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Models of q -Algebra Representations: Matrix Elements of $U_q(su_2)$

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Abstract. This paper continues a study of one and two variable function space models of irreducible representations of q-analogs of Lie enveloping algebras, motivated by recurrence relations satisfied by q-hypergeometric functions. Here we consider the quantum algebra $U_q(su_2)$. We show that various q-analogs of the exponential function can be used to mimic the exponential mapping from a Lie algebra to its Lie group and we compute the corresponding matrix elements of the "group operators" on these representation spaces. This "local" approach applies to more general families of special functions, e.g., with complex arguments and parameters, than does the quantum group approach. We show that the matrix elements themselves transform irreducibly under the action of the quantum algebra. We find an alternate and simpler derivation of a q-analog, due to Groza, Kachurik and Klimyk, of the Burchnall-Chaundy formula for the product of two hypergeometric functions ${}_2F_1$. It is interpreted here as the expansion of the matrix elements of a "group operator" (via the exponential mapping) in a tensor product basis in terms of the matrix elements in a reduced basis.

1. Introduction

This paper continues the study of function space models of irreducible representations of q-algebras [**1, 2**]. These algebras and models are motivated by recurrence relations satisfied by q-hypergeometric functions [**3**] and our treatment is an alternative to the theory of quantum groups. Here, we consider

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the irreducible representations of the quantum algebra $U_q(su_2)$. We replace the usual exponential function mapping from the Lie algebra to the Lie group by the q-exponential mappings E_q and e_q. In place of the usual matrix elements on the group (arising from an irreducible representation) which are expressible in terms of Jacobi polynomials, we find eight types of matrix elements expressible in terms of q-hypergeometric series. These q-matrix elements do not satisfy group homomorphism properties, so they do not lead to addition theorems in the usual sense. However, they do satisfy orthogonality relations. Furthermore, in analogy with true group representation theory we can show that each of the eight families of matrix elements determines a two-variable model for irreducible representations of $U_q(su_2)$. In §3 we show how this two-variable model leads to orthogonality relations for one class of matrix elements.

In §4 we find an alternate and simpler derivation of a q-analog, due to Groza, Kachurik and Klimyk [**4**], of the Burchnall-Chaundy formula for the product of two hypergeometric functions ${}_2F_1$. It is interpreted here as the expansion of the matrix elements of a "group operator" (via the exponential mapping) in a tensor product basis in terms of the matrix elements in a reduced basis.

Our approach to the derivation and understanding of q-series identities is based on the study of q-algebras as q-analogs of Lie algebras, [**5, 6**]. Essentially, we are attempting to find q-analogs of the theory relating Lie algebra and local Lie transformation groups [**7, 8**]. A similar approach has been adopted by Floreanini and Vinet [**9-12**]. This is an alternative to the elegant papers [**13-21**] which are based primarily on the theory of quantum groups. The main justification of the "local" approach is that it is more general; it applies to more general families of special functions than does the quantum group approach.

The notation used for q-series in this paper follows that of Gasper and Rahman [**22**].

2. Models of finite dimensional $U_q(su_2)$ representations

The quantum algebra $U_q(su_2)$ is the associative algebra generated by the elements H, E_+, E_- that obey the commutation relations

$$\begin{aligned} &[H, E_+] = E_+, \quad [H, E_-] = -E_-, \\ &[E_+, E_-] = \frac{q^H - q^{-H}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \end{aligned} \tag{2.1}$$

Here q is a real parameter such that $0 < q < 1$. In the limit as $q \to 1$ relations (2.1) go to the usual commutation relations for the complexification of the Lie algebra su_2. Finite dimensional irreducible representations of $U_q(su_2)$ are determined by the integral or half-integral number u: $2u = 0, 1, 2, \cdots$. The corresponding representation $D(2u)$ is defined on the $(2u+1)$-dimensional Hilbert space H_{2u} with orthonormal basis $\{e_m : m = -u, -u+1, \cdots, u\}$ such that

$$(2.2)\qquad \begin{aligned} E_+ e_m &= ([u-m]_q [u+m+1]_q)^{\frac{1}{2}}\, e_{m+1} \\ E_- e_m &= ([u+m]_q [u-m+1]_q)^{\frac{1}{2}}\, e_{m-1} \\ H e_m &= m e_m \end{aligned}$$

where

$$(2.3)\qquad [m]_q = \frac{q^{\frac{m}{2}} - q^{-\frac{m}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{-\frac{m-1}{2}} \left(\frac{1-q^m}{1-q} \right).$$

On this Hilbert space $E_+ = (E_-)^*$ and $H^* = H$. A second convenient basis for H_{2u} is the set $\{f_n : n = 0, 1, \cdots, 2u\}$ such that

$$(2.4)\qquad \begin{aligned} E_+ f_n &= -q^{-1}[2u-n]_q f_{n+1} \\ E_- f_n &= -q[n]_q f_{n-1} \\ H f_n &= (-u+n) f_n. \end{aligned}$$

Here

$$(2.5)\qquad f_n = \left[\frac{(-1)^n q^{(\frac{3}{2}-u)n} (q;q)_n}{(q^{-2u};q)_n} \right]^{\frac{1}{2}} e_{-u+n}.$$

Since the element

$$(2.6)\qquad C = E_+ E_- + \frac{q^{H-\frac{1}{2}} + q^{-H+\frac{1}{2}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}$$

commutes with each generator of $U_q(su_2)$ it corresponds to a multiple of the identity operator I on H_{2u}. Indeed,

$$(2.7)\qquad C = \frac{q^{u+\frac{1}{2}} + q^{-u-\frac{1}{2}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2} I.$$

Now we consider the discrete series $\uparrow_u$ of infinite dimensional representations of $U_q(su_2)$. This is defined on the Hilbert space H_0 with orthogonal basis $\{f_n : n = 0, 1, \cdots\}$ such that relations (2.4) hold. Now, however, u is a negative real number and

$$||f_n|| = \sqrt{\frac{q^{(\frac{3}{2}-u)n}(q;q)_n}{(q^{-2u};q)_n}}, \quad n = 0, 1, 2, \cdots.$$

On this Hilbert space $E_+ = -(E_-)^*$ and $H^* = H$. (In the limit as $q \to 1$ these representations correspond to the positive discrete series of unitary irreducible representations of the Lie algebra $su(1,1)$, [**18, 19**].)

Note, however, that for each complex number u such that $2u \neq 0, 1, 2, \cdots$ expressions (2.4) define an algebraically irreducible representation $\uparrow_u$ of $U_q(su_2)$ on an infinite dimensional vector space K consisting of all finite linear combinations of the basis vectors $\{f_n\}$.

We introduce a convenient one variable model of $D(2u)$. Here the vector space H_{2u} consists of polynomials $f(z)$ of maximum order $2u$ in the complex variable z. The action of $U_q(su_2)$ is defined by the operators

$$
\begin{aligned}
E_+ =&\frac{q^{-\frac{1}{2}}z}{1-q}(q^u T_z^{-\frac{1}{2}} - q^{-u}T_z^{\frac{1}{2}}) \\
E_- =&\frac{q^{\frac{3}{2}}}{z(1-q)}(T_z^{\frac{1}{2}} - T_z^{-\frac{1}{2}}) \\
H =& -u + z\frac{d}{dz}
\end{aligned}
\tag{2.8}
$$

where $T_z f(z) = f(qz)$. The basis functions $\{f_n = z^n : n = 0, 1 \cdots, 2u\}$ satisfy relations (2.4). Similarly, we can introduce a one variable model of $\uparrow_u$. A basis for the vector space consists of the functions $\{f_n(z) = z^n : n = 0, 1, 2\cdots\}$ in the complex variable z. The action of $U_q(su_2)$ on functions $f(z)$ is again defined by expressions (2.8). See [**1**] for derivations of the inner products corresponding to these models.

In analogy with a standard relationship between special functions and the representations of Lie groups we shall compute the "matrix elements" of q-analogs of the group operators $e^{\beta E_+}e^{\alpha E_-}$ with respect to the $\{f_n\}$ basis, in both the representations $D(2u)$ and $\uparrow_u$. Of course there are many q-analogs of the exponential mapping, none of which have all the properties needed to ensure that there is a true "group" associated with the q-algebra. Among the q-analogs we shall limit ourselves to the two that are most important, [**22**]:

$$
e_q(z) = \sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k}, \quad E_q(z) = \sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{(q;q)_k}z^k. \tag{2.9}
$$

If z is a complex number, the first series converges to $1/(z;q)_\infty$ for $|z| < 1$ and the second series converges to $(-z;q)_\infty$ for all z.

Using the model (2.4) we can define eight q-analogs of the matrix elements.

$$
\begin{aligned}
(e+,e-): &\quad e_q(\beta E_+)e_q(\alpha E_-)f_n = \sum_{n'} T^{(e+,e-)}_{n'n}(\alpha,\beta)f_{n'}, \\
(e-,e+): &\quad e_q(\beta E_-)e_q(\alpha E_+)f_n = \sum_{n'} T^{(e-,e+)}_{n'n}(\alpha,\beta)f_{n'}, \\
(e+,E-): &\quad e_q(\beta E_+)E_q(\alpha E_-)f_n = \sum_{n'} T^{(e+,E-)}_{n'n}(\alpha,\beta)f_{n'}, \\
(e-,E+): &\quad e_q(\beta E_-)E_q(\alpha E_+)f_n = \sum_{n'} T^{(e-,E+)}_{n'n}(\alpha,\beta)f_{n'}, \qquad (2.10) \\
(E+,e-): &\quad E_q(\beta E_+)e_q(\alpha E_-)f_n = \sum_{n'} T^{(E+,e-)}_{n'n}(\alpha,\beta)f_{n'},
\end{aligned}
$$

$$(E-,e+): \quad E_q(\beta E_-)e_q(\alpha E_+)f_n = \sum_{n'} T^{(E-,e+)}_{n'n}(\alpha,\beta)f_{n'},$$

$$(E+,E-): \quad E_q(\beta E_+)E_q(\alpha E_-)f_n = \sum_{n'} T^{(E+,E-)}_{n'n}(\alpha,\beta)f_{n'},$$

$$(E-,E+): \quad E_q(\beta E_-)E_q(\alpha E_+)f_n = \sum_{n'} T^{(E-,E+)}_{n'n}(\alpha,\beta)f_{n'}.$$

For the representations $D(2u)$ the sum on n' is from 0 to $2u$ and $n = 0,\cdots,2u$. For the representations $\uparrow_u$, the sum on n' is from 0 to ∞ and $n = 0,1,\cdots$. (Relations (2.10) and the relations to follow are well defined as formal power series in the variables α, β. We must look at each case separately to determine the domains of pointwise convergence of the series.)

Since $E_+^* = E_-$ for $D(2u)$ and $E_+^* = -E_-$ for $\uparrow_u$ the following relationships hold:

$$\begin{aligned}
&T^{(e+,e-)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(e+,e-)}_{nn'}(-\beta,-\alpha),\\
&T^{(e-,e+)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(e-,e+)}_{nn'}(-\beta,-\alpha),\\
&T^{(e+,E-)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(E+,e-)}_{nn'}(-\beta,-\alpha),\\
&T^{(e-,E+)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(E-,e+)}_{nn'}(-\beta,-\alpha),\\
&T^{(E+,E-)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(E+,E-)}_{nn'}(-\beta,-\alpha),\\
&T^{(E-,E+)}_{n'n}(\alpha,\beta)A_{n'n} = T^{(E-,E+)}_{nn'}(-\beta,-\alpha).
\end{aligned} \tag{2.11}$$

Here

$$A_{n'n} = \frac{q^{(3/2-u)(n'-n)}(q;q)_{n'}(q^{-2u};q)_n}{(q;q)_n(q^{-2u};q)_{n'}}.$$

Since $e_q(z)E_q(-z) = 1$, we have the identities

$$\begin{aligned}
&a)\ \sum_\ell T^{(e+,e-)}_{n'\ell}(\alpha,\beta)T^{(E-,E+)}_{\ell n}(-\beta,-\alpha) = \delta_{n'n}\\
&b)\ \sum_\ell T^{(e-,e+)}_{n'\ell}(\alpha,\beta)T^{(E+,E-)}_{\ell n}(-\beta,-\alpha) = \delta_{n'n}\\
&c)\ \sum_\ell T^{(E-,e+)}_{n'\ell}(\alpha,\beta)T^{(E+,e-)}_{\ell n}(-\beta,-\alpha) = \delta_{n'n}.
\end{aligned} \tag{2.12}$$

Again, the sum is finite or infinite, depending on the representation $D(2u)$ or $\uparrow_u$.

Using the model (2.4) to compute the matrix elements (which are model independent) we obtain the explicit results

$$
\begin{aligned}
&T^{(e+,e-)}_{n'n}(\alpha,\beta)\\
&=\frac{(q^{-2u};q)_{n'}\beta^{n'-n}q^{-(n'-n)(n'+n+1)/4}}{q^{u(n-n')}(q;q)_{n'-n}(q^{-2u};q)_n(1-q)^{n'-n}}\\
&\times {}_3\phi_1\left(\begin{matrix} q^{-n}, & q^{1-n+2u}, & 0\\ q^{n'-n+1} & & \end{matrix};q;\frac{\alpha\beta q^{n-u+1/2}}{(1-q)^2}\right)\\
&=\frac{(q;q)_n(-\alpha)^{n-n'}q^{-(n-n')(n'+n-5)/4}}{(q;q)_{n-n'}(q;q)_{n'}(1-q)^{n-n'}}\\
&\times {}_3\phi_1\left(\begin{matrix} q^{-n'}, & q^{1-n'+2u}, & 0\\ q^{n-n'+1} & & \end{matrix};q;\frac{\alpha\beta q^{n'-u+1/2}}{(1-q)^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(e-,e+)}_{n'n}(\alpha,\beta)\\
&=\frac{(q^{-2u};q)_{n'}\alpha^{n'-n}q^{-(n'-n)(n'+n+1)/4}}{q^{u(n-n')}(q;q)_{n'-n}(q^{-2u};q)_n(1-q)^{n'-n}}\\
&\times {}_3\phi_1\left(\begin{matrix} q^{n'+1}, & q^{n'-2u}, & 0\\ q^{n'-n+1} & & \end{matrix};q;\frac{\alpha\beta q^{u-n'+1/2}}{(1-q)^2}\right)\\
&=\frac{(q;q)_n(-\beta)^{n-n'}q^{-(n-n')(n+n'-5)/4}}{(q;q)_{n-n'}(q;q)_{n'}(1-q)^{n-n'}}\\
&\times {}_3\phi_1\left(\begin{matrix} q^{n+1}, & q^{n-2u}, & 0\\ q^{n-n'+1} & & \end{matrix};q;\frac{\alpha\beta q^{u-n+1/2}}{(1-q)^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(E+,e-)}_{n'n}(\alpha,\beta)\\
&=\frac{(q^{-2u};q)_{n'}\beta^{n'-n}q^{(n'-n)(n'-3n-3)/4}}{q^{u(n-n')}(q;q)_{n'-n}(q^{-2u};q)_n(1-q)^{n'-n}}\\
&\times {}_2\phi_1\left(\begin{matrix} q^{-n}, & q^{1-n+2u}\\ q^{n'-n+1} & \end{matrix};q;\frac{-\alpha\beta q^{n'-u+1/2}}{(1-q)^2}\right)\\
&=\frac{(q;q)_n(-\alpha)^{n-n'}q^{-(n-n')(n+n'-5)/4}}{(q;q)_{n-n'}(q;q)_{n'}(1-q)^{n-n'}}\\
&\times {}_2\phi_1\left(\begin{matrix} q^{-n'}, & q^{1-n'+2u}\\ q^{n-n'+1} & \end{matrix};q;\frac{-\alpha\beta q^{n'-u+1/2}}{(1-q)^2}\right)
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&T^{(E-,e+)}_{n'n}(\alpha,\beta)\\
&=\frac{(q^{-2u};q)_{n'}\alpha^{n'-n}q^{-(n'-n)(n'+n+1)/4}q^{u(n'-n)}(-\alpha\beta q^{n-u+1/2}/(1-q)^2;q)_\infty}{(q;q)_{n'-n}(q^{-2u};q)_n(1-q)^{n'-n}(-\alpha\beta q^{u-n'+1/2}/(1-q)^2;q)_\infty}\\
&\times {}_2\phi_1\left(\begin{matrix} q^{-n}, & q^{1-n+2u}\\ q^{n'-n+1} & \end{matrix};q;\frac{-\alpha\beta q^{n-u+1/2}}{(1-q)^2}\right)\\
&=\frac{(q;q)_n(-\beta)^{n-n'}q^{(n-n')(n-3n'+3)/4}(-\alpha\beta q^{n-u+1/2}/(1-q)^2;q)_\infty}{(q;q)_{n-n'}(q;q)_{n'}(1-q)^{n-n'}(-\alpha\beta q^{u-n'+1/2}/(1-q)^2;q)_\infty}\\
&\times {}_2\phi_1\left(\begin{matrix} q^{-n'}, & q^{1-n'+2u}\\ q^{n-n'+1} & \end{matrix};q;\frac{-\alpha\beta q^{n-u+1/2}}{(1-q)^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(E-,E+)}_{n'n}(\alpha,\beta)\\
&=\frac{(q;q)_n(-\beta)^{n-n'}q^{(n-n')(n-3n'+3)/4}}{(q;q)_{n-n'}(q;q)_{n'}(1-q)^{n-n'}}\\
&\times {}_2\phi_2\left(\begin{matrix} q^{n-2u}, & q^{n+1}\\ q^{n-n'+1}, & 0\end{matrix};q;\frac{\alpha\beta q^{u-n'+1/2}}{(1-q)^2}\right)\\
&=\frac{(q^{-2u};q)'_n\alpha^{n'-n}q^{(n'-n)(n'-3n-3)/4}}{q^{u(n-n')}(q;q)_{n'-n}(q^{-2u};q)_n(1-q)^{n'-n}}\\
&\times {}_2\phi_2\left(\begin{matrix} q^{n'-2u}, & q^{n'+1}\\ q^{n'-n+1}, & 0\end{matrix};q;\frac{\alpha\beta q^{u-n+1/2}}{(1-q)^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(E+,E-)}_{n'n}(\alpha,\beta)\\
&=\frac{(q^{-2u};q)_{n'}\beta^{n'-n}q^{-(n'-n)(n'-3n-3)/4}}{q^{u(n-n')}(q;q)_{n'-n}(q^{-2u};q)_n(1-q)^{n'-n}}\\
&\times {}_2\phi_2\left(\begin{matrix} q^{-n}, & q^{2u-n+1}\\ q^{n'-n+1}, & 0\end{matrix};q;\frac{\alpha\beta q^{n'-u+1/2}}{(1-q)^2}\right)\\
&=\frac{(q;q)_n(-\alpha)^{n-n'}q^{(n-n')(n-3n'+3)/4}}{(q;q)_{n-n'}(q;q)_{n'}(1-q)^{n-n'}}\\
&\times {}_2\phi_2\left(\begin{matrix} q^{-n'}, & q^{2u-n'+1}\\ q^{n-n'+1}, & 0\end{matrix};q;\frac{\alpha\beta q^{n-u+1/2}}{(1-q)^2}\right)
\end{aligned}
$$

For the finite dimensional representations $D(2u)$ all of these matrix elements are polynomials. For the infinite dimensional representations the elements $T^{(e-,e+)}$ diverge for $\alpha\beta \neq 0$, the $T^{(E-,e+)}$ converge for $|\alpha\beta q^{u-n'+1/2}/(1-q)^2| < 1$, and the remaining matrix elements are entire functions. Identities (2.12) are valid for all α, β corresponding to the finite dimensional representations $D(2u)$. Using the ratio test to determine the domains of convergence corresponding to $\uparrow_u$ we find that (2.12a) converges for all α, β, (2.12b) diverges, and (2.12c), multiplied by $(-\alpha\beta q^{u-n'+1/2}/(1-q)^2;q)_\infty$, converges for all α, β.

Each of these families of matrix elements determines models of the irreducible representations $D(2u)$ and $\uparrow_u$. This is a consequence of the commutation relations (2.1). To see this we make use of the following formal power series results for linear operators X, Y:

LEMMA 1.

$$E_q(\alpha X)Ye_q(-\alpha X) = \sum_{n=0}^{\infty} \frac{\alpha^n}{(q;q)_n}[X,Y]_n$$

where

$$[X,Y]_0 = Y, \quad [X,Y]_{n+1} = X[X,Y]_n q^n - [X,Y]_n X, \ n = 0, 1, \cdots .$$

LEMMA 2.

$$e_q(\beta X)YE_q(-\beta X) = \sum_{n=0}^{\infty} \frac{\beta^n}{(q;q)_n}[X,Y]'_n$$

where

$$[X,Y]'_0 = Y, \quad [X,Y]'_{n+1} = X[X,Y]'_n - q^n[X,Y]'_n X, \ n = 0, 1, \cdots .$$

As a consequence of these lemmas we have

(2.14)

$$a)\ E_q(\alpha E_-)E_+ = E_+E_q(\alpha E_-) - \frac{q^{1/2}\alpha}{(1-q)^2}\left(q^{-H}E_q(\alpha E_-) - q^H E_q(\alpha q E_-)\right)$$

$$b)\ e_q(\beta E_+)E_- = E_-e_q(\beta E_+) + \frac{q^{1/2}\beta}{(1-q)^2}\left(q^{-H}e_q(\beta E_+) - q^H e_q(\frac{\beta}{q}E_+)\right).$$

Note also the easily verified identities

$$E_q(-\beta E_+)q^{-H}e_q(\beta q E_+) = q^{-H},$$
$$E_q(-\beta q E_-)q^{-H}e_q(\beta E_-) = q^{-H}.$$

For the matrix elements (e+,e-) the operator identities

(2.15)

$$a)\ e_q(\beta E_+)e_q(\alpha E_-)E_- = \frac{1}{\alpha}(I - T_\alpha)e_q(\beta E_+)e_q(\alpha E_-)$$

$$b)\ e_q(\beta E_+)e_q(\alpha E_-)E_+ = \frac{1}{\beta}(I - T_\beta)e_q(\beta E_+)e_q(\alpha E_-)$$
$$+ \frac{\alpha q^{1/2}}{(1-q^2)}(q^{-u+n'}T_\beta^{-1} - q^{n'-u}T_\beta T_\alpha^{-1})e_q(\beta E_+)e_q(\alpha E_-)$$

$$c)\ [H, e_q(\beta E_+)e_q(\alpha E_-)] = (\beta\partial_\beta - \alpha\partial_\alpha)e_q(\beta E_+)e_q(\alpha E_-)$$

imply

$$
\begin{aligned}
&a)\ \ -q[n]_q T^{(e+,e-)}_{n',n-1}(\alpha,\beta)=\frac{1}{\alpha}(I-T_\alpha)T^{(e+,e-)}_{n'n}(\alpha,\beta)\\
&b)\ \ -q^{-1}[2u-n]_q T^{(e+,e-)}_{n',n+1}(\alpha,\beta)=\\
&\quad\left(\frac{1}{\beta}(I-T_\beta)+\frac{\alpha q^{1/2}}{(1-q)^2}(q^{-u+n'}T_\beta^{-1}-q^{u-n'}T_\beta T_\alpha^{-1})\right)T^{(e+,e-)}_{n'n}(\alpha,\beta)\\
&c)\ (-u+n)T^{(e+,e-)}_{n'n}(\alpha,\beta)=(-u+n'+\alpha\partial_\alpha-\beta\partial_\beta)\,T^{(e+,e-)}_{n'n}(\alpha,\beta)
\end{aligned}
$$

where $T_\alpha f(\alpha,\beta)=f(q\alpha,\beta)$.

For the matrix elements (e-,e+) we find

$$
\begin{aligned}
&a)\ \ -q[n]_q T^{(e-,e+)}_{n',n-1}(\alpha,\beta)=\\
&\quad\left(\frac{1}{\beta}(I-T_\beta)+\frac{\alpha q^{1/2}}{(1-q)^2}(q^{u-n'}T_\beta^{-1}-q^{-u+n'}T_\beta T_\alpha^{-1})\right)T^{(e-,e+)}_{n'n}(\alpha,\beta)\\
&b)\ \ -q^{-1}[2u-n]_q T^{(e-,e+)}_{n',n+1}(\alpha,\beta)=\frac{1}{\alpha}(I-T_\alpha)T^{(e-,e+)}_{n'n}(\alpha,\beta)\\
&c)\ (-u+n)T^{(e-,e+)}_{n'n}(\alpha,\beta)=(-u+n'-\alpha\partial_\alpha+\beta\partial_\beta)\,T^{(e-,e+)}_{n'n}(\alpha,\beta),
\end{aligned}
$$

for the matrix elements (E+,e-) we find

$$
\begin{aligned}
&a)\ \ -q[n]_q T^{(E+,e-)}_{n',n-1}(\alpha,\beta)=\frac{1}{\alpha}(I-T_\alpha)T^{(E+,e-)}_{n'n}(\alpha,\beta)\\
&b)\ \ -q^{-1}[2u-n]_q T^{(E+,e-)}_{n',n+1}(\alpha,\beta)=\\
&\quad\left(\frac{q}{\beta}(T_\beta^{-1}-I)+\frac{\alpha q^{1/2}}{(1-q)^2}(q^{-u+n'}T_\beta^{-1}-q^{u-n'}T_\beta T_\alpha^{-1})\right)T^{(E+,e-)}_{n'n}(\alpha,\beta)\\
&c)\ (-u+n)T^{(E+,e-)}_{n'n}(\alpha,\beta)=(-u+n'+\alpha\partial_\alpha-\beta\partial_\beta)\,T^{(E+,e-)}_{n'n}(\alpha,\beta),
\end{aligned}
\tag{2.16}
$$

for the matrix elements (E-,e+) we find

$$
\begin{aligned}
&a)\ \ -q[n]_q T^{(E-,e+)}_{n',n-1}(\alpha,\beta)=\\
&\quad\left(\frac{q}{\beta}(T_\beta^{-1}-I)+\frac{\alpha q^{1/2}}{(1-q)^2}(q^{u-n'}T_\beta^{-1}-q^{-u+n'}T_\beta T_\alpha^{-1})\right)T^{(E-,e+)}_{n'n}(\alpha,\beta)\\
&b)\ \ -q^{-1}[2u-n]_q T^{(E-,e+)}_{n',n+1}(\alpha,\beta)=\frac{1}{\alpha}(I-T_\alpha)T^{(E-,e+)}_{n'n}(\alpha,\beta)\\
&c)\ (-u+n)T^{(E-,e+)}_{n'n}(\alpha,\beta)=(-u+n'-\alpha\partial_\alpha+\beta\partial_\beta)\,T^{(E-,e+)}_{n'n}(\alpha,\beta),
\end{aligned}
$$

for the matrix elements (E-,E+) we find

$$\begin{aligned}
&a)\ -q[n]_q T^{(E-,E+)}_{n',n-1}(\alpha,\beta) = \\
&\quad \left(\frac{q}{\beta}(T_\beta^{-1}-I) + \frac{\alpha q^{1/2}}{(1-q)^2}(q^{u-n'}T_\beta^{-1}T_\alpha - q^{-u+n'}T_\beta)\right) T^{(E-,E+)}_{n'n}(\alpha,\beta)\\
&b)\ -q^{-1}[2u-n]_q T^{(E-,E+)}_{n',n+1}(\alpha,\beta) = \frac{q}{\alpha}(T_\alpha^{-1}-I)T^{(E-,E+)}_{n'n}(\alpha,\beta)\\
&c)\ (-u+n)T^{(E-,E+)}_{n'n}(\alpha,\beta) = (-u+n'-\alpha\partial_\alpha+\beta\partial_\beta)\,T^{(E-,E+)}_{n'n}(\alpha,\beta),
\end{aligned}$$

and for the matrix elements (E+,E-) we have

$$\begin{aligned}
&a)\ -q[n]_q T^{(E+,E-)}_{n',n-1}(\alpha,\beta) = \frac{q}{\alpha}(T_\alpha^{-1}-I)T^{(E+,E-)}_{n'n}(\alpha,\beta)\\
&b)\ -q^{-1}[2u-n]_q T^{(E+,E-)}_{n',n+1}(\alpha,\beta) = \\
&\quad \left(\frac{q}{\beta}(T_\beta^{-1}-I) + \frac{\alpha q^{1/2}}{(1-q)^2}(q^{-u+n'}T_\beta^{-1}T_\alpha - q^{u-n'}T_\beta)\right) T^{(E+,E-)}_{n'n}(\alpha,\beta)\\
&c)\ (-u+n)T^{(E+,E-)}_{n'n}(\alpha,\beta) = (-u+n'+\alpha\partial_\alpha-\beta\partial_\beta)\,T^{(E+,E-)}_{n'n}(\alpha,\beta).
\end{aligned}$$

These relations are equivalent to q-difference relations satisfied by various q-hypergeometric series. Furthermore, it is easy to verify from the series that the relations hold also for u and n' complex. Thus we have a wide variety of two variable models of algebraically irreducible representations of $U_q(su_2)$. We note that this approach is closely related to the factorization method of quantum mechanics [**23**].

Let X and Y be linear operators such that $YX = qXY$. A straightforward formal induction argument using this property, [**15**], [**22, page 28**], yields

Lemma 3.

$$(Y+X)^k = \sum_{\ell=0}^{k} \frac{(q;q)_k}{(q;q)_\ell (q;q)_{k-\ell}} X^\ell Y^{k-\ell},$$

$$e_q(X+Y) = e_q(X)e_q(Y), \quad E_q(X+Y) = E_q(Y)E_q(X).$$

Using these results and iterating relations (2.14) we find

$$\begin{aligned}
&E_q\left(\frac{\alpha\beta q^{1/2+H}}{(1-q)^2}\right) E_q(\beta E_+)e_q(\alpha E_-) = e_q(\alpha E_-)E_q(\beta E_+)E_q\left(\frac{\alpha\beta q^{1/2-H}}{(1-q)^2}\right),\\
&E_q\left(\frac{\alpha\beta q^{1/2-H}}{(1-q)^2}\right) E_q(\alpha E_-)e_q(\beta E_+) = e_q(\beta E_+)E_q(\alpha E_-)E_q\left(\frac{\alpha\beta q^{1/2+H}}{(1-q)^2}\right)
\end{aligned}$$

or

$$E_q\left(\frac{\alpha\beta q^{1/2-u+n'}}{(1-q)^2}\right)T^{(E+,e-)}_{n'n}(\alpha,\beta)=E_q\left(\frac{\alpha\beta q^{1/2+u-n}}{(1-q)^2}\right)T^{(e-,E+)}_{n'n}(\beta,\alpha),$$
$$E_q\left(\frac{\alpha\beta q^{1/2+u-n'}}{(1-q)^2}\right)T^{(E-,e+)}_{n'n}(\beta,\alpha)=E_q\left(\frac{\alpha\beta q^{1/2-u+n}}{(1-q)^2}\right)T^{(e+,E-)}_{n'n}(\alpha,\beta).$$

Identities (2.12a-c) yield orthogonality and biorthogonality relations for q-hypergeometric functions. For example, (2.12c) can be written in the form

(2.17)
$$\sum_\ell \frac{(zq^{\ell+1};q)_\infty(-q^{2u}z)^\ell(q^{-2u};q)_\ell}{q^{\ell(n+n')-\frac{\ell^2}{2}}}p_{n'}^{(\ell-n',2u-n'-\ell)}(zq^\ell)p_n^{(\ell-n,2u-n-\ell)}(zq^\ell)$$
$$=(zq^{1-n+2u};q)_\infty(-z)^n q^{2un-n/2-3n^2/2}(q;q)_n(q^{-2u};q)_n\delta_{nn'}.$$

Here the

$$p_n^{(\alpha,\beta)}(z)=\frac{(aq;q)_n}{(q;q)_n}{}_2\phi_1\left(\begin{matrix}q^{-n},q^{n+1}ab\\ aq\end{matrix};qz\right)$$
$$n=0,1,2,\cdots$$

are little q-Jacobi polynomials, [**24**], where $a=q^\alpha$, $b=q^\beta$. Note that the little q-Jacobi polynomials satisfy the identity

$$\frac{(q;q)_n q^{\frac{1}{2}(n'-n)(\frac{1}{2}-n)}}{(q^{\beta+1};q)_n(-z)^n}p_n^{(n'-n,\beta)}(z)=\frac{(q;q)_{n'}q^{\frac{1}{2}(n-n')(\frac{1}{2}-n')}}{(q^{\beta+1};q)_{n'}(-z)^{n'}}p_{n'}^{(n-n',\beta)}(z).$$

Since $F_n(y)=q^{-n\ell}p_n^{(\ell-n,2u-n-\ell)}(zq^\ell)$ is a polynomial of order n in the variable $y=q^{-\ell}$, we see that expression (2.17) is an orthogonality relation for this set of polynomials. See [**17**] for similar identities associated with q-Bessel functions.

A nontrivial extension of identity (2.12c) is

$$\sum_\ell T^{(E-,e+)}_{n'\ell}(\alpha,\gamma)T^{(E+,e-)}_{\ell n}(\beta,-\alpha)=S_{n'n}(\gamma,\beta)$$

where the matrix elements $S_{n'n}(\gamma,\beta)$ are defined by

$$e_q(\beta E_-)E_q(\gamma E_-)f_n=\sum_{n'}S_{n'n}(\gamma,\beta)f_{n'}.$$

Explicitly,

$$S_{n'n}(\gamma,\beta)=\begin{cases}\frac{\beta^{n-n'}(q^{-n};q)_{n-n'}}{(1-q)^{n-n'}(q;q)_{n-n'}}q^{(n-n')(7+n+n')/4}(-\frac{\gamma}{\beta};q)_{n-n'} & \text{if } n\ge n'\\ 0 & \text{if } n<n'.\end{cases}$$

In the special case $\gamma/\beta = -q^{n'-n+1}$ we find the result

$$\sum_\ell (zq^{n'+\ell+2};q)_\infty \frac{(q^{-2u};q)_\ell}{(q;q)_\ell}(-z)^\ell q^{\ell(\frac{3}{2}+2u-n)+\ell^2/2} p_{n'}^{(\ell-n',2u-n'-\ell)}(zq^{n'+\ell+1})\times$$
$$p_n^{(\ell-n,2u-n-\ell)}(zq^{n+\ell}) = (zq^{2u+2};q)_\infty(-z)^n q^{n(\frac{3}{2}+2u)-\frac{3}{2}n^2}(q;q)_n(q^{-2u};q)_n\delta_{nn'}.$$

(In the case of the Lie algebra of the Euclidean group in the plane, the analogous identities are the Hansen-Lommel identities for q-Bessel functions, [**25, 13, 2**].) There are similar extensions of (2.12a-b).

3. Orthogonality relations for matrix elements

The matrix elements of the form (E,e) and (e,E) satisfy orthogonality relations analogous to those arising from integration with respect to the Haar measure on $SU(2)$ in the $q=1$ case. Indeed these relations are directly derivable from the q-difference relations for the matrix elements $T^{(E+,e-)}_{n'n}(\alpha,\beta)$ in §2. We follow the derivation in [**3**] but now make clear the connection to the representation theory of $U_q(su_2)$.

Making the change of variable $z = -\alpha\beta q^{n'-u-1/2}/(1-q)^2$ and factoring out the dependence of β in the first equation (2.13) for $T^{(E+,e-)}_{n'n}(\alpha,\beta)$ and in the recurrence relations for these expressions, we obtain the (ordinary) difference equations

$$\begin{aligned} \tau^{(a,b)}p_n^{(\alpha,\beta)} &= -q^{1-n}(1-q^{n+1}ab)p_{n-1}^{(\alpha+1,\beta+1)} \\ \tau^{*(aq,bq)}p_{n-1}^{(\alpha+1,\beta+1)} &= -(1-q^n)p_n^{(\alpha,\beta)} \end{aligned} \tag{3.1}$$

where the $p_n^{(\alpha,\beta)}(z)$ are little q-Jacobi polynomials, a and b are complex numbers, $a=q^\alpha$, $b=q^\beta$ and

$$\tau^{(a,b)} = \frac{1}{z}(I-T_z), \quad \tau^{*(aq,bq)} = (z-1)T_z^{-1} + (aq-abq^2z).$$

Let $S_{a,b}$ be the space of all polynomials in z with the complex bilinear form

$$(f_1,f_2)_{a,b} = \frac{1}{2\pi i}\oint_C f_1(z)f_2(z)w_{a,b}(z)\,\frac{dz}{z},\ f_1,f_2\in S_{a,b} \tag{3.2}$$

where $C=\{z : |z| = 1+\epsilon,\quad \epsilon>0\}$ and $w_{a,b}$ is a weight function to be determined. We consider τ and τ^* as mappings

$$\tau^{(a,b)} : S_{a,b}\to S_{aq,bq}, \quad \tau^{*(aq,bq)} : S_{aq,bq}\to S_{a,b}$$

and look for a weight function so that τ^* is the adjoint of τ:

$$(\tau f,g)_{aq,bq} = (f,\tau^* g)_{a,b}, \quad \forall f\in S_{a,b}, \quad g\in S_{aq,bq}. \tag{3.3}$$

This yields the necessary and sufficient conditions

$$w_{aq,bq}(z/q) = \frac{z}{q}(1-z)w_{a,b}(z), \quad w_{aq,bq}(z) = aqz(1-bqz)w_{a,b}(z)$$

with the solution, unique to within a constant multiplier,

$$w_{a,b}(z) = \frac{(\frac{z}{a};q)_\infty(\frac{qa}{z};q)_\infty}{(zbq;q)_\infty(\frac{1}{z};q)_\infty(aq;q)_\infty(\frac{1}{a};q)_\infty(-aq;q)_\infty(-\frac{1}{a};q)_\infty}. \tag{3.4}$$

By construction $\tau^*\tau : S_{a,b} \to S_{a,b}$ is self-adjoint and

$$\tau^*\tau p_n^{(\alpha,\beta)} = \lambda_n p_n^{(\alpha,\beta)}, \quad \lambda_n = -q(1-q^{-n})(1-q^{n+1}ab).$$

Clearly $\lambda_n \neq \lambda_m$ if $n \neq m$ so we must have

$$(p_n^{(\alpha,\beta)}, p_m^{(\alpha,\beta)})_{a,b} = 0 \quad \text{if } m \neq n.$$

We have proved orthogonality for the little q-Jacobi polynomials. It remains to compute the normalization of these functions. Setting $f = p_n^{(\alpha,\beta)}$, $g = p_{n-1}^{(\alpha+1,\beta+1)}$ in the adjoint relation (3.3) we obtain the recurrence

$$||p_n^{(\alpha,\beta)}||^2_{a,b} = \frac{q(1-q^{n+1}ab)}{(1-q^n)}||p_{n-1}^{(\alpha+1,\beta+1)}||^2_{aq,bq}.$$

From the relation $(p_1^{(\alpha,\beta)}, p_0^{(\alpha,\beta)})_{a,b} = 0$ and the explicit expressions for p_1, p_0 we find

$$||1||^2_{a,bq} = \frac{(1-bq)}{(1-abq^2)}||1||^2_{a,b}. \tag{3.5}$$

(Here we use the fact that $(1-zbq, 1)_{a,b} = ||1||^2_{a,bq}$.)

To completely determine the norm we use the second equation (2.13) for $T_{n'n}^{(E+,e-)}(\alpha,\beta)$. Here the induced action of the recurrence operators on the basis $\{p_n^{(\alpha,\beta)}\}$ is

$$\mu^{(a,b)}p_n^{(\alpha,\beta)} = (1-aq^n)p_n^{(\alpha-1,\beta+1)}, \ \mu^{(aq^{-1},bq)}p_n^{(\alpha-1,\beta+1)} = \frac{q^{-n}}{a}(1-bq^{n+1})p_n^{(\alpha,\beta)}$$

where

$$\mu^{(a,b)} = 1 - aT_z, \quad \mu^{(a,b)} : S_{a,b} \to S_{aq^{-1},bq},$$

$$\mu^{*(aq^{-1},bq)} = \frac{z-1}{az}T_z^{-1} + \frac{1-bqz}{az}, \quad \mu^{*(aq^{-1},bq)} : S_{aq^{-1},bq} \to S_{a,b}.$$

It is straightforward to verify that

$$(\mu f, g)_{aq^{-1},bq} = (f, \mu^* g)_{a,b}, \quad \forall f \in S_{a,b},\ g \in S_{aq^{-1},bq}, \tag{3.6}$$

so μ^* is the adjoint of μ. Now set $f = p_0^{(\alpha,\beta)} = 1$, $g = p_0^{(\alpha-1,\beta+1)} = 1$ in (3.6) to obtain

$$||1||^2_{aq^{-1},bq} = \frac{(1-bq)}{a(1-a)}||1||^2_{a,b}. \tag{3.7}$$

The solution of the recurrences (3.5) and (3.7) is

$$||1||^2_{a,b} = \frac{(abq^2;q)_\infty K(q)}{(bq;q)_\infty(aq;q)_\infty(-\frac{1}{a};q)_\infty(-aq;q)_\infty}.$$

Thus we have

$$\frac{1}{2\pi i}\oint_C \frac{(\frac{z}{a};q)_\infty(\frac{qa}{z};q)_\infty}{(\frac{1}{a};q)_\infty(\frac{1}{z};q)_\infty(zbq;q)_\infty}\,dz = \frac{(abq^2;q)_\infty K(q)}{(bq;q)_\infty}.$$

To compute $K(q)$ we make particular choices of a and b such that the integral becomes trivial:

$$a > 1+\epsilon > 1 > qa, \quad abq = 1.$$

It follows easily that $K(q) = 1/(q;q)_\infty$. Thus we have derived the normalization for the little q-Jacobi polynomials entirely by elementary techniques and proved complex orthogonality with respect to the bilinear form (3.3), (3.4).

To obtain the orthogonality of the $\{p_n^{(\alpha,\beta)}\}$ with respect to a real positive definite inner product we expand the integral expression for the complex bilinear form $(p_n^{(\alpha,\beta)}, p_m^{(\alpha,\beta)})_{a,b}$ by residues at the poles $z = q^k$, $k = 0,1,2,\cdots$ of the weight function. The straightforward computation yields

(3.8)
$$\int_0^1 p_n^{(\alpha,\beta)}(x)p_m^{(\alpha,\beta)}(x)\rho_{\alpha,\beta}(x)\,d_qx = \frac{(aq)^n(abq^{n+1};q)_\infty(aq;q)_n(q;q)_\infty(1-q)}{(bq^{n+1};q)_\infty(aq;q)_\infty(q;q)_n(1-abq^{2m+1})}\delta_{m,n}$$

where

$$\int_0^1 f(x)\,d_qx = (1-q)\sum_{k=0}^{\infty} f(q^k)q^k$$

and $\rho_{\alpha,\beta}(x) = x^\alpha(xq;q)_\infty/(bxq;q)_\infty$. See [**26-28**] for related results on orthogonality of q-series and associated polynomials of a discrete variable.

4. A tensor product identity for $U_q(su_2)$ representations

Given the irreducible representations $D(2u_1)$ and $D(2u_2)$ on the spaces H_{2u_1} and H_{2u_2}, respectively, we define the tensor product representation $D(2u_1)\otimes D(2u_2)$ of $U_q(su_2)$ on the space $H_{2u_1}\otimes H_{2u_2}$ by the operators [**5**],

$$\begin{aligned} F_+ &= \Delta(E_+) = E_+\otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H}\otimes E_+ \\ F_- &= \Delta(E_-) = E_-\otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H}\otimes E_- \\ L &= \Delta(H) = H\otimes I + I\otimes H. \end{aligned} \tag{4.1}$$

The operators $F_\pm$, L satisfy the same commutation relations as the operators $E_\pm$, H:

$$[L, F_\pm] = \pm F_\pm, \quad [F_+, F_-] = \frac{q^L - q^{-L}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{4.2}$$

From Lemma 3 we have the formal identity

$$\begin{aligned} E_q(\beta F_+)e_q(\alpha F_-) &= E_q(\beta E_+ \otimes q^{\frac{1}{2}H})E_q(\beta q^{-\frac{1}{2}H} \otimes E_+) \\ &\times e_q(\alpha E_- \otimes q^{\frac{1}{2}H})e_q(\alpha q^{-\frac{1}{2}H} \otimes E_-) \\ &= E_q(\beta E_+ \otimes q^{\frac{1}{2}H})e_q(\alpha E_- \otimes q^{\frac{1}{2}H}) \\ &\times E_q(\beta q^{-\frac{1}{2}H} \otimes E_+)e_q(\alpha q^{-\frac{1}{2}H} \otimes E_-). \end{aligned}$$

Thus,

$$T^{(E+,e-)}_{n'm',nm}(\alpha,\beta) = T^{(E+,e-),u_1}_{n'n}(\alpha q^{\frac{m'}{2}}, \beta q^{\frac{m'}{2}})T^{(E+,e-),u_2}_{m'm}(\alpha q^{-\frac{n}{2}}, \beta q^{-\frac{n}{2}}) \tag{4.3}$$

where the functions $T^{(E+,e-)}_{n'm',nm}(\alpha,\beta)$ are the matrix elements of the operator $E_q(\beta F_+)e_q(\alpha F_-)$ in the tensor product basis $\{f^{u_1}_n \otimes f^{u_2}_m\}$. In other words, for this operator the matrix elements in a tensor product basis actually factor.

It is well known that the tensor product decomposes into a direct sum of irreducible representations [**5, 29, 1**],

$$D(2u_1) \otimes D(2u_2) \cong \sum_{v=|u_1-u_2|}^{u_1+u_2} \oplus D(2v).$$

Choosing canonical orthonormal basis vectors $\{e^v_m : m = v, v-1, \cdots, -v\}$ for the irreducible subspace corresponding to the representation $D(2v)$ we define the Clebsch-Gordan coefficients by the expansion formula

$$e^v_m = \sum_{n_1,n_2} \begin{bmatrix} u_1 & u_2 & v \\ n_1 & n_2 & m \end{bmatrix}_q e^{u_1}_{n_1} \otimes e^{u_2}_{n_2}. \tag{4.4}$$

A generating function for the coefficients is given by [**29, 1**]

$$\begin{aligned} & x_2^{u_3+u_2-u_1}(\frac{x_3}{x_2}q^{u_1-u_2-u_3};q)_{u_2+u_3-u_1} x_3^{u_3+u_1-u_2}(\frac{x_1}{x_3}q^{u_2-u_3-u_1};q)_{u_3+u_1-u_2} \\ & \quad \times x_1^{u_1+u_2-u_3}(\frac{x_2}{x_1}q^{1+u_3-u_1-u_2};q)_{u_1+u_2-u_3} \\ & = (-1)^{u_3+u_2-u_1} q^{\frac{1}{2}(u_1+u_2-u_3)+u_3(u_1+u_2)-\frac{1}{2}(u_1^2+u_2^2+u_3^2)} \times \\ & \quad \left[\frac{(-1)^{u_1+u_2-u_3}(q;q)_{u_1+u_2-u_3}(q^{-u_1-u_2-u_3-1};q)_{u_1+u_2-u_3}}{(q^{-2u_1};q)_{u_1+u_2-u_3}(q^{-2u_2};q)_{u_1+u_2-u_3}}\right]^{\frac{1}{2}} \times \\ & \sum_{n_1,n_2} x_3^{u_3-m} x_1^{u_1+n_1} x_2^{u_2+n_2} q^{\frac{1}{2}[m(u_2-u_1+u_3)+n_1(u_2-u_3+u_1)+n_2(u_3-u_1+u_2+2)]} \\ & \times (-1)^{u_3+m}\left[\frac{(q^{-2u_3};q)_{u_3+m}(q^{-2u_1};q)_{u_1+n_1}(q^{-2u_2};q)_{u_2+n_2}}{(q;q)_{u_3+m}(q;q)_{u_1+n_1}(q;q)_{u_2+n_2}(-1)^{u_1+u_2-u_3}}\right]^{\frac{1}{2}} \begin{bmatrix} u_1 & u_2 & u_3 \\ n_1 & n_2 & m \end{bmatrix}_q. \end{aligned}$$

It follows that

$$\begin{bmatrix} u_1 & u_2 & v \\ n_1 & n_2 & m \end{bmatrix}_q = \tag{4.5}$$

$$\left[\frac{(q^{-2u_1};q)_{u_1+u_2-v}(q^{-2u_2};q)_{u_1+u_2-v}(q^{-2u_1};q)_{u_1+n_1}(q^{-2u_2};q)_{u_2+n_2}}{(q;q)_{u_1+u_2-v}(q^{-u_1-u_2-v-1};q)_{u_1+u_2-v}(q^{-2v};q)_{v+m}(q;q)_{u_2+n_2}}\right]^{\frac{1}{2}}$$
$$\times\frac{[(q;q)_{v+m}(q;q)_{u_1+n_1}]^{\frac{1}{2}}(q^{n_1+v-u_2+1};q)_\infty}{(q;q)_\infty(q^{-2u_1};q)_{u_1+u_2-v}}$$
$$\times q^{\frac{1}{2}(u_1-u_2)(u_1+u_2+n_1-n_2)}q^{\frac{1}{2}[-u_1^2+u_2^2+2u_1u_2-2u_2v-n_1u_1+n_2u_1-n_1u_2-n_2u_2]}$$
$$\times {}_3\phi_2\left(\begin{matrix}q^{v-u_1-u_2}, & q^{u_1-u_2+v+1}, & q^{-u_2-n_2}\\ q^{v+n_1-u_2+1}, & q^{-2u_2} & \end{matrix};q;q^{-v+m}\right).$$

Note that the transformation [**1**]

$$x_j\to x_j^{-1},\quad u_j\to u_j,\quad q\to q^{-1},\quad j=1,2,3$$

followed by a multiplication by $x_1^{2u_1}x_2^{2u_2}x_3^{2u_3}$ maps the generating function to

$$(-1)^{u_1+u_2+u_3}q^{2(u_1^2+u_2^2+u_3^2)-\frac{1}{2}(u_1+u_2+u_3)^2+\frac{3}{2}u_3-\frac{1}{2}u_1-\frac{1}{2}u_2}$$

times itself. This leads to the transformation formula

$${}_3\phi_2\left(\begin{matrix}q^{-n}, & b, & c\\ d, & e & \end{matrix};q;\frac{deq^n}{bc}\right)=\frac{(e/c;q)_n}{(e;q)_n}{}_3\phi_2\left(\begin{matrix}q^{-n}, & c, & d/b\\ d, & cq^{1-n}/e & \end{matrix};q;q\right),$$

[**22**], so that the following alternate expression for the coefficients holds [**4, 29, 30**].

$$\begin{bmatrix}u_1 & u_2 & v\\ n_1 & n_2 & m\end{bmatrix}_q= \tag{4.6}$$

$$\left(\frac{[u_1-u_2+v]![v+m]![v-m]![u_1-n_1]![2v+1]}{[u_1+n_1]![u_1+u_2-v+1]![u_1+u_2-v]![v+u_2-u_1]![u_2+n_2]![u_2-n_2]!}\right)^{\frac{1}{2}}$$
$$\times q^{-\frac{1}{4}(v+u_1-u_2+1)(u_1+u_2-v)+\frac{1}{2}(v-u_2)n_2-\frac{1}{2}n_1u_2}\frac{(-1)^{v-u_1-u_2}[2u_2]!}{[v-u_2-n_1]!}$$
$$\times {}_3\phi_2\left(\begin{matrix}q^{v-u_1-u_2}, & q^{u_1-u_2+v+1}, & q^{-u_2+n_2}\\ q^{v-u_2-n_1+1}, & q^{-2u_2} & \end{matrix};q;q\right),$$

provided $v=u_1+u_2,u_1+u_2-1,\cdots,|u_1-u_2|$, $n_i=u_i,u_i-1,\cdots,-u_i$ and $m=n_1+n_2$. Otherwise, the coefficients vanish. Here,

$$[k]!=\frac{q^{-k(k-1)/4}}{(1-q)^k}(q;q)_k.$$

Defining normalized matrix elements for the representation $D(2u)$ by

$$\begin{aligned}S_{n_1n_2}^{(E+,e-)}(\alpha,\beta)&=<E_q(\beta E_+)e_q(\alpha E_-)e_{n_1},e_{n_2}>\\ &=\left[\frac{(-1)^{n-n'}(q;q)_{n'}(q^{-2u};q)_n}{q^{(3/2-u)(n-n')}(q;q)_n(q^{-2u};q)_{n'}}\right]T_{n'n}^{(E+,e-)}(\alpha,\beta),\end{aligned} \tag{4.7}$$

where $n_1 = -u + n$, $n_2 = -u + n'$, we note from (4.4), (4.6) and (4.7) that the following identity, relating the matrix elements of the operator $E_q(\beta F_+)e_q(\alpha F_-)$ in two distinct orthonormal bases, must hold:

$$S^{(E+,e-)}_{m_1m_2,n_1n_2}(\alpha,\beta) = \sum_v \begin{bmatrix} u_1 & u_2 & v \\ n_1 & n_2 & n_1+n_2 \end{bmatrix}_q S^{(E+,e-),v}_{m_1+m_2,n_1+n_2}(\alpha,\beta) \times \begin{bmatrix} u_1 & u_2 & v \\ m_1 & m_2 & m_1+m_2 \end{bmatrix}_q,$$

or,

$$\begin{aligned} &{}_2\phi_1\left(\begin{matrix} q^a, q^b \\ q^c \end{matrix}; q; x\right) {}_2\phi_1\left(\begin{matrix} q^\alpha, q^\beta \\ q^\gamma \end{matrix}; q; q^{a+b-c}x\right) = \\ &\sum_{r=0}^{\infty} q^{r(r+c-1)} \frac{(q^a;q)_r(q^b;q)_r(q^\gamma;q)_r x^r}{(q;q)_r(q^c;q)_r(q^{c+\gamma+r-1};q)_r} \times \\ &{}_3\phi_2\left(\begin{matrix} q^{-r}, q^\alpha, q^{1-r-c} \\ q^\gamma, q^{1-r-a} \end{matrix}; q; q\right) {}_3\phi_2\left(\begin{matrix} q^{-r}, q^\beta, q^{1-r-c} \\ q^\gamma, q^{1-r-b} \end{matrix}; q; q\right) \\ &\times {}_2\phi_1\left(\begin{matrix} q^{a+\alpha+r}, q^{b+\beta+r} \\ q^{c+\gamma+2r} \end{matrix}; q; x\right). \end{aligned} \tag{4.8}$$

The result (4.8) is established, intitially, only for negative integer values of the parameters, corresponding to the finite dimensional irreducible representations of $U_q(su_2)$. However, using the same arguments as in the interpretation of (2.12) we can establish it for all complex values of the parameters for which both sides of the equation are well defined. This result was first derived by Groza, Kachurik and Klimyk, [**4**] who used an argument in the spirit of quantum groups. The proof here is much simpler and the interpretation of the result is somewhat different.

As a final note we remark that Biedenharn and Tarlini, [**31**], have shown how to extend the notion of tensor operators for a Lie algebra to q-tensor operators for a quantum algebra in such a way that a generalized Wigner-Eckart theorem holds. This relates to our q-algebra models.

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