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Classification of superintegrable systems in three dimensions

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Abstract

A classical (or quantum) superintegrable system on an n-dimensional Riemannian manifold is an integrable Hamiltonian system with potential that admits $2n - 1$ functionally independent constants of the motion that are polynomial in the momenta, the maximum number possible. If these constants of the motion are all quadratic then the system is second order superintegrable, the most tractable case and the one we study here. Such systems have remarkable properties: multi-integrability and separability, a quadratic algebra of symmetries whose representation theory yields spectral information about the Schrödinger operator, and deep connections with expansion formulas relating classes of special functions. For $n = 2$ we have worked out the structure of these systems and classified all of the possible spaces and potentials. Here we discuss our recent and forthcoming work for the much more difficult case $n = 3$. We consider classical superintegrable systems with nondegenerate potentials in three dimensions and on a conformally flat real or complex space. We show that the quadratic algebra always closes at order 6. We describe the Stäckel transformation, an invertible conformal mapping between superintegrable structures on distinct spaces, and give evidence indicating that all our superintegrable systems are Stäckel transforms of systems on complex Euclidean space or the complex 3-sphere. Here, we announce the classification of all superintegrable systems that admit separation in generic coordinates. We find that there are 8 families of these systems.

1 Introduction

In this paper we report on recent and ongoing work to uncover the structure of second order superintegrable systems, both classical and quantum mechanical.

We concentrate on the basic ideas; the details of the proofs will be found elsewhere. The results on the quadratic algebra structure of 3D conformally flat systems with nondegenerate potential have appeared recently. The results on classification of generic superintegrable systems are announced here.

Superintegrable systems can lay claim to be the most symmetric solvable systems in mathematics. Here we consider only superintegrable systems on complex conformally flat spaces. It is easy to modify the results for real spaces. An n-dimensional complex Riemannian space is conformally flat if and only if it admits a set of local coordinates $x_1, \cdots, x_n$ such that the contravariant metric tensor takes the form $g^{ij} = \delta^{ij}/\lambda(\mathbf{x})$. Thus the metric is $ds^2 = \lambda(\mathbf{x})(\sum_{i=1}^n dx_i^2)$. A classical superintegrable system $\mathcal{H} = \sum_{ij} g^{ij} p_i p_j + V(\mathbf{x})$ on the phase space of this manifold is one that admits $2n-1$ functionally independent generalized symmetries (or constants of the motion) $\mathcal{S}_k, \quad k = 1, \cdots, 2n-1$ with $\mathcal{S}_1 = \mathcal{H}$ where the $\mathcal{S}_k$ are polynomials in the momenta p_j. That is, $\{\mathcal{H}, \mathcal{S}_k\} = 0$ where $\{f, g\} = \sum_{j=1}^n (\partial_{x_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{x_j} g)$ is the Poisson bracket for functions $f(\mathbf{x}, \mathbf{p}), g(\mathbf{x}, \mathbf{p})$ on phase space [1–6]. It is easy to see that $2n-1$ is the maximum possible number of functionally independent symmetries and, locally, such (in general nonpolynomial) symmetries always exist. The system is second order superintegrable if the $2n-1$ functionally independent symmetries can be chosen to be quadratic in the momenta. Usually a superintegrable system is also required to be integrable, i.e., it is assumed that n of the constants of the motion are in involution, though we do not make that assumption in this paper. Sophisticated tools such as R-matrix theory can be applied to the general study of superintegrable systems, e.g., [7–9]. However, the most detailed and complete results are known for second order superintegrable systems because separation of variables methods for the associated Hamilton-Jacobi equations can be applied. Standard orthogonal separation of variables techniques are associated with second-order symmetries, e.g., [10–15] and multiseparable Hamiltonian systems provide numerous examples of superintegrability. Thus here we concentrate on second-order superintegrable systems in which the symmetries take the form $\mathcal{S} = \sum a^{ij}(\mathbf{x}) p_i p_j + W(\mathbf{x})$, quadratic in the momenta.

There is an analogous definition for second-order quantum superintegrable systems with Schrödinger operator

$$H = \Delta + V(\mathbf{x}), \quad \Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} g^{ij}) \partial_{x_j},$$

the Laplace-Beltrami operator plus a potential function, [10]. Here there are $2n-1$ second-order symmetry operators

$$S_k = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} a^{ij}_{(k)}) \partial_{x_j} + W^{(k)}(\mathbf{x}), \quad k = 1, \cdots, 2n-1$$

with $S_1 = H$ and $[H, S_k] \equiv HS_k - S_k H = 0$. Again multiseparable systems yield many examples of superintegrability.

The basic motivation for studying superintegrable systems is that they can be solved explicitly and in multiple ways. It is the information gleaned from comparing the distinct solutions and expressing one solution set in terms of another that is a primary reason for their interest.

Two dimensional second order superintegrable systems have been completely classified recently [16–20]. Here we concentrate on three dimensional (3D) systems where new complications arise.

A typical structure for second order superintegrable systems is that of the quadratic algebra. Let $\{\mathcal{S}_j\}$ be a basis for the second order constants of the motion for the Hamiltonian $\mathcal{H}$. By the superintegrable assumption, the Poisson brackets $\{\mathcal{S}_i, \mathcal{S}_j\}$ must be functionally dependent on the basis symmetries $\mathcal{S}_k$, as are $\{\{\mathcal{S}_i, \mathcal{S}_j\}, \mathcal{S}_h\}$ and $\{\{\mathcal{S}_i, \mathcal{S}_j\}, \{\mathcal{S}_h, \mathcal{S}_s\}\}$. For the superintegrable systems with nondegenerate potentials that we study in this paper it is always true that the squares $\{\mathcal{S}_i, \mathcal{S}_j\}^2$ as well as $\{\{\mathcal{S}_i, \mathcal{S}_j\}, \mathcal{S}_h\}$ and $\{\{\mathcal{S}_i, \mathcal{S}_j\}, \{\mathcal{S}_h, \mathcal{S}_s\}\}$ are always uniquely expressible as *polynomials* in the $\{\mathcal{S}_k\}$. Similarly, each of these systems has a quantum extension with Poisson brackets replaced by commutators of symmetry operators that also has the quadratic algebra structure. This remarkable closure of the algebra generated by the second order symmetries leads to the very special properties enjoyed by the classical and quantum superintegrable systems.

Observed common features of these superintegrable systems are that they are usually multiseparable and that the eigenfunctions of one separable system can be expanded in terms of the eigenfunctions of another. This is the source of nontrivial special function expansion theorems in the quantum case [21]. The quantum symmetry operators are in formal self-adjoint form and suitable for spectral analysis. Also, the quadratic algebra identities allow us to relate eigenbases and eigenvalues of one symmetry operator to those of another. The representation theory of the abstract quadratic algebra can be used to derive spectral properties of the second order generators in a manner analogous to the use of Lie algebra representation theory to derive spectral properties of quantum systems that admit Lie symmetry algebras, [21–24].

The structure theory of classical superintegrable systems is simpler than for the quantum case, so we study it first. However, in a paper under preparation we shall show that each of the classical superintegrable systems with nondegenerate potential as studied here has a unique extension to a quantum superintegrable system.

For a classical 3D system on a conformally flat space we can always choose local coordinates x, y, z, not unique, such that the Hamiltonian takes the form $\mathcal{H} = p_1^2 + p_2^2 + p_3^2/\lambda(x,y,z) + V(x,y,z)$. This system is *second order superintegrable* with *nondegenerate* potential $V = V(x,y,z,\alpha,\beta,\gamma,\delta)$ if it admits 5 functionally independent quadratic constants of the motion (i.e., generalized symmetries) $\mathcal{S}_k = \sum_{ij} a^{ij}_{(k)} p_i p_j + W_{(k)}(x,y,\alpha,\beta,\gamma)$. As described in [18], the potential V is *nondegenerate* if it satisfies a system of coupled PDEs of the form

$$V_{22} = V_{11} + A^{22}(x,y,z)V_1 + B^{22}(x,y,z)V_2 + C^{22}(x,y,z)V_3, \qquad (1)$$

$$V_{33} = V_{11} + A^{33}(x,y,z)V_1 + B^{33}(x,y,z)V_2 + C^{33}(x,y,z)V_3,$$

$$V_{12} = A^{12}(x,y,z)V_1 + B^{12}(x,y,z)V_2 + C^{12}(x,y,z)V_3,$$
$$V_{13} = A^{13}(x,y,z)V_1 + B^{13}(x,y,z)V_2 + C^{13}(x,y,z)V_3,$$
$$V_{23} = A^{23}(x,y,z)V_1 + B^{23}(x,y,z)V_2 + C^{23}(x,y,z)V_3,$$

whose integrability conditions are satisfied identically. The analytic functions A^{ij}, B^{ij}, C^{ij} are determined uniquely from the Bertrand-Darboux equations for the 5 constants of the motion and are analytic except for a finite number of poles. At any regular point $\mathbf{x}_0 = (x_0, y_0, z_0)$, i.e., a point where the A^{ij}, B^{ij}, C^{ij} are defined and analytic and the constants of the motion are functionally independent, we can prescribe the values of $V(\mathbf{x}_0)$, $V_1(\mathbf{x}_0)$,$V_2(\mathbf{x}_0)$,$V_3(\mathbf{x}_0)$,$V_{11}(\mathbf{x}_0)$ arbitrarily and obtain a unique solution of (1). Here, $V_1 = \partial V/\partial x$, $V_2 = \partial V/\partial y$, etc. The significance of the 4 parameters for a nondegenerate potential (in addition to the usual additive constant) is that it is the maximum number of parameters that can appear in a superintegrable system. If the number of parameters is fewer than 4, we say that the superintegrable potential is *degenerate*.

We employ a theoretical method based on integrability conditions to derive structure common to all 3D superintegrable systems, with a view to complete classification. For 2D nondegenerate superintegrable systems we earlier showed that the $3 = 2(2) - 1$ functionally independent constants of the motion were (with one exception) also linearly independent, so at each regular point we could find a unique constant of the motion that matches a quadratic expression in the momenta at that point [16, 17]. However, for 3D systems we have only $5 = 2(3) - 1$ functionally independent constants of the motion and the quadratic forms span a 6 dimensional space. This is a major problem. However, for nondegenerate potentials we have proved the "$5 \Longrightarrow 6$ Theorem" to show that the space of second order constants of the motion is in fact 6 dimensional: there is a symmetry that is functionally dependent on the symmetries that arise from superintegrability, but linearly independent of them. With that result established, the treatment of the 3D case can proceed in analogy with the nondegenerate 2D case treated in [16]. Though the details are quite complicated, we can construct explicit bases for the 4th and 6th order constants in terms of products of the 2nd order constants. This means that there is a quadratic algebra structure [18].

The 3D Stäckel transform is a conformal transformation of a superintegrable system on one conformally flat 3D space to a superintegrable system on another such space. We discuss some of the properties of this transform for a classical system and then prove the fundamental result that every superintegrable system with nondegenerate potential is multiseparable. We give strong evidence that, as in the 2D case, all nondegenerate 3D superintegrable systems are Stäckel transforms of constant curvature systems, but we don't settle the issue. This suggests that to obtain all nondegenerate conformally flat superintegrable systems, it is sufficient to classify those in complex Euclidean space and on the complex 3-sphere. These statements rest on results obtained in [18].

Finally, we use the results of the first part of this paper and our explicit knowledge of all separable coordinate systems on 3D constant curvature spaces to make a major advance in the classification of all separable systems with nondegenerate potential on a conformally flat space. Among the separable systems for 3D complex Euclidean space there are 7 that are "generic". Essentially this means that the coordinates belong to a multiparameter family. The ultimate generic coordinates are the Jacobi elliptic coordinates from which all others can be obtained by limiting processes [25, 26]. We show that each of the generic separable systems uniquely determines a nondegenerate superintegrable system that contains it. We obtain a similar result for the 5 generic separable systems on the complex 3-sphere. However, 4 of these turn out to be Stäckel transforms of Euclidean generic systems. Thus we find 8 Stäckel inequivalent generic systems on constant curvature spaces and all generic systems on 3D conformally flat spaces must be Stäckel equivalent to one of these. (In addition there are 2 nondegenerate superintegrable systems in Euclidean space that are only weakly functionally independent and these give rise to similar systems on a variety of conformally flat spaces.) This doesn't solve the classification problem completely, but it is a major advance. Any remaining nondegenerate superintegrable systems must be multiseparable but separate *only* in degenerate separable coordinates. This remaining problem is still complicated, but much less so than the original problem.

1.1 Second order constants of the motion

Suppose $\mathcal{S} = \sum_{ij} a^{ij}_{(k)} p_i p_j + W$ is a constant of the motion for the conformally flat Hamiltonian system $\mathcal{H} = (p_1^2+p_2^2+p_3^2)/\lambda + V$. This means that $\{\mathcal{H}, \mathcal{S}\} = 0$. The conditions are thus

$$\begin{aligned} a^{ii}_i &= -G_1 a^{1i} - G_2 a^{2i} - G_3 a^{3i} \\ 2a^{ij}_i + a^{ii}_j &= -G_1 a^{1j} - G_2 a^{2j} - G_3 a^{3j}, \quad i \neq j \\ a^{ij}_k + a^{ki}_j + a^{jk}_i &= 0, \quad i,j,k \text{ distinct} \end{aligned} \tag{2}$$

and

$$W_k = \lambda \sum_{s=1}^{3} a^{sk} V_s, \quad k = 1,2,3, \tag{3}$$

where $G = \ln \lambda$. (Here a subscript j denotes differentiation with respect to x_j.) The requirement that $\partial_{x_\ell} W_j = \partial_{x_j} W_\ell$, $\ell \neq j$ leads from (3) to the second order Bertrand-Darboux partial differential equations for the potential.

$$\sum_{s=1}^{3} \left[V_{sj} \lambda a^{s\ell} - V_{s\ell} \lambda a^{sj} + V_s \left((\lambda a^{s\ell})_j - (\lambda a^{sj})_\ell \right) \right] = 0. \tag{4}$$

For second order superintegrabilty in 3D there must be five functionally independent constants of the motion (including the Hamiltonian itself). Thus the Hamilton-Jacobi equation admits four additional constants of the motion:

$$\mathcal{S}_h = \sum_{j,k=1}^{3} a^{jk}_{(h)} p_k p_j + W_{(h)} = \mathcal{L}_h + W_{(h)}, \qquad h = 1, \cdots, 4. \tag{5}$$

We assume that the four functions $\mathcal{S}_h$ together with $\mathcal{H}$ are functionally independent in the six-dimensional phase space, i.e., that the differentials $d\mathcal{S}_h,\ d\mathcal{H}$ are linearly independent.

Assume that we have a 3D superintegrable position with nondegenerate potential V, i.e., we can prescribe the values of V, V_x, V_y, V_z, V_xx arbitrarily at any regular point and this uniquely determines the potential. (Thus is the maximal possible number of parameters for a superintegrable potential.) Then the system admits 5 functionally independent second order symmetries S_k. From the Poisson bracket relations (2) the Bertrand-Darboux equations (4) and the nondegenerate potential conditions (1) we can obtain a system of equations for each of the partial derivatives a_k^{ij} that is in involution:

$$
\begin{aligned}
a_1^{11} &= -G_1a^{11} - G_2a^{12} - G_3a^{13} \qquad (6)\\
a_2^{22} &= -G_1a^{12} - G_2a^{22} - G_3a^{23},\\
a_3^{33} &= -G_1a^{13} - G_2a^{23} - G_3a^{33},\\
3a_1^{12} &= a^{12}A^{22} - (a^{22} - a^{11})A^{12} - a^{23}A^{13} + a^{13}A^{23}\\
&\quad +G_2a^{11} - 2G_1a^{12} - G_2a^{22} - G_3a^{23},\\
3a_2^{11} &= -2a^{12}A^{22} + 2(a^{22} - a^{11})A^{12} + 2a^{23}A^{13} - 2a^{13}A^{23}\\
&\quad -2G_2a^{11} + G_1a^{12} - G_2a^{22} - G_3a^{23},\\
3a_3^{13} &= -a^{12}C^{23} + (a^{33} - a^{11})C^{13} + a^{23}C^{12} - a^{13}C^{33}\\
&\quad -G_1a^{11} - G_2a^{12} - 2G_3a^{13} + G_1a^{33},\\
3a_1^{33} &= 2a^{12}C^{23} - 2(a^{33} - a^{11})C^{13} - 2a^{23}C^{12} + 2a^{13}C^{33}\\
&\quad -G_1a^{11} - G_2a^{12} + G_3a^{13} - 2G_1a^{33},\\
3a_2^{23} &= a^{23}(B^{33} - B^{22}) - (a^{33} - a^{22})B^{23} - a^{13}B^{12} + a^{12}B^{13}\\
&\quad -G_1a^{13} - 2G_2a^{23} - G_3a^{33} + G_3a^{22},\\
3a_3^{22} &= -2a^{23}(B^{33} - B^{22}) + 2(a^{33} - a^{22})B^{23} + 2a^{13}B^{12} - 2a^{12}B^{13}\\
&\quad -G_1a^{13} + G_2a^{23} - G_3a^{33} - 2G_3a^{22},\\
3a_1^{13} &= -a^{23}A^{12} + (a^{11} - a^{33})A^{13} + a^{13}A^{33} + a^{12}A^{23}\\
&\quad -2G_1a^{13} - G_2a^{23} - G_3a^{33} + G_3a^{11},\\
3a_3^{11} &= 2a^{23}A^{12} + 2(a^{33} - a^{11})A^{13} - 2a^{13}A^{33} - 2a^{12}A^{23}\\
&\quad +G_1a^{13} - G_2a^{23} - G_3a^{33} - 2G_3a^{11},\\
3a_2^{33} &= -2a^{13}C^{12} + 2(a^{22} - a^{33})C^{23} + 2a^{12}C^{13} - 2a^{23}(C^{22} - C^{33})\\
&\quad -G_1a^{12} - G_2a^{22} + G_3a^{23} - 2G_2a^{33},\\
3a_3^{23} &= a^{13}C^{12} - (a^{22} - a^{33})C^{23} - a^{12}C^{13} - a^{23}(C^{33} - C^{22})\\
&\quad -G_1a^{12} - G_2a^{22} - 2G_3a^{23} + G_2a^{33},\\
3a_2^{12} &= -a^{13}B^{23} + (a^{22} - a^{11})B^{12} - a^{12}B^{22} + a^{23}B^{13}\\
&\quad -G_1a^{11} - 2G_2a^{12} - G_3a^{13} + G_1a^{22},\\
3a_1^{22} &= 2a^{13}B^{23} - 2(a^{22} - a^{11})B^{12} + 2a^{12}B^{22} - 2a^{23}B^{13}\\
&\quad -G_1a^{11} + G_2a^{12} - G_3a^{13} - 2G_1a^{22},
\end{aligned}
$$

$$
\begin{aligned}
3a^{23}_1 &= a^{12}(B^{23}+C^{22})+a^{11}(B^{13}+C^{12})-a^{22}C^{12}-a^{33}B^{13}\\
&\quad +a^{13}(B^{33}+C^{23})-a^{23}(C^{13}+B^{12})-2G_1a^{23}+G_2a^{13}\\
&\quad +G_3a^{12}.\\
3a^{12}_3 &= a^{12}(-2B^{23}+C^{22})+a^{11}(C^{12}-2B^{13})-a^{22}C^{12}+2a^{33}B^{13}\\
&\quad +a^{13}(-2B^{33}+C^{23})+a^{23}(-C^{13}+2B^{12})-2G_3a^{12}+G_2a^{13}\\
&\quad +G_1a^{23}.\\
3a^{13}_2 &= a^{12}(B^{23}-2C^{22})+a^{11}(B^{13}-2C^{12})+2a^{22}C^{12}-a^{33}B^{13}\\
&\quad +a^{13}(B^{33}-2C^{23})+a^{23}(2C^{13}-B^{12})-2G_2a^{13}+G_1a^{23}\\
&\quad +G_3a^{12},
\end{aligned}
$$

plus the linear relations

$$
A^{23}=B^{13}=C^{12},\quad B^{23}-A^{31}-C^{22}=0, \tag{7}
$$

$$
B^{12}-A^{22}+A^{33}-C^{13}=0,\quad B^{33}+A^{12}-C^{23}=0.
$$

Using the linear relations we can express $C^{12}, C^{13}, C^{22}, C^{23}$ and B^{13} in terms of the remaining 10 functions.

If the integrabilty conditions for (6) were identically satisfied then at any regular point $\mathbf{x}_0$ (a point where the 10 functions $A^{12},\cdots,C^{33}$ have no singularities), we could prescribe the 6 values $a^{ij}(\mathbf{x}_0)$ arbitrarily and find a unique solution. However, the superintegrabiltiy conditions guarantee only a 5-parameter family of solutions, not 6. In [16] we found a way around this difficulty by showing that the superintegrability conditions plus the integrability conditions for the nondegeneracy conditions (1) implied that the integrability conditions for (6) were identically satisfied. This is the $5 \Longrightarrow 6$ Theorem. Thus the assumption of 5 functionally independent second order symmetries leads to the existence of 6 linearly independent symmetries. In [16] we also showed explicitly that polynomials in the basis of 6 second order constants of the motion spanned the space of all fourth and sixth order constants of the motion, so that the quadratic algebra always closed. Also the fundamental quadratic identities for the 10 independent functions:

$$
\begin{aligned}
a)\quad & -A^{23}B^{33}-A^{12}A^{23}+A^{13}B^{12}+B^{22}A^{23}+B^{23}A^{33}\\
&+\tfrac12A^{22}G_3-\tfrac12A^{33}G_3-\tfrac12B^{12}G_3-\tfrac12G_1G_3\\
&-\tfrac12A^{13}G_1+\tfrac32G_{13}-\tfrac12A^{23}G_2-A^{22}B^{23} &= 0,\\
b)\quad & \left(A^{33}\right)^2+B^{12}A^{33}-A^{33}A^{22}-B^{33}A^{12}-C^{33}A^{13}+B^{22}A^{12}\\
&-B^{12}A^{22}+A^{13}B^{23}-\left(A^{12}\right)^2+\\
&\tfrac32G_{22}-\tfrac12G_y^2-\tfrac32G_{33}+\tfrac12A^{13}G_3+\tfrac12B^{33}G_2+\\
&-\tfrac12A^{22}G_1+\tfrac12A^{33}G_1-\tfrac12B^{23}G_3-\tfrac12B^{22}G_2+\tfrac12C^{33}G_3+\tfrac12(G_3)^2 &= 0,\\
c)\quad & -\left(B^{33}\right)^2-B^{33}A^{12}+B^{33}B^{22}+B^{12}A^{33}+B^{23}C^{33}-\left(B^{23}\right)^2\\
&+\left(B^{12}\right)^2+\tfrac12(G_1)^2-\tfrac32G_{11}+\tfrac32G_{33}\\
&-\tfrac12B^{33}G_2-\tfrac12A^{33}G_1-\tfrac12(G_3)^2-\tfrac12C^{33}G_3 &= 0,\\
&-B^{12}A^{23}-A^{33}A^{23}+A^{13}B^{33}+A^{12}B^{23}\\
&+\tfrac32G_{23}-\tfrac12A^{23}G_1-\tfrac12A^{12}G_3\\
&-\tfrac12B^{23}G_2-\tfrac12G_2G_3-\tfrac12B^{33}G_3 &= 0,
\end{aligned}
$$

$$
\begin{array}{ll}
d)e) & A^{12}B^{12}+C^{33}A^{23}-A^{23}B^{23}+B^{33}A^{22}-B^{33}A^{33} \\
& \qquad +\frac{3}{2}G_{12}-\frac{1}{2}G_1G_2-\frac{1}{2}A^{12}G_1 \\
& \qquad\qquad -\frac{1}{2}B^{12}G_2-\frac{1}{2}A^{23}G_3 \quad =0
\end{array}
$$

followed as a consequence of the integrabilty conditions for (6) holding identically.

2 The Stäckel transform for 3D systems

The Stäckel transform [27] or coupling constant metamorphosis [28] plays a fundamental role in relating superintegrable systems on different manifolds. Suppose we have a superintegrable system

$$
\mathcal{H}=\frac{p_1^2+p_2^2+p_3^2}{\lambda(x,y,z)}+V(x,y,z) \tag{8}
$$

in local orthogonal coordinates, with nondegenerate potential $V(x,y,z)$, i.e., the general solution of equations (1) and suppose $U(x,y,z)$ is a particular solution of equations (1), nonzero in an open set. Then it is straightforward to show that the transformed system $\tilde{\mathcal{H}}=(p_1^2+p_2^2+p_3^2)/\tilde{\lambda}+\tilde{V}$ with nondegenerate potential $\tilde{V}(x,y,z)$:

$$
\begin{array}{lllll}
\tilde{V}_{33} & = & \tilde{V}_{11} & + & \tilde{A}^{33}\tilde{V}_1+\tilde{B}^{33}\tilde{V}_2+\tilde{C}^{33}\tilde{V}_3, \\
\tilde{V}_{22} & = & \tilde{V}_{11} & + & \tilde{A}^{22}\tilde{V}_1+\tilde{B}^{22}\tilde{V}_2+\tilde{C}^{22}\tilde{V}_3, \\
\tilde{V}_{23} & = & & & \tilde{A}^{23}\tilde{V}_1+\tilde{B}^{23}\tilde{V}_2+\tilde{C}^{23}\tilde{V}_3, \\
\tilde{V}_{13} & = & & & \tilde{A}^{13}\tilde{V}_1+\tilde{B}^{13}\tilde{V}_2+\tilde{C}^{13}\tilde{V}_3, \\
\tilde{V}_{12} & = & & & \tilde{A}^{12}\tilde{V}_1+\tilde{B}^{12}\tilde{V}_2+\tilde{C}^{12}\tilde{V}_3,
\end{array} \tag{9}
$$

is also superintegrable, where

$$
\tilde{\lambda}=\lambda U,\ \tilde{V}=\frac{V}{U},\ \tilde{A}^{33}=A^{33}+2\frac{U_1}{U},\ \tilde{B}^{33}=B^{33},\ \tilde{C}^{33}=C^{33}-2\frac{U_3}{U},
$$

$$
\tilde{A}^{22}=A^{22}+2\frac{U_1}{U},\ \tilde{B}^{22}=B^{22}-2\frac{U_2}{U},\ \tilde{C}^{22}=C^{22},\ \tilde{A}^{23}=A^{23},
$$

$$
\tilde{B}^{23}=B^{23}-\frac{U_3}{U},\ \tilde{C}^{23}=C^{23}-\frac{U_2}{U},\ \tilde{A}^{13}=A^{13}-\frac{U_3}{U},\ \tilde{B}^{13}=B^{13},
$$

$$
\tilde{C}^{13}=C^{13}-\frac{U_1}{U},\ \tilde{A}^{12}=A^{12}-\frac{U_2}{U},\ \tilde{B}^{12}=B^{12}-\frac{U_1}{U},\ \tilde{C}^{12}=C^{12}.
$$

Let $\mathcal{S}=\sum a^{ij}p_ip_j+W=\mathcal{S}_0+W$ be a second order symmetry of $\mathcal{H}$ and $\mathcal{S}_U=\sum a^{ij}p_ip_j+W_U=\mathcal{S}_0+W_U$ be the special case that is in involution with $(p_1^2+p_2^2+p_3^2)/\lambda+U$. Then $\tilde{\mathcal{S}}=\mathcal{S}_0-(W_U/U)\mathcal{H}+(1/U)\mathcal{H}$ is the corresponding symmetry of $\tilde{\mathcal{H}}$. Since one can always add a constant to a nondegenerate potential, it follows that $1/U$ defines an inverse Stäckel transform of $\tilde{\mathcal{H}}$ to $\mathcal{H}$. See [27] for many examples of this transform.

3 Multiseparability and Stäckel equivalence

From the general theory of variable separation for Hamilton-Jacobi equations, e.g., [14, 15] we know that second order symmetries $\mathcal{S}_1, \mathcal{S}_2$ define a separable system for the equation $\mathcal{H} = E$, where $\mathcal{H}$ is given by (8), if and only if 1) the symmetries $\mathcal{H}, \mathcal{S}_1, \mathcal{S}_2$ form a linearly independent set as quadratic forms, 2) $\{\mathcal{S}_1, \mathcal{S}_2\} = 0$, and 3) the three quadratic forms have a common eigenbasis of differential forms. This last requirement means that, expressed in coordinates x, y, z, at least one of the matrices $\mathcal{A}_{(j)}(\mathbf{x})$ (of the quadratic form associated with $\mathcal{S}_j$) can be diagonalized by conjugacy transforms in a neighborhood of a regular point and that $[\mathcal{A}_{(2)}(\mathbf{x}), \mathcal{A}_{(1)}(\mathbf{x})] = 0$. However, for nondegenerate superintegrable potentials in a conformally flat space the intrinsic conditions for the existence of a separable coordinate system can be greatly simplified to yield:

Theorem 1 *Let V be a superintegrable nondegenerate potential in a 3D conformally flat space. Then V defines a multiseparable system.*

The details of the proof can be found in [18]. In [26] the following result was obtained.

Theorem 2 *Let u_1, u_2, u_3 be an orthogonal separable coordinate system for a 3D conformally flat space with metric $d\tilde{s}^2$ Then there is a function f such that $fd\tilde{s}^2 = ds^2$ where ds^2 is a constant curvature space metric and ds^2 is orthogonally separable in exactly these same coordinates u_1, u_2, u_3. The function f is called a Stäckel multiplier with respect to this coordinate system.*

It follows that the possible separable coordinate systems for a conformally flat space are all obtained, via a Stäckel multiplier, from separable systems on 3D flat space or on the 3-sphere, [18]. This suggests that, just as in the 2D case, every nondegenerate conformally flat superintegrable system is Stäckel equivalent to a constant curvature superintegrable system, although though we have not yet been able to establish this. We have shown that the result is true for superintegrable systems permitting separation in a family of generic coordinates such that the constants of the motion characterizing the family spans a 5-dimensional subspace.

4 Classification of 3D conformally flat systems with nondegenerate potential

It is a difficult task to list all 3D conformally flat superintegrable systems with nondegenerate potential. However, we now have tools to simplify the problem. First, we have strong evidence for general systems (and a proof for generic systems) that every conformally flat system is Stäckel equivalent to a system on Euclidean space or the complex sphere. This suggests that we can restrict ourselves to those two spaces. Second, since every such system is multiseparable, we can bring to bear our knowledge of all orthogonal separable coordinates on these spaces. These results can be gleaned from the books [14, 25] and many papers of the authors, e.g., [26]. Thus in principle, we have enough information to accomplish our task, though the details are complicated.

We begin by summarizing the full list of orthogonal separable systems in complex Euclidean space and the associated symmetry operators. Here, a "natural" basis for first order symmetries is given by

$$p_1 \equiv p_x,\ p_2 \equiv p_y,\ p_3 \equiv p_z,\ J_1 = yp_z - zp_y,\ J_2 = zp_x - xp_z,\ J_3 = xp_y - yp_x$$

in the classical case and

$$p_1 = \partial_x,\ p_2 = \partial_y,\ p_3 = \partial_z,\ J_1 = y\partial_z - z\partial_y,\ J_2 = z\partial_x - x\partial_z,\ J_3 = x\partial_y - y\partial_x$$

in the quantum case. (In the operator characterizations for the quantum case, the classical product of two constants of the motion is replaced by the symmetrized product of the corresponding operator symmetries.) The Hamiltonian is $\mathcal{H} = p_1^2 + p_2^2 + p_3^2$. We use the bracket notation [2111] for the separable coordinates, a notation that goes back to Bôcher [25], and here denotes the fact that, to obtain this system, two of the 5 parameters e_j have been made to coincide. In the case [23] we list the coordinates followed by the constants of the motion that characterize them. To save space, in the other cases we list only the coordinates and refer to [19] for the details about the constants of the motion that characterize them.

$$[2111] \quad x^2 = c^2\frac{(u-e_1)(v-e_1)(w-e_1)}{(e_1-e_2)(e_1-e_3)},$$

$$y^2 = c^2\frac{(u-e_2)(v-e_2)(w-e_2)}{(e_2-e_1)(e_2-e_3)},\ z^2 = c^2\frac{(u-e_3)(v-e_3)(w-e_3)}{(e_3-e_1)(e_3-e_2)}$$

$$[221] \quad x^2+y^2 = -c^2\left[\frac{(u-e_1)(v-e_1)(w-e_1)}{(e_1-e_2)^2}\right]$$

$$-\frac{c^2}{e_1-e_2}\left[(u-e_1)(v-e_1)+(u-e_1)(w-e_1)+(v-e_1)(w-e_1)\right],$$

$$(x-iy)^2 = c^2\frac{(u-e_1)(v-e_1)(w-e_1)}{e_1-e_2}, \quad z^2 = c^2\frac{(u-e_2)(v-e_2)(w-e_2)}{(e_2-e_1)^2}.$$

$$[23] \quad x-iy = \frac{1}{2}c(\frac{u^2+v^2+w^2}{uvw} - \frac{1}{2}\frac{u^2v^2+u^2w^2+v^2w^2}{u^3v^3w^3}),$$

$$z = \frac{1}{2}c(\frac{uv}{w}+\frac{uw}{v}+\frac{vw}{u}), \quad x+iy = cuvw.$$

$$L_1 = J_1^2+J_2^2+J_3^2+2c^2(p_1+ip_2)p_3, \quad L_2 = -2J_3(J_1+iJ_2)+c^2(p_1+ip_2)^2.$$

$$[311] \quad x = \frac{c}{4}(u^2+v^2+w^2+\frac{1}{u^2}+\frac{1}{v^2}+\frac{1}{w^2})+\frac{3}{2}c,$$

$$y = -\frac{c}{4}\frac{(u^2-1)(v^2-1)(w^2-1)}{uvw}, \quad z = i\frac{c}{4}\frac{(u^2+1)(v^2+1)(w^2+1)}{uvw}.$$

$$[32] \quad x+iy = uvw, \quad x-iy = -(\frac{uv}{w}+\frac{uw}{v}+\frac{vw}{u}), \quad z = \frac{1}{2}(u^2+v^2+w^2).$$

$$[41] \quad x+iy = u^2v^2+u^2w^2+v^2w^2-\frac{1}{2}(u^4+v^4+w^4),\ x-iy = c^2(u^2+v^2+w^2),$$
$$z = 2icuvw.$$
$$[5] \quad x-iy = \frac{c}{4}(u-v-w)(u+v-w)(u+w-v),$$
$$x+iy = c(u+v+w), \quad z = -\frac{c}{4}(u^2+v^2+w^2-2(uv+uw+vw)).$$

We summarize the remaining degenerate orthogonal separable coordinates:

Euclidean coordinates. All of these have one symmetry $L_1 = p_3^2$. The 7 systems are, polar, Cartesian, light cone, elliptic, parabolic, hyperbolic and semihyperbolic.

Complex sphere coordinates. These all have one symmetry $L_1 = J_1^2 + J_2^2 + J_3^2$. The 5 systems are spherical, horospherical, elliptical, hyperbolic, and semi-circular parabolic.

Rotational types of coordinates. There are 3 of these systems, each of which is characterized by the fact that one defining symmetry is a perfect square.

The first 7 separable systems are "generic," i.e., they occur in one, two or three - parameter families, whereas the remaining systems are special limiting cases of the generic ones. Each of the 7 "generic" Euclidean separable systems depends on a scaling parameter c and up to three parameters e_1, e_2, e_3. For each such set of coordinates there is exactly one nondegenerate superintegrable system that admits separation in these coordinates *simultaneously for all values of the parameters* c, e_j. Consider the system [23], for example. If a nondegenerate superintegrable system separates in these coordinates for all values of the parameter c, then the space of second order symmetries must contain the 5 symmetries

$$\mathcal{H} = p_x^2 + p_y^2 + p_z^2 + V, \quad \mathcal{S}_1 = J_1^2 + J_2^2 + J_3^2 + f_1, \quad \mathcal{S}_2 = J_3(J_1 + iJ_2) + f_2,$$
$$\mathcal{S}_3 = (p_x + ip_y)^2 + f_3, \quad \mathcal{S}_4 = p_z(p_x + ip_y) + f_4.$$

It is straightforward to check that the 12×5 matrix of coefficients of the second derivative terms in the 12 Bertrand-Darboux equations associated with symmetries $\mathcal{S}_1, \cdots, \mathcal{S}_4$ has rank 5 in general. Thus, there is at most one nondegenerate superintegrable system admitting these symmetries. Solving the Bertrand-Darboux equations for the potential we find the unique solution

$$V(\mathbf{x}) := \alpha(x^2+y^2+z^2) + \frac{\beta}{(x+iy)^2} + \frac{\gamma z}{(x+iy)^3} + \frac{\delta(x^2+y^2-3z^2)}{(x+iy)^4}.$$

Finally, we can use the symmetry conditions for this potential to obtain the full 6-dimensional space of second order symmetries. This is superintegrable system [III] on the table to follow. The other six cases yield corresponding results.

Theorem 3 *Each of the* 7 *"generic" Euclidean separable systems determines a unique nondegenerate superintegrable system that permits separation simultaneously for all values of the scaling parameter c and any other defining parameters* e_j. *For each of these systems there is a basis of* 5 *(strongly) functionally independent and* 6 *linearly independent second order symmetries. The* *corresponding nondegenerate potentials and basis of symmetries are (the f_j are functions of x_1, x_2, x_3):*

$$\text{I [2111]} \qquad V = \frac{\alpha_1}{x^2} + \frac{\alpha_2}{y^2} + \frac{\alpha_3}{z^2} + \delta(x^2+y^2+z^2), \tag{10}$$

$$\mathcal{P}_i = \partial^2_{x_i} + \delta x_i^2 + \frac{\alpha_i}{x_i^2}, \qquad \mathcal{J}_{ij} = (x_i p_{x_j} - x_j p_{x_i})^2 + \alpha_i^2 \frac{x_j^2}{x_i^2} + \alpha_j^2 \frac{x_i^2}{x_j^2}, \quad i \geq j.$$

$$\text{II [221]} \qquad V = \alpha(x^2+y^2+z^2) + \beta\frac{x-iy}{(x+iy)^3} + \frac{\gamma}{(x+iy)^2} + \frac{\delta}{z^2}, \tag{11}$$

$$\mathcal{S}_1 = J\cdot J + f_1, \quad \mathcal{S}_2 = p_z^2 + f_2, \quad \mathcal{S}_3 = J_3^2 + f_3,$$
$$\mathcal{S}_4 = (p_x+ip_y)^2 + f_4, \quad L_5 = (J_2 - iJ_1)^2 + f_5.$$

$$\text{III [23]} \qquad V = \alpha(x^2+y^2+z^2) + \frac{\beta}{(x+iy)^2} + \frac{\gamma z}{(x+iy)^3} + \frac{\delta(x^2+y^2-3z^2)}{(x+iy)^4}, \tag{12}$$

$$\mathcal{S}_1 = J\cdot J + f_1, \quad \mathcal{S}_2 = (J_2 - iJ_1)^2 + f_2, \quad \mathcal{S}_3 = J_3(J_2 - iJ_1) + f_3,$$
$$\mathcal{S}_4 = (p_x+ip_y)^2 + f_4, \quad \mathcal{S}_5 = p_z(p_x+ip_y) + f_5.$$

$$\text{IV [311]} \qquad V = \alpha(4x^2+y^2+z^2) + \beta x + \frac{\gamma}{y^2} + \frac{\delta}{z^2}, \tag{13}$$

$$\mathcal{S}_1 = p_x^2 + f_1, \quad \mathcal{S}_2 = p_y^2 + f_2, \quad \mathcal{S}_3 = p_z J_2 + f_3,$$
$$\mathcal{S}_4 = p_y J_3 + f_4, \quad \mathcal{S}_5 = J_1^2 + f_5.$$

$$\text{V [32]} \qquad V = \alpha(4x^2+y^2+z^2) + \beta x + \frac{\gamma}{(y+iz)^2} + \frac{\delta(y-iz)}{(y+iz)^3}, \tag{14}$$

$$\mathcal{S}_1 = p_x^2 + f_1, \quad \mathcal{S}_2 = J_1^2 + f_2, \quad \mathcal{S}_3 = (p_z - ip_y)(J_2 + iJ_3) + f_3,$$
$$\mathcal{S}_4 = p_z J_2 - p_y J_3 + f_4, \quad \mathcal{S}_5 = (p_z - ip_y)^2 + f_5.$$

$$\text{VI [41]} \tag{15}$$

$$V = \alpha\left(z^2 - 2(x-iy)^3 + 4(x^2+y^2)\right) + \beta\left(2(x+iy) - 3(x-iy)^2\right)$$
$$+\gamma(x-iy) + \frac{\delta}{z^2},$$

$$\mathcal{S}_1 = (p_x - ip_y)^2 + f_1, \quad \mathcal{S}_2 = p_z^2 + f_2, \quad \mathcal{S}_3 = p_z(J_2 + iJ_1) + f_3,$$

$$\mathcal{S}_4 = J_3(p_x - ip_y) - \frac{i}{4}(p_x+ip_y)^2 + f_4, \quad \mathcal{S}_5 = (J_2 + iJ_1)^2 + 4ip_z J_1 + f_5.$$

$$\text{VII [5]} \qquad V = \alpha(x+iy) + \beta(\frac{3}{4}(x+iy)^2 + \frac{1}{4}z) + \gamma((x+iy)^3 + \frac{1}{16}(x-iy) \tag{16}$$
$$+\frac{3}{4}(x+iy)z) + \delta(\frac{5}{16}(x+iy)^4 + \frac{1}{16}(x^2+y^2+z^2) + \frac{3}{8}(x+iy)^2 z),$$

$$\mathcal{S}_1 = (J_1 + iJ_2)^2 + 2iJ_1(p_x+ip_y) - J_2(p_x+ip_y) + \frac{1}{4}(p_y^2 - p_z^2) - iJ_3 p_z + f_1,$$

$$\mathcal{S}_2 = J_2p_z - J_3p_y + i(J_3p_x - J_1p_z) - \frac{i}{2}p_yp_z + f_2, \quad \mathcal{S}_3 = (p_x + ip_y)^2 + f_4,$$

$$\mathcal{S}_4 = J_3p_z + iJ_1p_y + iJ_2p_x + 2J_1p_x + \frac{i}{4}p_z^2 + f_3, \quad \mathcal{S}_5 = p_z(p_x + ip_y) + f_5.$$

In [19] we used the detailed structure conditions (6) and (8) to prove:

Theorem 4 *A 3D Euclidean nondegenerate superintegrable system admits separation in a special case of the generic coordinates [2111], [221], [23], [311], [32], [41] or [5], respectively, if and only if it is equivalent via Euclidean transformation to system [I], [II], [III], [IV], [V], [VI] or [VII], respectively.*

Thus each of the Euclidean generic separable coordinate system determines one and only one superintegrable system associated with it. This does not settle the problem of classifying all 3D nondegenerate superintegrable systems in complex Euclidean space, for we have not excluded the possibility of such systems that separate only in degenerate separable coordinates. In fact we have already studied two such systems in [18]:

$$[O] \quad V(x,y,z) = \alpha x + \beta y + \gamma z + \delta(x^2 + y^2 + z^2).$$

$$[OO] \quad V(x,y,z) = \frac{\alpha}{2}(x^2 + y^2 + \frac{1}{4}z^2) + \beta x + \gamma y + \frac{\delta}{z^2}. \qquad (17)$$

An investigation of other possible Euclidean systems is in progress.

4.1 "Generic" superintegrable systems on the 3-sphere

An important task remaining is to classify the possible systems on the 3-sphere (particularly those 3-sphere systems not Stäckel equivalent to a flat space system). We proceed in analogy with the Euclidean case.

In reference [29] we have determined all orthogonal separable coordinate systems on the complex unit 3-sphere $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$. Of the 21 systems listed 5 are "generic", in the sense we used for Euclidean separable systems. However, 4 of these are Stäckel equivalent to generic systems on Euclidean space. (Here we take the Hamiltonian as $\mathcal{L}_0 = I_{12}^2 + I_{13}^2 + I_{14}^2 + I_{23}^2 + I_{24}^2 + I_{34}^2$, where $I_{jk} = -I_{kj} = s_jp_k - s_kp_j$ and we recall the identity $I_{23}I_{41} + I_{31}I_{42} + I_{12}I_{43} = 0$.) The only new generic system is
[1111]] (system (17) in [29])

$$s_1^2 = \frac{(x_1 - e_1)(x_2 - e_1)(x_3 - e_1)}{(e_1 - e_2)(e_1 - e_3)(e_1 - e_4)}, \quad s_2^2 = \frac{(x_1 - e_2)(x_2 - e_2)(x_3 - e_2)}{(e_2 - e_1)(e_2 - e_3)(e_2 - e_4)},$$

$$s_3^2 = \frac{(x_1 - e_3)(x_2 - e_3)(x_3 - e_3)}{(e_3 - e_1)(e_3 - e_2)(e_3 - e_4)}, \quad s_4^2 = \frac{(x_1 - e_4)(x_2 - e_4)(x_3 - e_4)}{(e_4 - e_2)(e_4 - e_3)(e_4 - e_1)}.$$

$$\mathcal{L}_1 = (e_1 + e_2)I_{12}^2 + (e_1 + e_3)I_{13}^2 + (e_1 + e_4)I_{14}^2 + (e_2 + e_3)I_{23}^2 + (e_2 + e_4)I_{24}^2 + (e_3 + e_4)I_{34}^2,$$

$$\mathcal{L}_2 = e_1e_2I_{12}^2 + e_1e_3I_{13}^2 + e_1e_4I_{14}^2 + e_2e_3I_{23}^2 + e_2e_4I_{24}^2 + e_3e_4I_{34}^2.$$

Again, each generic separable system on the 3-sphere uniquely determines a superintegrable system with nondegenerate potential. The proof is, in most part, analogous to that for the Euclidean case. Consider system [1111], for example. If we have a superintegrable system that admits the symmetries $\mathcal{L}_1, \mathcal{L}_2$ for all values of the parameters $e_1, \cdots, e_4$ then it must have the basis of symmetries

$$\text{VIII} \qquad \mathcal{S}_0 = I_{12}^2 + f_0, \quad \mathcal{S}_1 = I_{13}^2 + f_1, \quad \mathcal{S}_2 = I_{14}^2 + f_2, \quad \mathcal{S}_3 = I_{32}^2 + f_3,$$
$$\mathcal{S}_4 = I_{24}^2 + f_4, \quad \mathcal{S}_5 = I_{34}^2 + f_5.$$

The system of Bertrand-Darboux equations associated with these symmetries has rank 5 so the potential is uniquely determined. Solving the Bertrand-Darboux equations we obtain the nondegenerate potential on the 3-sphere

$$V(\mathbf{s}) = \frac{\alpha}{s_1^2} + \frac{\beta}{s_2^2} + \frac{\gamma}{s_3^2} + \frac{\delta}{s_4^2}. \tag{18}$$

Just as for the Euclidean case, the 3-sphere generic coordinates each uniquely determine a superintegrable system with nondegenerate potential to which it belongs.

Theorem 5 *A 3-sphere nondegenerate superintegrable system admits separation in a special case of the generic coordinates [1111] if and only if it is equivalent via a complex rotation to system [VIII].*

5 Discussion and conclusions

All classical superintegrable systems with nondegenerate potential on real or complex 3D conformally flat spaces admit 6 linearly independent second order constants of the motion (even though only 5 functionally independent second order constants are assumed) and the spaces of fourth order and sixth order symmetries are spanned by polynomials in the second order symmetries. This implies that a quadratic algebra structure always exists for such systems. Such systems are always multiseparable, more precisely they permit separation of variables in at least three orthogonal coordinate systems.

We studied the Stäckel transform, a conformal invertible mapping from a superintegrable system on one space to a system on another space. Using prior results from the theory of separation of variables on conformally flat spaces we have evidence, but no proof as yet, that, just as in the 2D case, every nondegenerate superintegrable system on such a space is Stäckel equivalent to a superintegrable system on complex Euclidean space or on the complex 3-sphere. Thus to classify all such superintegrable systems it appears that we can restrict attention to these two constant curvature spaces, and then obtain all other cases via Stäckel transforms. We are making considerable progress on the classification theory [19], though the problem is complicated. All of our 2D and 3D classical results can be extended to quantum systems and the Schrödinger equation and we are in the process of writing these up.

An interesting set of issues comes from the consideration of 3D superintegrable systems with degenerate, but multiparameter, potentials. In some cases such as the extended Kepler-Coulomb potential there is no quadratic algebra, whereas in other cases the quadratic algebra exists. Understanding the underlying structure of these systems is a major challenge. Finally there is the challenge of generalizing the 2D and 3D results to higher dimensions.

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