c}, & x, & y \\ ax, & by & \end{matrix}; q, c\right),$$

see [31], we obtain

$$\begin{aligned}
K(\lambda,\mu;\alpha,\beta,\gamma,\delta) &= \frac{1}{2i\sin\pi(\lambda-\mu)(-\frac{\beta\delta\ell^2}{1-q};q)_\infty} \times \\
&\Bigg[(-\delta\ell)^{\mu-\lambda}\frac{(-\frac{\alpha}{\delta},q^{\mu-\lambda+1},\frac{\gamma\delta\ell^2}{1-q};q)_\infty}{(q,-\frac{\alpha}{\delta}q^{\mu-\lambda};q)_\infty} {}_3\phi_2\left(\begin{matrix} q^{-\lambda+1}, & -\frac{\delta}{\alpha}q, & -\frac{\gamma}{\beta} \\ q^{\mu-\lambda+1}, & \frac{\gamma\delta\ell^2}{1-q} & \end{matrix}; q, \frac{\alpha\beta\ell^2}{1-q}q^{\mu-1}\right) \\
&- \left(\frac{\beta\ell}{1-q}\right)^{\lambda-\mu}\frac{(-\frac{\gamma}{\beta},q^{\mu},q^{\lambda-\mu+1},\frac{\gamma\delta\ell^2}{1-q}q^{\lambda-\mu};q)_\infty}{(q,q^{\lambda},-\frac{\gamma}{\beta}q^{\lambda-\mu};q)_\infty}\times \\
&{}_3\phi_2\left(\begin{matrix} q^{-\mu+1}, & -\frac{\delta}{\alpha}q^{\lambda-\mu+1}, & -\frac{\gamma}{\beta}q^{\lambda-\mu} \\ q^{\lambda-\mu+1}, & \frac{\gamma\delta\ell^2}{1-q}q^{\lambda-\mu} & \end{matrix}; q, \frac{\alpha\beta\ell^2}{1-q}q^{\mu-1}\right)\Bigg].
\end{aligned} \tag{5.7}$$

(The apparent singularities at the zeros of $(-\frac{\beta\delta\ell^2}{1-q};q)_\infty$ are removable.) The following special cases of (5.7) are of interest:

$$\begin{aligned}
K(\lambda,\mu;0,\beta,0,\delta) &= \frac{1}{2i\sin\pi(\lambda-\mu)(-\frac{\beta\delta\ell^2}{1-q};q)_\infty}\times \\
&\Bigg[(-\delta\ell)^{\mu-\lambda}\frac{(q^{\mu-\lambda+1};q)_\infty}{(q;q)_\infty} {}_1\phi_1\left(\begin{matrix} q^{-\lambda+1} \\ q^{\mu-\lambda+1}\end{matrix}; q, -\frac{\beta\delta\ell^2}{1-q}q^{\mu}\right) \\
&- \left(\frac{\beta\ell}{1-q}\right)^{\lambda-\mu}\frac{(q^{\mu},q^{\lambda-\mu+1};q)_\infty}{(q;q)_\infty} {}_1\phi_1\left(\begin{matrix} q^{-\mu+1} \\ q^{\lambda-\mu+1}\end{matrix}; q, -\frac{\beta\delta\ell^2}{1-q}q^{\lambda}\right)\Bigg],
\end{aligned} \tag{5.8}$$

$$|q^\lambda| < 1,$$

$$K(\lambda,\mu;0,\beta,\gamma,0) = \frac{-1}{2i\sin\pi(\lambda-\mu)}\left(\frac{\beta\ell}{1-q}\right)^{\lambda-\mu}\frac{(-\frac{\gamma}{\beta},q^{\mu},q^{\lambda-\mu+1};q)_\infty}{(q,q^{\lambda},-\frac{\gamma}{\beta}q^{\lambda-\mu};q)_\infty} \tag{5.9}$$

$$|\gamma/\beta| < |q^\mu| < |q^\lambda| < 1,$$

$$K(\lambda,\mu;\alpha,0,0,\delta) = \frac{1}{2i\sin\pi(\lambda-\mu)}(-\delta\ell)^{\mu-\lambda}\frac{(-\frac{\alpha}{\delta},q^{\mu-\lambda+1};q)_\infty}{(q,-\frac{\alpha}{\delta}q^{\mu-\lambda};q)_\infty}, \tag{5.10}$$

$$|q^\lambda| < |q^\mu| < |\delta q^\lambda/\alpha|.$$

Setting $\alpha = (1-q)\alpha', \cdots, \delta = (1-q)\delta'$ in (5.7) we see that in the limit as $q \to 1-$,

$$K(\lambda,\mu;\alpha,\beta,\gamma,\delta) \to \frac{1}{2i\sin\pi(\lambda-\mu)}\left[\frac{[-\ell(\alpha'+\delta')]^{\mu-\lambda}}{\Gamma(\mu-\lambda+1)}{}_1F_1\left(\begin{matrix}-\lambda+1\\ \mu-\lambda+1\end{matrix};\ell^2(\alpha'+\delta')(\beta'+\gamma')\right)\right.$$
$$\left. -\frac{[\ell(\beta'+\gamma')]^{\lambda-\mu}\Gamma(\lambda)}{\Gamma(\mu)\Gamma(\lambda-\mu+1)}{}_1F_1\left(\begin{matrix}-\mu+1\\ \lambda-\mu+1\end{matrix};\ell^2(\alpha'+\delta')(\beta'+\gamma\right)\right], \tag{5.11}$$

in agreement with [24, Chapter 8; 25].

From the expression (5.3) we see that the "addition formula"

$$\mathbf{T}(\alpha,\beta,\gamma,\delta)\mathbf{T}(\alpha',\beta',-\beta,-\alpha) = \mathbf{T}(\alpha',\beta',\gamma,\delta) \tag{5.12}$$

holds rigorously when both sides are applied to $f \in \mathcal{G}$, provided δ is not positive and α,β,β' are not negative. Furthermore, one can show that for $|\gamma/\beta| < 1$, the function $h(x) = \mathbf{T}(\alpha,\beta,\gamma,\delta)f(x)$ has the properties that $x^2h(x), x^2h'(x), x^2h''(x) \to 0$ as $x \to 0+$. Thus, the Mellin transform $H(\lambda)$ of h, (5.4), has the properties that 1) $H(\lambda)$ is an analytic function of λ for $|q^\lambda| < 1$, 2) $|F(\lambda)| < Ce^{k|\mathrm{Re}\ \lambda|}$ for some positive constants C,k with$|q^{\lambda-1}| < 1$, and 3) $\lim_{t\to\infty}|t|^2|F(c+it)| = 0$ for $|q^{\lambda-2}| < 1$.

For the kernel functions, (5.12) takes the form

(5.13)
$$\int_{b-i\infty}^{b+i\infty} K(\lambda,\mu;\alpha',\beta',\gamma,\delta)F(\mu)\ d\mu =$$
$$\int_{b-i\infty}^{b+i\infty} K(\lambda,\nu;\alpha,\beta,\gamma,\delta)\ d\nu \int_{a-i\infty}^{a+i\infty} K(\nu,\mu;\alpha',\beta',-\beta,-\alpha)F(\mu)\ d\mu.$$

Then, if the integrals in (5.13) are absolutely convergent we have the functional relations

$$K(\lambda,\mu;\alpha',\beta',\gamma,\delta) = \int_{a-i\infty}^{a+i\infty} K(\lambda,\nu;\alpha,\beta,\gamma,\delta)K(\nu,\mu;\alpha',\beta',-\beta,-\alpha)\ d\nu, \tag{5.14}$$

$$|\alpha q^\mu/\alpha'|, |\delta q^\lambda/\gamma| < |q^a| < |\alpha q^\lambda/\beta|, |\gamma' q^\mu/\gamma|.$$

An interesting special case is

$$K(\lambda,\mu;\alpha,\beta,\gamma,\delta) = \frac{1}{(\frac{\alpha\beta\ell^2q^{\mu-1}}{1-q};q)_\infty}\int_{a-i\infty}^{a+i\infty} K(\lambda,\nu;\alpha,0,0,\delta)K(\nu,\mu;0,\beta,\gamma,0)\ d\nu,$$

or

$$K(\lambda,\mu;\alpha,\beta,\gamma,\delta) = \frac{(-\frac{\alpha}{\delta},-\frac{\gamma}{\beta},q^\mu;q)_\infty}{4(q,q,\frac{\alpha\beta\ell^2q^{\mu-1}}{1-q};q)_\infty}\times$$
$$\int_{a-i\infty}^{a+i\infty}\frac{(-\delta\ell)^{\nu-\lambda}(\frac{\beta\ell}{1-q})^{\mu-\nu}(q^{\nu-\lambda+1},q^{\nu-\mu+1},q^\mu,-\frac{\alpha\gamma\ell^2}{1-q}q^{\nu-1};q)_\infty}{\sin\pi(\lambda-\nu)\sin\pi(\nu-\mu)(q^\nu,-\frac{\alpha}{\delta}q^{\nu-\lambda},-\frac{\gamma}{\beta}q^{\nu-\mu};q)_\infty}\ d\nu,$$

$$|\gamma/\beta| < |q^\mu| < |q^a| < 1, \qquad |q^\lambda| < |q^a| < |\delta q^\lambda/\alpha|.$$

We conclude by studying a model of an alternate q-analog of the pseudo-oscillator algebra. Here the generators of our algebra are q^{-H}, E_+, E_-, $\mathcal{E}$ which obey the relations

$$E_+q^{-H} = qq^{-H}E_+, \quad qE_-q^{-H} = q^{-H}E_-, \quad [E_+, E_-] = -q^{-H}\mathcal{E}, \tag{5.15}$$

$$[\mathcal{E}, E_\pm] = [\mathcal{E}, q^{-H}] = 0.$$

We consider the following class of irreducible representations $\uparrow_{\ell,0}$ for this algebra, characterized by the positive number ℓ. The Hilbert space consists of complex functions $f(x)$ with domain $x = q^n$, $n = 0, \pm1, \pm2, \cdots$ and such that $(f, f) < \infty$, where the inner product is

$$(f, g) = \sum_{n=-\infty}^{\infty} f(q^n)\overline{g}(q^n). \tag{5.16}$$

The action of the algebra on this Hilbert space is given by the operators

$$E_+ = \ell x, \quad E_- = -\frac{\ell}{(1-q)x}(1 - T_x^{-1}), \quad q^{-H}f(x) = f(q^{-1}x). \tag{5.17}$$

To define these operators rigorously we can restrict their action to, say, the dense subspace $\mathcal{L}$ of all functions in the Hilbert space that are nonzero at only a finite number of points. We define the (inverse) Fourier transform $\mathcal{F}$ of $f \in \mathcal{L}$ by

$$\mathcal{F}(\lambda) \equiv \mathcal{F}\,[z] = (f, x^{\overline{\lambda}}) = \sum_{n=-\infty}^{\infty} f(q^n)q^{n\lambda}, \quad \lambda \in \mathbb{C}, \tag{5.18}$$

where $z = q^\lambda$. Then the induced action of the algebra on the transform space $\hat{\mathcal{L}}$ is

$$q^{-H}\mathcal{F}\,[z] = z\mathcal{F}\,[z], \quad E_+\mathcal{F}\,[z] = \ell\mathcal{F}\,[qz],$$
$$E_-\mathcal{F}\,[z] = -\frac{\ell}{1-q}(1 - \frac{z}{q})\mathcal{F}\,[q^{-1}z],$$

so the operator q^{-H} is diagonalized in the transform space. Clearly, every $\mathcal{F} \in \hat{\mathcal{L}}$ is analytic for all $z \neq 0$. We can recover f from its transform $\mathcal{F}$ via the formula

$$f(q^m) = \frac{1}{2\pi i}\oint \mathcal{F}\,[z]z^{-m-1}\,dz \tag{5.19}$$

where the integration path is a simple closed curve around the origin in the z-plane.

The action of the "pseudo-oscillator group" is given by the operator $\mathbf{T}(\alpha, \beta, \gamma, \delta)$, (5.3) for $f \in \mathcal{L}$. The induced action of $\mathbf{T}$ on $\hat{\mathcal{L}}$ is given by

$$\mathbf{T}(\alpha, \beta, \gamma, \delta)\mathcal{F}\,[z] = \oint K(z, w; \alpha, \beta, \gamma, \delta)\mathcal{F}\,[w]\,\frac{dw}{w}, \tag{5.20}$$

where

$$K(z,w;\alpha,\beta,\gamma,\delta)=\frac{1}{2\pi i(1-q)}\int_0^{\infty}\frac{\left(-\alpha\ell x,-\frac{\alpha\gamma\ell^2 w}{q(1-q)},;q\right)_\infty}{\left(\delta\ell x,\frac{\alpha\beta\ell^2 w}{q(1-q)};q\right)_\infty}$$
$$\times\ {}_2\phi_1\left(\begin{matrix}w, & -\frac{\gamma}{\beta}\\ -\frac{\alpha\gamma\ell^2 w}{q(1-q)} & \end{matrix};q,-\frac{\beta\ell}{(1-q)x}\right)x^{\lambda-\mu-1}\,d_qx,$$

$|\gamma/\beta|<|w|<|\delta z/\alpha|$, $|z|<1$, $|w|<|q(1-q)/\alpha\beta\ell^2|$, $\delta,-\beta$ nonpositive, $w=q^\mu, z=q^\lambda$.

To evaluate the q-integral expression for the kernel function K we use the identity

$${}_2\phi_1\left(\begin{matrix}A, & B\\ C & \end{matrix};q,z\right)=\frac{(\frac{ABz}{C};q)_\infty}{(\frac{Bz}{C};q)_\infty}{}_3\phi_2\left(\begin{matrix}A, & \frac{C}{B}, & 0\\ \frac{Cq}{Bz}, & C & \end{matrix};q,q\right)$$
$$+\frac{(\frac{C}{B},A,Bz;q)_\infty}{(C,z,\frac{C}{Bz};q)_\infty}{}_3\phi_2\left(\begin{matrix}z, & \frac{ABz}{C}, & 0\\ \frac{Bzq}{C}, & Bz & \end{matrix};q,q\right),$$

[29, page 245, III.34] and obtain

$$K(z,w;\alpha,\beta,\gamma,\delta)=\frac{(-\alpha\ell,\delta\ell z;q)_\infty}{2\pi i(\delta\ell;q)_inf ty}\left[\frac{(-\frac{\alpha\gamma\ell^2 w}{q(1-q)},-\frac{q}{\alpha\ell},q,\frac{q}{\delta\ell z},-\frac{\alpha w}{\delta};q)_\infty}{(\frac{\alpha\beta\ell^2 w}{q(1-q)},\frac{q}{\delta\ell},-\alpha\ell w,-\frac{q}{\alpha\ell w},z,-\frac{\alpha w}{\delta z};q)_\infty}\right.$$
$$\times\ {}_3\phi_2\left(\begin{matrix}w, & \frac{\alpha\beta\ell^2 w}{q(1-q)}, & -\frac{\alpha w}{\delta z}\\ -\frac{\alpha\gamma\ell^2 w}{q(1-q)}, & -\frac{\alpha w}{\delta} & \end{matrix};q,q\right)$$
$$\left.+\frac{(w,\frac{\gamma\ell}{1-q};q)_\infty}{(z,-\frac{\alpha\ell w}{q},-\frac{\beta\ell}{1-q};q)_\infty}{}_3\psi_3\left(\begin{matrix}-\frac{\beta\ell}{1-q}, & -\frac{q}{\alpha\ell}, & \frac{q}{\delta\ell z}\\ -\frac{q^2}{\alpha\ell w}, & \frac{\gamma\ell}{1-q}, & \frac{q}{\delta\ell}\end{matrix};q,q\right)\right]. \tag{5.21}$$

Here

$${}_3\psi_3\left(\begin{matrix}a_1, & a_2, & a_3\\ b_1, & b_2, & b_3\end{matrix};q,z\right)=\sum_{n=-\infty}^{\infty}\frac{(a_1,a_2,a_3;q)_n}{(b_1,b_2,b_3;q)_n}z^n,\quad |\frac{b_1b_2b_3}{a_1a_2a_3}|<|z|<1.$$

Now the formula

$$\mathbf{T}(\alpha,\beta,\gamma,\delta)\mathbf{T}(\alpha',\beta',-\beta,-\alpha)=\mathbf{T}(\alpha',\beta',\gamma,\delta)$$

leads to the functional relation

$$K(z,w;\alpha',\beta,\gamma',\delta)=\oint K(z,y;\alpha,\beta,\gamma,\delta)K(y,w;\alpha',\alpha,\gamma',\gamma)\,\frac{dy}{y}, \tag{5.22}$$

where, choosing the integration path as the unit circle $|y|=1$, we have the requirements

$$|\alpha w/\alpha'|,|\delta z/\gamma|<1<|\alpha z/\beta|,|\gamma' w/\gamma|.$$

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