SEPARABILITY OF WAVE EQUATIONS

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1. Introduction

The method of separation of variables has proved a useful tool with which to address various aspects of the physics of black holes. In this article we will review the progress made using this method in solving various systems of equations in black hole space-time backgrounds. The equations of important physical interest that we shall discuss consist primarily of the linear perturbation equations for various fields. In particular: Maxwell's equations, Dirac's equation and the Weyl neutrino equation and the gravitational perturbation equations. We will also discuss generalised Hertz potentials the Rarita-Schwinger equation and the properties of Teukolsky functions.

Separation of variables is a well known method of solution for problems in classical mechanics. Many of the fundamental (scalar) partial differential equations of mathematical physics in one unknown function can be reduced to ordinary differential equations using the method. The Hamilton-Jacobi, Helmholtz and Schrodinger equations are often amenable to such an approach. A comprehensive theory of separation of variables exists for these scalar wave equations a review of which is given by Miller (1988). The development of a corresponding theory for the non-scalar equations of mathematical physics has however proved more elusive. A theory of separation of variables has two principle aims. The first is to find separable solutions to a set of equations where such solutions exist. The second is to intrinsically characterise (i.e. in a frame independent way) the tetrad, coordinates and operators associated with the separation. A physical or geometrical interpretation of the separation constants, operators and frames may then be possible.

In the Kerr space-time the first separable solutions of significance were obtained by Carter (1968) for the Hamilton-Jacobi geodesic equations and Schrodinger's equation. The most significant results however were those of Teukolsky (1972) who obtained separable solutions for some components of the linearly perturbed electromagnetic and gravitational fields in the Kerr space-time. With such solutions available it became possible to consider electromagnetic and gravitational perturbations of the Kerr black hole. These ideas were extended to the Weyl neutrino field in the Kerr background independently by Teukolsky (1973) and Unruh (1973). A crucial development was the solution of the Dirac equation in the Kerr background via a separation of variables method. This was achieved by Chandrasekhar (1976) and later extended by Page (1976) to the Kerr-Newman background.

2. Black Hole Space-times

The Kerr-Newman solution is a Petrov type D electrovac solution. It appears to be the most general physically reasonable space-time for a stable isolated black hole. It is stationary, axisymmetric and asymptotically flat. (For a precise definition of these concepts see, for example, Heusler (1996)). The line element of the Kerr-Newman space-time in Boyer-Lindquist coordinates is

$$\begin{aligned} ds^2 &= \rho'\bar{\rho}'\left(\frac{1}{\Delta}dr^2 + d\theta^2\right) + (r^2+a^2)\sin^2\theta\, d\varphi^2 - dt^2 \\ &\quad + \frac{2Mr}{\rho'\bar{\rho}'}(a\sin^2\theta\, d\varphi - dt)^2. \end{aligned} \tag{1}$$

where

$$\rho' = r + ia\cos\theta \quad and \quad \Delta = r^2 - 2Mr + a^2 + q^2. \tag{2}$$

This corresponds to a charged rotating black hole having mass M, angular momentum $J = aM$ and charge q. When $q = 0$ or $a = 0$ we recover the metrics of the Kerr or Reissner-Nordström space-times respectively. When both are zero we recover the Schwarzschild metric. With rotation absent ($a = 0$) the space-time has spherical symmetry.

In this article we will use the Newman-Penrose and abstract index formalisms as found in Penrose and Rindler (1984). We also use the Kinnersley null tetrad, the corresponding directional derivatives of which are

$$\begin{aligned} l^a\nabla_a &= D = \frac{1}{\Delta}((r^2+a^2)\partial_t + \Delta\partial_r + a\partial_\varphi) \\ m^a\nabla_a &= \delta = \frac{1}{\rho'\sqrt{2}}(ia\sin\theta\partial_t + \partial_\theta + i\csc\theta\partial_\varphi) \\ n^a\nabla_a &= \Delta = \frac{1}{2\rho'\bar{\rho}'}((r^2+a^2)\partial_t - \Delta\partial_r + a\partial_\varphi). \end{aligned} \tag{3}$$

This is precisely the same tetrad as used by Chandrasekhar (1983). The Kinnersley tetrad is a canonical one, the vectors l^a and n^a being aligned with the principal null directions of the space-time. Note that we use the symbol Δ is used in two different roles. Which it represents should be clear from context.

In the Kinnersley tetrad the non-zero spin coefficients are

$$\begin{aligned} &\rho = -\frac{1}{\bar{\rho}'}, \quad \pi = \frac{ia\sin\theta}{\sqrt{2}\bar{\rho}'^2}, \quad \tau = -\frac{ia\sin\theta}{\sqrt{2}\rho'^2}, \quad \mu = -\frac{\Delta}{2\rho'^2\bar{\rho}'}, \\ &\gamma = \mu + \frac{r-M}{2\rho'\bar{\rho}'}, \quad \beta = \frac{\cot\theta}{2\sqrt{2}\bar{\rho}'}, \quad \alpha = \pi - \bar{\beta}. \end{aligned} \tag{4}$$

The only non-zero components of the Weyl, Ricci and Maxwell spinors are

$$\Psi_2 = -\frac{M}{\bar{\rho}'^3} + \frac{q^2}{\rho'\bar{\rho}'^3}, \quad \Phi_{11} = \frac{q^2}{2\rho'^2\bar{\rho}'^2}, \quad \phi_1 = \frac{q}{2\bar{\rho}'^2}. \tag{5}$$

When considering the linear perturbation equations for a given field in the Kerr-Newman background one can make use of the stationarity and the axisymmetry to decompose each component of a perturbation into a superposition of modes of the form

$$f(r,\theta)e^{i\omega t + im\varphi} \tag{6}$$

where ω is real and m is a half-integer or integer depending on whether the perturbed field or wave equation under consideration is fermionic or bosonic respectively. When acting on quantities of the form (6) the directional derivatives (3) can be succinctly written in terms of the following radial and angular differential operators

$$\begin{aligned} \mathcal{D}_n &= \partial_r + i\frac{K}{\Delta} + 2n\frac{r-M}{\Delta}, \\ \mathcal{D}_n^\dagger &= \partial_r - i\frac{K}{\Delta} + 2n\frac{r-M}{\Delta}, \\ \mathcal{L}_n &= \partial_\theta + Q + n\cot\theta, \\ \mathcal{L}_n^\dagger &= \partial_\theta - Q + n\cot\theta, \end{aligned} \tag{7}$$

where

$$K = \omega(r^2 + a^2) + am \quad and \quad Q = \omega a \sin\theta + m \csc\theta. \tag{8}$$

When rotation is not present the Kerr-Newman space-time degenerates to one having spherical symmetry. Group theory then guarantees that any linearised perturbation equations can have all the angular dependence separated out (see for instance Gel'fand *et al.* (1963)). Solution of a given perturbation problem in the Schwarzschild or Reissner-Nordström space-times will therefore amount to solving for the remaining radial dependence of the various quantities. This is by no means a trivial problem however. Solution of the combined electromagnetic and gravitational perturbation equations for the Reissner-Nordström space-time, for example, proceeds via an unexpected and not at all obvious decoupling of the radial equations (see Section 6).

The Kerr and Kerr-Newman space-times lack spherical symmetry. One cannot therefore expect, on the basis of an appeal to group theory, to effect a separation of variables for a given set of perturbation equations. Surprisingly most of the linearised perturbation equations in these two space-times, with one notable exception, continue to yield to separation of variables methods. Associated with each of the separable perturbation equations one typically finds a symmetry operator constructed from a Killing-Yano tensor.

A valence p Killing-Yano tensor is any totally anti-symmetric tensor $K_{a_1 a_2 \ldots a_p}$ which satisfies

$$\nabla_{(b} K_{a_1) a_2 \ldots a_p} = 0. \tag{9}$$

A valence p Killing spinor is a totally symmetric spinor $K_{A_1 A_2 \ldots A_p}$ satisfying

$$\nabla_{(BB'} K_{A_1 A_2 \ldots A_p)} = 0. \tag{10}$$

In the Kerr-Newman space-time there is only one valence two Killing-Yano tensor and one valence two Killing spinor. They are related by

$$K_{ab} = \frac{1}{2}(\epsilon_{A'B'} K_{AB} - \epsilon_{AB} \bar{K}_{A'B'}). \tag{11}$$

In addition the Killing spinor in the Kerr-Newman space-time also satisfies

$$\nabla_{BA'} K_A{}^B - \nabla_{AB'} \bar{K}_{A'}{}^{B'} = 0. \tag{12}$$

The vector

$$M_{AA'} = \frac{1}{3} \nabla_{BA'} K_A{}^B \tag{13}$$

is a Killing vector, that is, it satisifes

$$\nabla_{(b} M_{a)} = 0. \tag{14}$$

In the spin frame naturally associated with the Kinnersley tetrad the Killing spinor K_{AB} has components

$$K_{00} = 0, \quad K_{01} = -\bar{\rho}', \quad K_{11} = 0. \tag{15}$$

3. Maxwell's equations in the Kerr space-time

Maxwell's equations for a source free electromagnetic field are, in spinor form,

$$\nabla^A_{A'}\phi_{AB} = 0 \tag{16}$$

where ϕ_{AB} is a symmetric spinor. Expressed in the Newman-Penrose formalism Maxwell's equations read

$$\begin{aligned}
(D - 2\rho)\phi_1 &= (\bar{\delta} + \pi - 2\alpha)\phi_0 - \kappa\phi_2, \\
(\delta - 2\tau)\phi_1 &= (\Delta + \mu - 2\gamma)\phi_0 - \sigma\phi_2, \\
(\bar{\delta} + 2\pi)\phi_1 &= (D - \rho + 2\epsilon)\phi_2 + \lambda\phi_0, \\
(\Delta + 2\mu)\phi_1 &= (\delta - \tau + 2\beta)\phi_2 + \nu\phi_0.
\end{aligned} \tag{17}$$

If we write

$$\phi_k = (\rho'\sqrt{2})^{-k}\xi_k(r,\theta)e^{i\omega t + im\varphi} \tag{18}$$

then in the Kerr space-time and the Kinnersley tetrad Maxwell's equations become

$$\begin{aligned}
\left(\mathcal{D}_0 + \frac{1}{\bar{\rho}'}\right)\xi_1 &= \left(\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}'}\right)\xi_0, \\
\left(\mathcal{L}_0 + \frac{ia\sin\theta}{\bar{\rho}'}\right)\xi_1 &= \left(\mathcal{D}_0 - \frac{1}{\bar{\rho}'}\right)\xi_2, \\
\Delta\left(\mathcal{D}_0^\dagger + \frac{1}{\bar{\rho}'}\right)\xi_1 &= -\left(\mathcal{L}_1^\dagger - \frac{ia\sin\theta}{\bar{\rho}'}\right)\xi_2, \\
\left(\mathcal{L}_0^\dagger + \frac{ia\sin\theta}{\bar{\rho}'}\right)\xi_1 &= -\Delta\left(\mathcal{D}_1^\dagger - \frac{1}{\bar{\rho}'}\right)\xi_0.
\end{aligned} \tag{19}$$

It is reasonably straightforward to obtain decoupled second order equations for ξ_0 and ξ_2. These second order equations admit separable solutions of the form

$$\begin{aligned}
\xi_0 &= R_{+1}(r)S_{+1}(\theta), \\
\xi_2 &= R_{-1}(r)S_{-1}(\theta).
\end{aligned} \tag{20}$$

Here $R_{\pm1}(r)$ and $S_{\pm1}(\theta)$ are Teukolsky functions (see Section 7). Having obtained solutions for ϕ_0 and ϕ_2 one can then obtain (Chandrasekhar, 1983)

$$\phi_1 = \frac{1}{\sqrt{2}\mathcal{C}_1\bar{\rho}'}\left[\mathcal{D}_0\mathcal{L}_1 - \frac{1}{\bar{\rho}'}(\mathcal{L}_1 + ia\sin\theta\mathcal{D}_0)\right]R_{-1}(r)S_{+1}(\theta)e^{i\omega t + im\varphi} \tag{21}$$

where

$$\mathcal{C}_1 = \sqrt{\lambda^2 - 4a\omega(a\omega + m)}. \tag{22}$$

The separation constant λ and the Starobinsky constant $\mathcal{C}_1$ have been shown by Kalnins *et al.* (1989a) to be characterised by symmetry operators

$$\Lambda^C_{(A}\phi_{B)C} = \frac{1}{2}\lambda\phi_{AB} \quad \textit{and} \quad C^{AB}_{A'B'}\phi_{AB} = \mathcal{C}_1\bar{\phi}_{A'B'} \tag{23}$$

where

$$\Lambda^C_A = \left(K_A{}^{A'EE'}\nabla_{EE'} - M_A{}^{A'}\right)\left(K^C{}_{A'}{}^{DD'}\nabla_{DD'} + 2M^C{}_{A'}\right) \tag{24}$$

and

$$C^{AB}_{A'B'} = \left(K^A{}_{(A'|}{}^{EE'}\nabla_{EE'} + M^A{}_{(A'|}\right)\left(K^B{}_{|B')}{}^{DD'}\nabla_{DD'} + 2M^B{}_{|B')}\right). \tag{25}$$

As can be seen the Killing-Yano tensor plays a crucial role in the above separation of Maxwell's equations.

Kalnins *et al.* (1986) have shown that the vectors defined by the null tetrad are in fact eigenvectors of the Killing-Yano tensor K_{ab} considered as a matrix. It should also be mentioned at this point that Torres del Castillo (1989c) has observed the relation between Killing spinors and the separability of Maxwell's equations and also extended the separability of Maxwell's equations to type D space-time backgrounds (Torres del Castillo, 1988a). Maxwell's equations can also be solved for the Robertson-Walker type space-times (Kalnins and Miller, 1991). Decoupled equations for Debye potentials of electromagnetic fields in space-times with local rotational symmetry have been obtained by Dhurandhar *et al.* (1980). The most general linear second order symmetry operators of Maxwell's equations in curved space-time have been found by Kalnins *et al.* (1992). Implicitly included in this study are the operators characterising λ and $\mathcal{C}_1$.

4. Dirac's equation, the Weyl Neutrino equation and the Rarita-Schwinger equation

Dirac's equation for a massive charged spin half particle in a curved space-time background with a background electromagnetic field has, in spinor notation, the form

$$\begin{aligned}
(\nabla^{B'}_B - iq_e A^{B'}_B)\chi_{B'} &= -\frac{im_e}{\sqrt{2}}\phi_B, \\
(\nabla^B_{B'} - iq_e A^B_{B'})\phi_B &= \frac{im_e}{\sqrt{2}}\chi_{B'}.
\end{aligned} \tag{26}$$

where $A_{BB'}$ is the vector potential of the background electromagnetic field. Setting $m_e = 0$, $q_e = 0$ and $\chi_{A'} = \bar{\phi}_{A'}$ we recover two copies of the Weyl neutrino equation. In the Newman-Penrose formalism Dirac's equations have the form

$$\begin{aligned}
(\bar{\delta} - iq_e A_{10'} + \pi - \alpha)\phi_0 - (D - iq_e A_{00'} + \epsilon - \rho)\phi_1 &= \frac{im_e}{\sqrt{2}}\chi_{0'} \\
(\Delta - iq_e A_{11'} + \mu - \gamma)\phi_0 - (\delta - iq_e A_{01'} + \beta - \tau)\phi_1 &= \frac{im_e}{\sqrt{2}}\chi_{1'} \\
(D - iq_e A_{00'} + \bar{\epsilon} - \bar{\rho})\chi_{1'} - (\delta - iq_e A_{01'} + \bar{\pi} - \bar{\alpha})\chi_{0'} &= \frac{im_e}{\sqrt{2}}\phi_0 \qquad (27) \\
(\bar{\delta} - iq_e A_{10'} + \bar{\beta} - \bar{\tau})\chi_{1'} - (\Delta - iq_e A_{11'} + \bar{\mu} - \bar{\gamma})\chi_{0'} &= \frac{im_e}{\sqrt{2}}\phi_1
\end{aligned}$$

In contrast to the procedure carried out for Maxwell's equations where decoupling occured first followed by separation, the solution of Dirac's equation proceeds by postulating a separable ansatz for the solutions first and then subsequently decoupling the resulting equations for the functions of one variable. In the Kerr-Newman space-time the equations above have the solutions

$$\begin{aligned}
\phi_0 &= R_{+\frac{1}{2}} S_{+\frac{1}{2}} e^{i\omega t + im\varphi}, \qquad \phi_1 = -\frac{1}{\bar{\rho}'} R_{-\frac{1}{2}} S_{-\frac{1}{2}} e^{i\omega t + im\varphi}, \\
\chi_{0'} &= R_{+\frac{1}{2}} S_{-\frac{1}{2}} e^{i\omega t + im\varphi}, \qquad \chi_{1'} = \frac{1}{\rho'} R_{-\frac{1}{2}} S_{+\frac{1}{2}} e^{i\omega t + im\varphi}. \qquad (28)
\end{aligned}$$

where the functions $R_{\pm\frac{1}{2}}$ and $S_{\pm\frac{1}{2}}$ satisfy the equations

$$\begin{aligned}
\left(\mathcal{D}_0 - \frac{iqq_e r}{\sqrt{2}\Delta}\right) R_{-\frac{1}{2}} &= (\lambda + im_e r) R_{+\frac{1}{2}} \\
\left(\Delta \mathcal{D}^\dagger_{\frac{1}{2}} + \frac{iqq_e r}{\sqrt{2}}\right) R_{+\frac{1}{2}} &= (\lambda - im_e r) R_{-\frac{1}{2}} \\
\mathcal{L}^\dagger_{\frac{1}{2}} S_{-\frac{1}{2}} &= (\lambda + am_e \cos\theta) S_{+\frac{1}{2}} \qquad (29) \\
\mathcal{L}_{\frac{1}{2}} S_{+\frac{1}{2}} &= -(\lambda - am_e \cos\theta) S_{-\frac{1}{2}}
\end{aligned}$$

These equations are readily decoupled to give four second order equations for $R_{\pm\frac{1}{2}}$ and $S_{\pm\frac{1}{2}}$. The functions $R_{\pm\frac{1}{2}}$ and $S_{\pm\frac{1}{2}}$ are generalisations of Teukolsky functions. The separation parameter λ appearing above has been characterised by Carter and McLenaghan (1979) in terms of a non-trivial symmetry operator constructed from the Killing-Yano tensor $K_{AA'BB'}$. Their result expressed in spinor form is

$$(K^{BB'}{}_A{}^{A'} \nabla_{BB'} + M_A{}^{A'})\chi_{A'} = -\frac{\lambda}{\sqrt{2}}\phi_A,$$

$$(K^{BB'A}{}_{A'}\nabla_{BB'} - M^{A}{}_{A'})\phi_A = -\frac{\lambda}{\sqrt{2}}\chi_{A'}. \tag{30}$$

Physically this can be interpreted as a generalised angular momentum operator. As with Maxwell's equations the Killing-Yano tensor plays a central role in the separation of the Dirac equation. The most general first order formally self-adjoint operator that commutes with the Dirac operator has been found by McLenaghan and Spindel (1979). The general second order symmetry operator of the Weyl neutrino equation on curved space-time has been found by McLenaghan and Walker (1996).

Separation and solution of the Dirac equation in a class of Petrov type D space-times has been achieved by Güven (1977) and in a wide class of quite general space-times by Debever *et al.* (1984). Kamran and McLenaghan (1984a) have shown that the Weyl neutrino equation is separable in the class of solutions which are Petrov type D electrovac with non-singular aligned Maxwell field satisfying the generalised Goldberg-Sachs theorem and that the Dirac equations is separable in a sub-class of these solutions. They have also characterised the separable solution of the Dirac equation in an Einstein-Maxwell space-time (Kamran and McLenaghan, 1984b). Separation of the Dirac equation in space-times with local rotational symmetry has been studied by Iyer and Vishveshwara (1985), (1987) and Iyer and Kamran (1991).

The manner of separation that occurs with the Dirac equation in the Kerr space-time can be described using the idea of factorisable systems as formulated by Shapovalov and Ekle (1974a), (1974b) and Miller (1988). The notion of factorisable systems is not however enough for a comprehensive treatment of solutions of the Dirac equation. Fels and Kamran (1990) exhibit a space-time in which characterisation of the separation constants as the eigenfunctions of first order symmetry operators is not possible. Rather a complete characterisation requires the use of one second order symmetry operator. For a general discussion of mechanisms of separation for the Dirac equation in curved space-times see Rudiger (1984).

Explicit solutions of the Dirac equation have been studied extensively in Minkowski space-time. For a description of this work see Bagrov and Gitman (1990) and Shishkin and Villalba (1989). An analysis of the properties of the separable solutions appearing for the Dirac equation can be found in Cosgrove *et al.* (1983) and Kalnins and Miller (1992).

The separation and solution of the equations for various scalars associated with a Rarita-Schwinger field in a Petrov type D space-time was first achieved by Güven (1980). In proving the no-hair conjecture for the uncharged black holes of supergravity Güven showed that all solutions of the Rarita-Schwinger equation in the Kerr space-time that cannot be transformed away via a supersymmetry transformation are expressible in terms of the Teukolsky functions for $s = \pm\frac{3}{2}$. Kamran (1985) later extended the separability of the Rarita-Schwinger equation to all type D vacuum space-times (see also Torres del Castillo (1989a)). The Rarita-Schwinger equation can be written in spinor form as

$$\nabla^{A}_{A'}\theta_{ABB'} = 0 \tag{31}$$

where $\theta_{ABB'} = \theta_{(AB)B'}$. A related system of coupled equations can be constructed from the Rarita-Schwinger equation. Forming a new field ϕ_{ABC} from the Rarita-Schwinger field by

$$\phi_{ABC} = \nabla_{(AA'}\theta_{BC)}{}^{A'}. \tag{32}$$

it is not difficult to show that in a vacuum ϕ_{ABC} must satisfy

$$\nabla^{A}_{A'}\phi_{ABC} = \Psi_{BCKL}\theta^{KL}{}_{A'}. \tag{33}$$

Taken together the last two equations admit solutions for ϕ_{000} of the form

$$\phi_{000} = R_{-\frac{3}{2}}(r)S_{-\frac{3}{2}}(\theta)e^{i\omega t + im\varphi} \tag{34}$$

where $R_{-\frac{3}{2}}$ and $S_{-\frac{3}{2}}$ are Teukolsky functions.

5. Gravitational perturbations of the Kerr space-time

Teukolsky (1972) found separable solutions for two of the components of the perturbed Weyl spinor in the Kerr space-time. Given a perturbation of the Kerr space-time the following relations hold true to first order in the perturbation. ¿From the Bianchi identities and the Newman-Penrose equations we have

$$\begin{aligned}(\bar{\delta} - 4\alpha + \pi)\hat{\Psi}_0 - (D - 4\bar{\rho})\hat{\Psi}_1 &= 3\hat{\kappa}\Psi_2 \\ (\Delta - 4\gamma + \mu)\hat{\Psi}_0 - (\delta - 4\tau - 2\beta)\hat{\Psi}_1 &= 3\hat{\sigma}\Psi_2 \\ (D - \rho - \bar{\rho})\hat{\sigma} - (\delta - \tau + \bar{\pi} - \bar{\alpha} - 3\beta)\hat{\kappa} &= \hat{\Psi}_0\end{aligned} \tag{35}$$

In these equations $\hat{\Psi}_0$, $\hat{\Psi}_1$, $\hat{\sigma}$ and $\hat{\kappa}$ are first order perturbations to the components of the Weyl spinor and the spin coefficients respectively. Making use of the identities that result from the commutation relations of the directional derivatives one can show that $\hat{\Psi}_0$ satisfies the decoupled equation

$$\begin{aligned}&[(D - 2\rho - \bar{\rho})(\Delta - 4\gamma + \mu) - (\delta - 2\tau + \bar{\pi} - \bar{\alpha} - 3\beta)(\bar{\delta} - 4\alpha + \pi)]\hat{\Psi}_0 \\ &\quad = \Psi_2\hat{\Psi}_0\end{aligned} \tag{36}$$

writing $\hat{\Psi}_0 = \xi_0(r,\theta)e^{i\omega t + im\varphi}$ this equation reduces to

$$(\Delta \mathcal{D}_1 \mathcal{D}_2^\dagger + \mathcal{L}_{-1}^\dagger \mathcal{L}_2 - 6i\omega\rho')\xi_0 \quad = \quad 0 \tag{37}$$

which has solution $\xi_0 = R_{+2}(r)S_{+2}(\theta)$. By considering the analogous set of equations for $\hat{\Psi}_4$ one obtains the following separable solution

$$\hat{\Psi}_4 = \frac{1}{4\bar{\rho}'^4} R_{-2}(r) S_{-2}(\theta) e^{i\omega t + im\varphi}. \tag{38}$$

The functions $R_{\pm 2}(r)$ and $S_{\pm 2}(\theta)$ are Teukolsky functions. The remaining components of the perturbed Weyl spinor are most easily found using the method of generalised Hertz potentials. The question as to the operator which characterises the separation constant and the interpretation of that operator remains unclear (see Kalnins *et al.* (1996)).

The use of generalised Hertz potentials to produce solutions to various equations in algebraically special space-times is due to Cohen and Kegeles (1975) and Kegeles and Cohen (1979). For an elegant explanation of the reasons behind the efficacy of this technique see Wald (1978). A generalised Hertz potential is a totally symmetric spinor $\bar{P}^{A_1' A_2' \ldots A_{2s}'}$ satisfying

$$\begin{aligned} \nabla^{B(A_1'} \left[\nabla_{BB'} \bar{P}^{A_2' \ldots A_{2s}')B'} - G_B{}^{A_2' \ldots A_{2s}')} \right] & \\ - (2s-1)(s-1)\bar{\Psi}_{B'C'}{}^{(A_1' A_2'} \bar{P}^{A_3' \ldots A_{2s}')B'C'} &= 0 \end{aligned} \tag{39}$$

where $G_B{}^{A_2' \ldots A_{2s}'}$ is an arbitrary gauge field. One can then construct a spin s field from the potential and gauge field as follows

$$\begin{aligned} \phi_{A_1 A_2 \ldots A_{2s}} \quad = \quad & \nabla_{(A_1 A_1'} \nabla_{A_2 A_2'} \ldots \nabla_{A_{2s-1} A_{2s-1}'} \\ & \left[\nabla_{A_{2s}) A_{2s}'} \bar{P}^{A_1' A_2' \ldots A_{2s}'} - G_{A_{2s})}{}^{A_1' A_2' \ldots A_{2s-1}'} \right] \end{aligned} \tag{40}$$

In a flat space-time the resulting field $\phi_{A_1 \ldots A_{2s}}$ will satisfy the massless spin s field equation

$$\nabla_{B'}^B \phi_{BA_2 \ldots A_{2s}} = 0. \tag{41}$$

When the space-time is curved the field $\phi_{A_1 \ldots A_{2s}}$ satisfies the massless spin s field equations for $s = \frac{1}{2}$ and $s = 1$ only. For $s = 2$ the method of generalised Hertz potentials can be used to construct a solution for the linear gravitational perturbations of the Kerr space-time.

Solving the equation for the generalised Hertz potential via separation of variables requires one make the right choice of gauge field. For the case of Maxwell's equations we have

$$\phi_{AB} = \nabla_{(AA'} \left[\nabla_{B)B'} \bar{P}^{A'B'} - G_{B)}^{A'} \right]. \tag{42}$$

The new field satisfies Maxwell's equations (16). In the Kinnersley tetrad and with the specific choice of gauge field $G_C{}^{B'} = -2\bar{U}_{CC'}\bar{P}^{C'B'}$ where $U_{CC'}$ has components

$$U_{00'} = \bar{\rho}, \quad U_{01'} = -\bar{\pi}, \quad U_{10'} = \bar{\tau}, \quad and \quad U_{11'} = -\bar{\mu}, \tag{43}$$

the Hertz potentials have solutions of the form

$$\bar{P}^{0'0'} = R_{-1}S_{+1}e^{i\omega t+im\varphi}, \quad \bar{P}^{0'1'} = 0, \quad \bar{P}^{1'1'} = 0 \tag{44}$$

or

$$\bar{P}^{1'1'} = \bar{\rho}'^2 R_{+1}S_{-1}e^{i\omega t+im\varphi}, \quad \bar{P}^{0'1'} = 0, \quad \bar{P}^{0'0'} = 0 \tag{45}$$

Both these forms of the Hertz potential lead to the same Maxwell spinors (by virtue of the Teukolsky-Starobinsky identities amongst the Teukolsky functions).

The interesting feature of the Hertz potential equation is that for the choice of gauge function

$$G_{BA'_2\ldots A'_{2s}} = -2sU_{BB'}\bar{P}^{B'}{}_{A'_2\ldots A'_{2s}} \tag{46}$$

solutions exist whose only non-zero components are of the form

$$\bar{P}^{0'\ldots0'} = R_{-s}S_{+s}e^{i\omega t+im\varphi} \quad or \quad \bar{P}^{1'\ldots1'} = \rho'^{2s}R_{+s}S_{-s}e^{i\omega t+im\varphi} \tag{47}$$

where $R_{\pm s}$ and $S_{\pm s}$ are the spin s Teukolsky functions.

These potential functions for $s = 2$ have been used by Kegeles and Cohen (1979) to obtain components of the perturbed metric tensor corresponding to gravitational perturbations of a vacuum space-time. For $s = \frac{3}{2}$ they have been used by Torres del Castillo (1989b) in the context of Rarita-Schwinger fields. For gravitational perturbations the perturbed Einstein vacuum field equations $\delta R_{ab} = 0$ take the form

$$\nabla_k\nabla^k h_{ab} - 2\nabla_k\nabla_{(a}h_{b)}{}^k + \nabla_a\nabla_b h_k{}^k = 0 \tag{48}$$

where h_{ab} is the perturbation to the metric tensor. This can be thought of also as a spin 2 wave equation. In terms of the Hertz potential and gauge spinor h_{ab} is given by

$$\begin{aligned} h_{ABA'B'} &= \left[\nabla_{(AK'}\nabla_{B)L'}\bar{P}^{K'L'}{}_{A'B'} - \nabla_{(AK'}G^{K'}{}_{B)A'B'}\right] \\ &\quad + \left[\nabla_{K(A'}\nabla_{LB')}P^{KL}{}_{AB} - \nabla_{K(A'}G^{K}{}_{B')AB}\right]. \end{aligned} \tag{49}$$

It should of course be understood that all that has been done for potentials with primed indices can also be repeated for potentials with unprimed indices. Hence the appearance of $P^{KL}{}_{AB}$ in (49) presents no problem.

The first order perturbation of the Weyl spinor is given by

$$\hat{\Psi}_{ABCD} = \nabla_{(AK'}\nabla_{BL'}h_{CD)}{}^{K'L'}. \tag{50}$$

Making one or other of the choices (47) results in explicit expressions for the perturbed metric in either the ingoing or outgoing radiation gauges ($h_{ab}l^b = 0$ or $h_{ab}n^b = 0$). An alternative method of solution of the gravitational perturbation problem has been given by Chandrasekhar (1983). In his solution the expressions for the perturbed components of the metric are more complex.

6. Combined electromagnetic and gravitational perturbations of charged black holes

The Kerr-Newman space-time has a background electromagnetic field. A first order pertubation in this field gives rise to first order perturbations in the energy-momentum tensor resulting in first order gravitational perturbations. Consequently in the Kerr-Newman and Reissner-Nordström space-time linear electromagnetic or gravitational perturbations cannot be considered separately. This makes the solution of the corresponding perturbation equations more difficult. In fact the method of separation of variables appears to fail in the Kerr-Newman case. In what follows we will distinguish the value of a quantity in the perturbed space-time from its value in the background space-time by placing a circumflex above it.

Central to the solution of the Kerr-Newman perturbation problem are the following equations. From the Newman-Penrose equations we have, to first order in the perturbation,

$$(\Delta + \mu + \bar{\mu} + 3\gamma - \bar{\gamma})\hat{\lambda} - (\bar{\delta} + \pi + 3\alpha + \beta - \tau)\hat{\nu} = -\hat{\Psi}_4. \tag{51}$$

¿From the Bianchi identities we have, to first order

$$\begin{aligned}
&(\bar{\delta} + 2\alpha + 4\pi)\hat{\Psi}_3 - (D + 4\epsilon - \rho)\hat{\Psi}_4 + (\bar{\delta} + 2\alpha - 2\bar{\tau})\hat{\Phi}_{21} \\
&\qquad = \hat{\lambda}(3\Psi_2 + 2\Phi_{11}), && (52)\\
&(\Delta + 2\gamma + 4\mu)\hat{\Psi}_3 - (\delta + 4\beta - \tau)\Psi_4 - (\Delta + 2\gamma + 2\bar{\mu})\hat{\Phi}_{21} \\
&\qquad = \hat{\nu}(3\Psi_2 - 2\Phi_{11}). && (53)
\end{aligned}$$

The fourth perturbation equation arises from considering what is essentially one of the Ricci identities, namely

$$\begin{aligned}
[\hat{\bar{\delta}} + 2\hat{\pi}, \hat{\Delta} + 2\hat{\mu}]\hat{\phi}_1 \;=\; &(\hat{\bar{\tau}} - \hat{\bar{\beta}} - \hat{\alpha})\hat{\Delta}\hat{\phi}_1 + (\hat{\bar{\mu}} - \hat{\bar{\gamma}} + \hat{\gamma})\hat{\bar{\delta}}\hat{\phi}_1 \\
&- \hat{\nu}\hat{D}\hat{\phi}_1 + \hat{\lambda}\hat{\delta}\hat{\phi}_1 - 2\hat{\phi}_1\hat{\Delta}\hat{\pi} + 2\hat{\phi}_1\hat{\bar{\delta}}\hat{\mu} \qquad (54)
\end{aligned}$$

A similar set of four equations exist between the quantities $\hat{\Psi}_0,\ \hat{\Psi}_1,\ \hat{\Phi}_{01},\ \hat{\kappa}$ and $\hat{\sigma}$. Using Maxwell's equations to substitute for the derivatives of $\hat{\phi}_1$ in equation (54), applying two of the Newman-Penrose equations, then swapping the order of differentiation in each of the two second order differential operators occuring on the left hand side of the resulting equation, one can obtain, to first order

$$\begin{aligned}(\delta+3\beta-\bar{\alpha}+\bar{\pi})[(\bar{\delta}+2\alpha+3\pi)\hat{\phi}_2-2\hat{\lambda}\phi_1] & \\ -(D+3\epsilon+\bar{\epsilon}-\bar{\rho})[(\Delta+2\gamma+3\mu)\hat{\phi}_2-2\hat{\nu}\phi_1] &= 2(2\hat{\Psi}_3\phi_1-3\Psi_2\hat{\phi}_2) \quad (55)\end{aligned}$$

By making the following change of variables

$$\begin{aligned} u_3 &= \tfrac{1}{4}(3\Psi_2\hat{\phi}_2-2\hat{\Psi}_3\phi_1), & u_4 &= -\hat{\Psi}_4\phi_1, \\ v_0 &= (\bar{\delta}+2\alpha+3\pi)\hat{\phi}_2-2\hat{\lambda}\phi_1, & v_1 &= (\Delta+2\gamma+3\mu)\hat{\phi}_2-2\hat{\nu}\phi_1, \quad (56)\end{aligned}$$

equations (51) - (54) can be cast in a simpler and more symmetric form, becoming respectively

$$\begin{aligned}(\delta+3\beta-\bar{\alpha}+\bar{\pi})v_0-(D+3\epsilon+\bar{\epsilon}-\bar{\rho})v_1 &= -8u_3, \quad (57)\\ (\bar{\delta}+3\pi+3\alpha+\bar{\beta}-\bar{\tau})v_1-(\Delta+3\mu+3\gamma+\bar{\mu}-\bar{\gamma})v_0 &= 2u_4, \quad (58)\\ -2(\bar{\delta}+2\alpha+6\pi)u_3+(D+4\epsilon-3\rho)u_4 &= -\frac{1}{2}v_0\Omega_+, \quad (59)\\ -2(\Delta+2\gamma+6\mu)u_3+(\delta+4\beta-3\tau)u_4 &= -\frac{1}{2}v_1\Omega_-, \quad (60)\end{aligned}$$

where $\Omega_\pm = 3\Psi_2 \pm 2\Phi_{11}$. The quantities $u_3,\ u_4,\ v_0$ and v_1 are also invariant under a change in the tetrad of first order of smallness (see Wald (1979) and Torres del Castillo (1987)).

The four equations (57) - (60) do not appear to admit separable solutions. The presence of terms in the Ricci spinor on the right hand side of equations (59) and (60) prevents an obvious decoupling of the equations as has occured in the case of the Kerr metric gravitational perturbations. Torres del Castillo (1988c) has shown that the Hertz potential methods used in the Kerr case can be extended to the problem of the Kerr-Newman perturbation. However instead of one potential two are now required. The equations satisfied by these potentials are closely related to equations (57) - (60). This observation implies the existence of a symmetry operator for the complex potential equations (Kalnins and Williams, 1997a). Consequently if we have just one solution to the equations we can generate more by application of the symmetry operator. Indeed the equations (57) - (60) can be reformulated in terms of integral equations using the Green's function of the Teukolsky operator for spins 1 and 2. The resulting formulation then provides a solution in principle by repeated iteration of the resulting integral operator (Kalnins and Williams, 1997a).

Chandrasekhar in his computations with these equations sought a decoupling analogous to that of the Kerr case. He was unsuccessful but computer algebra computations have shown that this is indeed possible (Kalnins and Williams, 1997b) resulting in partial differential equations of eighth order. However the condition appears to be without utility.

In the Reissner-Nordström space-time solution of equations (57) - (60) is possible. By writing each of the four functions u_3, u_4, v_0 and v_1 as a product of a radial function and an angular function and by making use of the spherical symmetry of the Reissner-Nordström space-time one obtains a solution for these four functions of the form

$$\begin{aligned} v_0 &= -iu^l_{m2}(\theta)V_0(r), \qquad & v_1 &= u^l_{m1}(\theta)V_1(r), \\ u_3 &= u^l_{m1}(\theta)U_3(r), \qquad & u_4 &= -iu^l_{m2}(\theta)U_4(r), \end{aligned} \tag{61}$$

where the functions $u^l_{mn}(\theta)$ are generalised spherical functions (Gel'fand *et al.*, 1963). The functions U_3, U_4, V_0 and V_1 satisfy four radial equations which can be written in matrix form as

$$\begin{bmatrix} (\mathcal{D}_0 - 3r^{-1}) & \mu & 0 & -2r^{-1} \\ \mu & \Delta(\mathcal{D}^\dagger_{-1} + 3r^{-1}) & -r^{-1} & 0 \\ 0 & 8q^2r^{-1} - 6M & (\mathcal{D}_0 - 3r^{-1}) & -\mu \\ 4q^2r^{-1} - 6M & 0 & -\mu & \Delta(\mathcal{D}^\dagger_{-1} + 3r^{-1}) \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_0 \\ U_4 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{62}$$

where $\mu = \sqrt{(l+2)(l-1)}$. It remains to decouple these equations. Following Chandrasekhar (1979) we make the change of variables

$$\begin{bmatrix} V_1 \\ V_0 \\ U_4 \\ U_3 \end{bmatrix} = \begin{bmatrix} \mu & 0 & -\mu & 0 \\ 0 & -\mu & 0 & \mu \\ -q_1 & 0 & q_0 & 0 \\ 0 & q_0 & 0 & -q_1 \end{bmatrix} \begin{bmatrix} F_0 \\ G_0 \\ F_1 \\ G_1 \end{bmatrix} \tag{63}$$

in equation (62) and then multiply on the left by

$$\frac{1}{\mu(b_0 - b_1)} \begin{bmatrix} b_0 & 0 & \mu & 0 \\ 0 & b_1 & 0 & \mu \\ b_1 & 0 & \mu & 0 \\ 0 & b_0 & 0 & \mu \end{bmatrix} \tag{64}$$

where b_0 and b_1 are defined by the relations

$$b_0 + b_1 = 6M \quad and \quad b_0 b_1 = -4\mu^2 q^2. \tag{65}$$

The four equations decouple into the following two pairs (here $i = 0,\ 1$)

$$\begin{aligned}\left(\mathcal{D}_0 - \frac{3}{r}\right)F_i - \left(\mu + \frac{2b_i}{\mu r}\right)G_i &= 0,\\ \Delta\left(\mathcal{D}^\dagger_{-1} + \frac{3}{r}\right)G_i - \left(\mu + \frac{b_{1-i}}{\mu r}\right)F_i &= 0.\end{aligned} \tag{66}$$

If we make the substitutions

$$F_i = r^3 Y_i \quad and \quad G_i = \frac{\Delta}{r^3}X_i \tag{67}$$

then these two pairs of equations become,

$$\begin{aligned}r^3\mathcal{D}_0 Y_i - \frac{\Delta}{r^3}\left(\mu + \frac{2b_i}{\mu r}\right)X_i &= 0,\\ \frac{\Delta^2}{r^3}\mathcal{D}^\dagger_0 X_i - r^3\left(\mu + \frac{b_{1-i}}{\mu r}\right)Y_i &= 0.\end{aligned} \tag{68}$$

Eliminating X_i from the second of equations (68) results in the second order ordinary differential equation

$$\Delta\left(\mathcal{D}^\dagger_{-1} + \frac{6}{r}\right)\left[\left(\mu + \frac{2b_i}{\mu r}\right)^{-1}\mathcal{D}_0 Y_i\right] - \left(\mu + \frac{b_{1-i}}{\mu r}\right)Y_i = 0. \tag{69}$$

7. Teukolsky Functions and Teukolsky-Starobinsky Identities

The spin s Teukolsky functions $R_{\pm s}(r)$ and $S_{\pm s}(\theta)$ satisfy the second order differential equations (here $\Delta = r^2 - 2Mr + a^2$)

$$\begin{aligned}(\Delta\mathcal{D}^\dagger_{1-s}\mathcal{D}_0 + 2(2s-1)i\omega r)R_{-s} &= \lambda R_{-s}\\ (\Delta\mathcal{D}_1\mathcal{D}^\dagger_s - 2(2s-1)i\omega r)R_{+s} &= \lambda R_{+s}\\ (\mathcal{L}^\dagger_{1-s}\mathcal{L}_s + 2(2s-1)\omega a\cos\theta)S_{+s} &= -\lambda S_{+s}\\ (\mathcal{L}_{1-s}\mathcal{L}^\dagger_s - 2(2s-1)\omega a\cos\theta)S_{-s} &= -\lambda S_{-s}\end{aligned} \tag{70}$$

The Teukolsky functions $R_{\pm s}$ and $S_{\pm s}$ have some interesting properties. In fact the equations satisfied by the $S_{\pm s}$ functions are a form of confluent Heun equation. Fackerell and Crossman (1977) and Leaver (1986) have studied in detail expansion theorems and the spectrum of the parameter λ and its various asymptotic forms.

Kalnins *et al.* (1989b) have shown the Teukolsky-Starobinsky identities hold for $s = \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \frac{5}{2},\ \ldots$. The identities are

$$\begin{aligned}&\Delta^s\mathcal{D}_0^{2s}(\Delta\mathcal{D}^\dagger_{1-s}\mathcal{D}_0 + 2(2s-1)i\omega r)\\ &\quad = (\Delta\mathcal{D}_{1-s}\mathcal{D}^\dagger_0 - 2(2s-1)i\omega r)\Delta^s\mathcal{D}_0^{2s},\end{aligned} \tag{71}$$

$$\begin{aligned}&\mathcal{L}_{1-s}\mathcal{L}_{2-s}\ldots\mathcal{L}_{s-1}\mathcal{L}_s(\mathcal{L}^\dagger_{1-s}\mathcal{L}_s + 2(2s-1)\omega a\cos\theta)\\ &\quad = (\mathcal{L}_{1-s}\mathcal{L}^\dagger_s - 2(2s-1)\omega a\cos\theta)\mathcal{L}_{1-s}\mathcal{L}_{2-s}\ldots\mathcal{L}_{s-1}\mathcal{L}_s.\end{aligned} \tag{72}$$

Two very similar identities are obtained by taking the complex conjugate of the first identity ($\mathcal{D} \longleftrightarrow \mathcal{D}^\dagger$) and by letting $\theta \to \pi - \theta$ in the second identity ($\mathcal{L} \longleftrightarrow -\mathcal{L}^\dagger$). They read

$$\begin{aligned}
&\Delta^s \mathcal{D}^{\dagger 2s}_0 (\Delta \mathcal{D}_{1-s} \mathcal{D}^\dagger_0 - 2(2s-1)i\omega r) \\
&\quad = (\Delta \mathcal{D}^\dagger_{1-s} \mathcal{D}_0 + 2(2s-1)i\omega r)\Delta^s \mathcal{D}^{\dagger 2s}_0, && (73) \\
&\mathcal{L}^\dagger_{1-s}\mathcal{L}^\dagger_{2-s}\dots\mathcal{L}^\dagger_{s-1}\mathcal{L}^\dagger_s(\mathcal{L}_{1-s}\mathcal{L}^\dagger_s - 2(2s-1)\omega a\cos\theta) \\
&\quad = (\mathcal{L}^\dagger_{1-s}\mathcal{L}_s + 2(2s-1)\omega a\cos\theta)\mathcal{L}^\dagger_{1-s}\mathcal{L}^\dagger_{2-s}\dots\dots\mathcal{L}^\dagger_{s-1}\mathcal{L}^\dagger_s. && (74)
\end{aligned}$$

Note that for any function $f(r)$

$$\mathcal{D}_n(\Delta^s f(r)) = \Delta^s \mathcal{D}_{n+s} f(r). \tag{75}$$

Note also that R_{-s} and $\Delta^s R_{+s}$ satisfy complex conjugate equations. Similarly $S_{+s}(\pi-\theta)$ satisfies the same equation as $S_{-s}(\theta)$.

A direct consequence of the Teukolsky-Starobinsky identities is the following: If one acts on the function R_{-s} with both sides of (71) then one finds that $\Delta^s\mathcal{D}^{2s}_0 R_{-s}$ is a solution of the Teukolsky equation for $\Delta^s R_{+s}$. Similarly by acting on the function $\Delta^s R_{+s}$ with both sides of (73) one finds that $\Delta^s \mathcal{D}^{\dagger 2s}_0 \Delta^s R_{+s}$ is a solution of the Teukolsky equation for R_{-s}. By suitably choosing the relative normalisation of the functions R_{-s} and R_{+s} we can write

$$\begin{aligned}
\Delta^s \mathcal{D}^{2s}_0 R_{-s} &= \mathcal{C}_s \Delta^s R_{+s}, \\
\Delta^s \mathcal{D}^{\dagger 2s}_0 \Delta^s R_{+s} &= \bar{\mathcal{C}}_s R_{-s}
\end{aligned} \tag{76}$$

where $\mathcal{C}_s$ is a complex constant. A similar treatment of the angular identities results in the relations

$$\begin{aligned}
\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_s S_{+s} &= \mathcal{B}_s S_{-s}, \\
\mathcal{L}^\dagger_{1-s}\mathcal{L}^\dagger_{2-s}\dots\mathcal{L}^\dagger_{s-1}\mathcal{L}^\dagger_s S_{-s} &= (-1)^{2s}\mathcal{B}_s S_{+s}.
\end{aligned} \tag{77}$$

where $\mathcal{B}_s$ is a real constant. The constants $\mathcal{C}_s$ and $\mathcal{B}_s$ are referred to as Starobinsky constants.

By use of the mapping $r \longleftrightarrow \xi = ia\cos\theta$ one can make the above radial and angular identities (76) and (77) almost formally identical. By such means one can show that

$$\mathcal{B}^2_s = (-1)^{2s}|\mathcal{C}_s|^2\,|_{M=0} \tag{78}$$

Computation of $|\mathcal{C}_s|^2$ can be achieved by examining the identity

$$\Delta^s \mathcal{D}^{\dagger 2s}_0 \Delta^s \mathcal{D}^{2s}_0 R_{-s} = |\mathcal{C}_s|^2 \tag{79}$$

and replacing the occurences of second order and higher derivatives of R_{-s} with first and zeroth order terms by means of the Teukolsky equation that R_{-s} must satisfy. Such a procedure is very protracted for all but the first few small values of s. Chandrasekhar (1990) has shown how to determine the general expression for the magnitude of the Teukolsky-Starobinsky constant $|\mathcal{C}_s|^2$ in the form of a determinant resulting from considerations following from the notion of algebraically special solutions. Torres del Castillo (1988b) has also investigated aspects of the applicability of Teukolsky-Starobinsky identities in type D vacuum backgrounds with cosmological constant. The magnitude of the Starobinsky constants for spin $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2 are

$$
\begin{aligned}
|\mathcal{C}_{\frac{1}{2}}|^2 &= \lambda, && (80)\\
|\mathcal{C}_1|^2 &= \lambda^2 - 4\omega^2\tilde{\alpha}^2, && (81)\\
|\mathcal{C}_{\frac{3}{2}}|^2 &= \lambda^2(\lambda+1) - 16\omega^2(\lambda\tilde{\alpha}^2 - a^2), && (82)\\
|\mathcal{C}_2|^2 &= \lambda^2(\lambda+2)^2 - 8\omega^2\tilde{\alpha}^2\lambda(5\lambda+6) + 96\omega^2a^2\lambda \\
&\quad + 144\omega^4\tilde{\alpha}^4 + 144\omega^2M^2 && (83)
\end{aligned}
$$

where

$$\tilde{\alpha}^2 = a^2 + \frac{am}{\omega}. \tag{84}$$

We also note here that it is possible to invent a relativistically invariant system of equations which has the general spin s Teukolsky functions appearing in their solution. In fact the coupled system of equations

$$
\begin{aligned}
&\nabla^{KA'}\phi_{KA_2\ldots A_{2s}} = (2s-1)(s-1)\Psi^{KL}{}_{(A_2A_3}\theta^{A'}{}_{A_4\ldots A_{2s})KL},\\
&I\nabla_{(A_1K'}\theta^{K'}{}_{A_2\ldots A_{2s})} - \frac{1}{6}(2s-3)\theta^{K'}{}_{(A_2\ldots A_{2s}}\nabla_{A_1)K'}I = I\phi_{A_1A_2\ldots A_{2s}},
\end{aligned} \tag{85}
$$

where $I = \Psi_{ABCD}\Psi^{ABCD}$, typically has solutions of the form

$$
\begin{aligned}
\phi_{0\ldots0} &= R_{+s}S_{+s}e^{i\omega t + im\varphi},\\
\theta^{0'}_{0\ldots0} &= -\frac{1}{\sqrt{2}\rho'^2(2s-1)(s-1)\Psi_2}\Delta\left(\mathcal{D}^\dagger_s - \frac{2s-1}{\bar{\rho}'}\right)\phi_{0\ldots0},\\
\theta^{1'}_{0\ldots0} &= -\frac{1}{\sqrt{2}\bar{\rho}'(2s-1)(s-1)\Psi_2}\left(\mathcal{L}^\dagger_s - \frac{(2s-1)ia\sin\theta}{\bar{\rho}'}\right)\phi_{0\ldots0}.
\end{aligned} \tag{86}
$$

with similar expressions existing for components having indices consisting of all 1's.

8. Comments

We have seen that the solution of systems of differential equations that arise in General Relativity via some sort of separation of variables contains elegant examples but as yet no theoretical basis. More work at a mathematical level needs to be done to improve this theoretical situation as well as solving the problems specific to General Relativity itself e.g. the combined perturbation problem for the Kerr-Newman black hole. A theory of separation of variables for the non-scalar wave equations of mathematical physics needs to explain the separability observed in these important physical examples.

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