Here $K, K'$ are defined by

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2} \, d\theta, \quad K' = K(k'). \quad (C.3)$$

Special relations:

$$\begin{align*}
\text{sn}(-z) &= -\text{sn}(z), \quad \text{cn}(-z) = \text{cn}z, \quad \text{dn}(-z) = \text{dn}z, \\
\text{sn}^2 z + \text{cn}^2 z &= 1, \quad k'^2 \text{sn}^2 z + \text{dn}^2 z = 1.
\end{align*} \quad (C.4)$$

Special values:

$$\begin{align*}
\text{sn}0 &= 0, \quad \text{sn}K = 1, \quad \text{sn}(K + iK') = 1/k, \\
\text{cn}0 &= 1, \quad \text{cn}K = 0, \quad \text{cn}(K + iK') = -ik'/k, \\
\text{dn}0 &= 1, \quad \text{dn}K = k', \quad \text{dn}(K + iK') = 0. 
\end{align*} \quad (C.5)$$

The elliptic functions all have simple poles at $z = iK'$. As $z$ increases from 0 to $K$, $\text{sn}z$ increases from 0 to 1, $\text{cn}z$ decreases from 1 to 0, and $\text{dn}z$ decreases from 1 to $k'$. As $z$ varies from $K$ to $K + iK'$, $\text{sn}z$ increases from 1 to $k^{-1}$, $\text{cn}z$ is pure imaginary and varies from 0 to $-ik'/k$, and $\text{dn}z$ decreases from $k'$ to 0. As $z$ varies from $K + iK'$ to $iK'$, $\text{sn}z$ increases from $1/k$ to $+\infty$, $\text{cn}z$ is pure imaginary and varies from $-ik'/k$ to $-i\infty$, and $\text{dn}z$ is pure imaginary and varies from 0 to $-i\infty$.

Derivatives:

$$\begin{align*}
\frac{d}{dz} \text{sn}z &= \text{cn}z \, \text{dn}z, \quad \frac{d}{dz} \text{cn}z = -\text{sn}z \, \text{dn}z, \quad \frac{d}{dz} \text{dn}z = -k^2 \text{sn}z \, \text{cn}z. 
\end{align*} \quad (C.6)$$

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