Variable separation and second order superintegrability

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Abstract

In this talk we shall first describe clearly what “separation of variables" means in general, the mechanism of variable separation and its (intrinsic) group-theoretic significance for some fundamental scalar PDEs of mathematical physics: Hamilton-Jacobi, Helmholtz, Schrödinger, etc. We conclude with an application of these techniques to second order superintegrable systems.

SEPARABILITY: Intuitive

Partial differential equation:

$$H(x^i, u, u_i, u_{ij}, u_{ijk}, \cdots) = E \qquad 1 \leq i, j, k, \cdots \leq N$$

Seek solution of form:

$$u = \sum_{i=1}^{N} S^{(i)}(x^i)$$

or

$$v = \prod_{i=1}^{N} T^{(i)}(x^i)$$

Note: Can set $v = \exp u$

SEPARABILITY: Naive

Write equation

$$H(x^i, u, u_i, u_{ii}, u_{iii}, \cdots) = E \qquad 1 \leq i \leq N$$

in the form

$$K^{(1)}(x^1, u_1, u_{11}, \cdots) = K(x^j, u_j, u_{jj}, \cdots), \qquad 2 \leq j \leq N.$$

THIS DEFINITION IS TOO RESTRICTIVE

SEPARABILITY: Technical-1

Postulate separation equations (ODEs).
Example: Orthogonal separation for the Hamilton-Jacobi equation.

$$(1) \qquad \sum_{i=1}^{N} H_i^{-2} u_i^2 = E, \qquad ds^2 = \sum_{i=1}^{N} H_i^2 (dx^i)^2.$$

Assume additive separation so that $\partial_j u_i = \partial_j \partial_i u = 0$ for $i \neq j$.
Separation equations:

$$(2) \qquad u_i^2 + \sum_{j=1}^{N} s_{ij}(x^i)\lambda^j = 0, \qquad i = 1, \cdots, N \quad \lambda^1 = -E.$$

SEPARABILITY: Technical-2

Here $\partial_k s_{ij}(x^i) = 0$ for $k \neq i$ and $\det(s_{ij}) \neq 0$. We say that $S = (s_{ij})$ is a *Stäckel matrix*. Then (1) can be recovered from (2) provided

$$H_j^{-2} = (S^{-1})^{1j}.$$

The quadratic forms

$$\mathcal{H}^\ell = \sum_{i=1}^{N} (S^{-1})^{\ell j} u_j^2$$

satisfy

$$\mathcal{H}^\ell = -\lambda^\ell$$

for a separable solution.

SEPARABILITY: Technical-3

Furthermore, setting $u_i = p_i$, we have

$$\{\mathcal{H}^\ell, \mathcal{H}^j\} = 0, \qquad \ell \neq j$$

where

$$\{\mathcal{H}, \mathcal{K}\} = \sum_{i=1}^{N} (\partial_{x^i}\mathcal{H}\partial_{p_i}\mathcal{K} - \partial_{x^i}\mathcal{K}\partial_{p_i}\mathcal{H})$$

is the Poisson Bracket. Thus the $\mathcal{H}^\ell, \quad 2 \leq \ell \leq N$, are *constants of the motion* for the *Hamiltonian* $\mathcal{H}^{(1)}$.

Similar constructions apply to 2nd order linear PDE's and lead to 2nd order symmetry operators, i.e., operators mapping solutions to solutions.

COMPROMISE APPROACH -1

$$H(x^i, u, u_i, u_{ii}, u_{iii}, \cdots) = E \qquad 1 \leq i \leq N$$

Look for solution of form:

$$u = \sum_{i=1}^{N} S^{(i)}(x^i)$$

Let

$$u_{i,1} \equiv u_i, \quad u_{i,j+1} \equiv \partial_{x^i} u_{i,j}, \qquad j = 1, 2, \cdots$$

Let m_i be the largest integer ℓ such that $\partial_{u_{i,\ell}} H = H_{u_{i,\ell}} \not\equiv 0$.
Let D_i be the total differentiation operator

$$D_i = \partial_{x^i} + u_{i,1}\partial_u + u_{i,2}\partial_{u_{i,1}} + \cdots + u_{i,m_i+1}\partial_{u_{i,m_i}} + \cdots$$

COMPROMISE APPROACH -2

Then the equation

$$D_i H(x,u) = 0$$

implies

$$u_{i,m_i+1} = -\frac{\tilde{D}_i H}{H_{u_{i,m_i}}}, \qquad i = 1, 2, \cdots, N,$$

where

$$\tilde{D}_i = \partial_{x^i} + u_{i,1}\partial_u + u_{i,2}\partial_{u_{i,1}} + \cdots + u_{i,m_i}\partial_{u_{i,m_i-1}}.$$

COMPROMISE APPROACH -3

It follows that u satisfies the integrability conditions

$$D_j u_{i,m_i+1} = 0, \qquad j \neq i,$$

or

$$(3) \qquad H_{u_{i,m_i}} H_{u_{j,m_j}} (\tilde{D}_i \tilde{D}_j H) + H_{u_{i,m_i} u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H)$$

$$= H_{u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H_{u_{i,m_i}}) + H_{u_{i,m_i}} (\tilde{D}_j H)(\tilde{D}_i H_{u_{j,m_j}}).$$

Theorem: If conditions (3) are satisfied identically in the dependent variables $u, u_{k,\ell}$, then the partial differential equation $H = E$ admits a $\sum_{i=1}^{N} m_i + 1$ parameter family of separable solutions.

Example 1.

$H = (x_1 + x_2)(u_{11} + u_{22}) - 2(u_1 + u_2)$. Equations (3) are satisfied identically so $\{x_1, x_2\}$ is a regular separable system. The general separable solution depends on five parameters and is given by

$$u = (\alpha x_1^3 + \beta x_1^2 + \gamma x_1 - \frac{1}{2} E x_1) + (-\alpha x_2^3 + \beta x_2^2 - \gamma x_2 + \delta).$$

DIFFERENTIAL STÄCKEL FORM-1

The appropriate separation equations are

$$\begin{aligned}
\partial_1 u + E/2 - \gamma - 2\beta x_1 - 3\alpha x_1^2 &= 0, \\
\partial_{11} u - 2\beta - 6\alpha x_1 &= 0, \\
\partial_2 u + \gamma - 2\beta x_2 + 3\alpha x_2^2 &= 0, \\
\partial_{22} u - 2\beta + 6\alpha x_2 &= 0.
\end{aligned}$$

DIFFERENTIAL STÄCKEL FORM-2

The associated “Stäckel matrix” responsible for the separation is

$$\begin{bmatrix} \frac{1}{2} & -1 & -2x_1 & -3x_1^2 \\ 0 & 0 & -2 & -6x_1 \\ 0 & 1 & -2x_2 & 3x_2^2 \\ 0 & 0 & -2 & 6x_2 \end{bmatrix}.$$

Not a true Stäckel matrix since more than one row depends on a given variable x_i. The second and fourth rows are the derivatives of the first and third rows, respectively. It is a nontrivial example of a differential-Stäckel matrix. All additive separation for n^{th} order linear equations is of this form.

Example 2.

$H = u_{11}^2 + u_1 + u_{22}$. Here we have $u_{111} = -\frac{1}{2}$ (provided $u_{11} \neq 0$) and $u_{222} = 0$ so equations (*) are satisfied identically and $\{x_1, x_2\}$, is a regular separable system. The general separable solution depends on five parameters:

$$u = (-\frac{1}{12}x_1^3 + \alpha x_1^2 + \beta x_1) + \left(\frac{1}{2}(E - 4\alpha^2 - \beta)x_2^2 + \gamma x_2 + \delta\right).$$

Example 3.

$H = x_2 u_{11} + x_1 u_{22} + u_1 + u_2$. Equations (3) reduce to the requirement $u_{11} + u_{22} = 0$. The general separable solution depends on four parameters:

$$u = (\alpha x_1^2 + \beta x_1) + (-\alpha x_2^2 + (E - \beta) x_2 + \gamma).$$

This is a nonregular separable system.

Example 4.

$H = (u_{11} + u_{22})/u$. Equations (3) are satisfied identically for $u \neq 0$. The general separable solution depends on five parameters:

$$u = \alpha \exp(x_1\sqrt{E}) + \beta \exp(-x_1\sqrt{E}) + \gamma \exp(x_2\sqrt{E}) + \delta \exp(-x_2\sqrt{E})$$

for $E > 0$, with obvious modifications for $E \leq 0$.

Laplace-like equations-1

There is a similar theory of additive separation for partial differential equations with $E = 0$, i.e., equations not depending on a parameter. We make the same assumptions on H as before and take the equation

$$H(x_i, u, u_i, u_{ii}, \ldots) = 0$$

Then a separable solution u of must satisfy the usual integrability conditions. In case the integrability conditions are identities in the sense that there exist functions $P_{i,j}(x_k, u, u_{k,\ell})$, polynomials in $u_{k,\ell}$ such that

$$\begin{aligned} \mathcal{F}_{ij} &\equiv H_{u_{i,m_i}} H_{u_{j,m_j}} (\tilde{D}_i \tilde{D}_j H) + H_{u_{i,m_i}, u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H) \\ &- H_{u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H_{u_{i,m_i}}) - H_{u_{i,m_i}} (\tilde{D}_j H)(\tilde{D}_i H_{u_{j,m_j}}) \\ &= P_{i,j} H, \qquad i \neq j, \end{aligned}$$

Laplace-like equations-2

Theorem: If $\{x_k\}$ is a regular separable system for $H=0$ then for every set of $m_1+m_2+\ldots+m_n+1$ constants $\{v^0, v^0_{i,j}\}$ with $H(x^0,v^0)=0$ and $H_{u_{j,m_j}}(x^0,v^0)\neq 0$, there is a unique separable solution u of $H(x,u)=0$ such that $u(x^0)=v^0$, $u_{i,j}(x^0)=v^0$, $u_{i,j}(x^0)=v^0_{i,j}$, $1\leq i\leq n$, $1\leq j\leq m_i$.

Again we observe that if equations (3) are not satisfied identically, separable solutions still may exist but will depend on fewer that $\sum_{i=0}^{n} m_i+1$ independent parameters. This is *non-regular* separation. Examples 1-4 above for $E=0$ are instances of regular and nonregular separation.

Example 5. (less trivial)

$$H = (x_2 - x_3)u_{11} + (x_3 - x_1)u_{22} + (x_1 - x_2)u_{33}.$$

Equations (3) are satisfied with $P_{i,j} \neq 0$, so $\{x_k\}$ is a regular separable system for $H = 0$, though not for $H = E$. The general separable solution depends on six parameters and is given by

$$u = \frac{1}{6}\alpha(x_1^3 + x_2^3 + x_3^3) + \frac{1}{2}\beta(x_1^2 + x_2^2 + x_3^2) + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \delta.$$

Example 6. Orthogonal R-separation

Consider the Helmholtx equation

$$(4) \qquad \Delta_N \Psi(x) + V(x)\Psi(x) = E\Psi(x)$$

Here, Δ_N is the Laplacian on a pseudo-Riemannian manifold, written in an orthogonal coordinate system x^i:

$$\Delta_N = \frac{1}{\sqrt{H_1 \cdots H_N}} \sum_{i=1}^{N} \partial_{x^i}(H_1 \cdots H_N H_i^{-2} \partial_{x^i})$$

where

$$ds^2 = \sum_{i=1}^{N} H_i^2 (dx^i)^2.$$

Orthogonal R-separation-2

Look for multiplicative R-separation:

$$\Psi = \exp R(x) \prod_{i=1}^{N} \Psi_{(i)}(x^i).$$

Set $u = R - \ln\Psi$ to get standard PDE:

$$\sum_{i=1}^{N} [H_i^{-2}(u_{ii}+u_i^2)+(2H_i^{-2}\partial_i R+s_i)u_i+H_i^{-2}(\partial_{ii}R+(\partial_i R)^2+s_i\partial_i R)]$$

$$+V = E.$$

Here,

$$s_i = \frac{1}{H_1\cdots H_N}\partial_i(H_1\cdots H_N H_i^{-2}).$$

Orthogonal R-separation -3

Require regular separation. Substitute into integrability conditions(4) and equate coefficients:

- Coeff. of u_i^2: The H_i^{-2} are in Stäckel form.

$$\partial_{jk} H_i^{-2} = \partial_j H_i^{-2} \partial_k \ln H_j^{-2} + \partial_k H_i^{-2} \partial_j \ln H_k^{-2}, \quad j \neq k.$$

  Levi-Civita separability conditions.

- Coeff. of u_{ii}: Determines R.

$$R = -\frac{1}{2} \ln \frac{g}{S} + \sum_{i=1}^{n} L^{(i)}(x^i)$$

  where the functions $L^{(i)}$ are arbitrary.

- Coeff. of 1: Generalized Robertson conditions for the potential $\tilde{V}$.

Orthogonal R-separation -4

The conditions are

$$\partial_{ik}\tilde{V} - \partial_k \ln H_j^{-2}\partial_j\tilde{V} - \partial_j \ln H_k^{-2}\partial_k\tilde{V} = 0, \quad j \neq k.$$

This means precisely that the potential function can be expressed in the form

$$\tilde{V} = \sum_{i=1}^{n} f^{(i)}(x^i)H_i^{-2}.$$

All R-separable solutions of (4) follow from the Stäckel construction.

Orthogonal R-separation -5

It follows that every orthogonal coordinate system permitting product separation of the Helmholtz equation corresponds to a Stäckel form; hence it permits additive separation of the Hamilton-Jacobi equation. Eisenhart has shown that the additional Robertson condition so that $R = 1$ for product separation is equivalent to the requirement $R_{ij} = 0$ for $i \neq j$, where R_{ij} is the Ricci tensor of V^n expressed in the Stäckel coordinates $\{x^i\}$. It follows that the Robertson condition is automatically satisfied in Euclidean space, a space of constant curvature or any Einstein space.

Orthogonal R-separation -6

The question arises whether nontrivial R-separation necessarily occurs. From Eisenhart's formulation of Robertson's condition as $R_{ij} = 0$, $i \neq j$, we see that only trivial orthogonal R-separation can occur in an Einstein space. However, nontrivial R-separation can occur, even in conformally flat spaces. An example is

$$
\begin{aligned}
ds^2 &= (x+y+z)[(x-y)(x-z)dx^2 + (y-z)(y-x)dy^2 \\
&+ (z-x)(z-y)dz^2], \\
e^R &= (x+y+z)^{-\frac{1}{4}}.
\end{aligned}
$$

Results for scalar PDE's:

Equation	Type of Separation
Hamilton-Jacobi	additive sep.
Helmholtz (Klein-Gordon)	multiplicative R-sep.
Laplace or wave	multiplicative R-sep.
heat/time-dependent Schrödinger	multiplicative R-sep.

- All separation is determined via the Stäckel procedure.
- Separation can be characterized via the symmetry operators for the equation.
- All separable systems can (in principle) be classified.
- Applies to N-dimensional pseudo-Riemannian manifolds and both orthogonal and non-orthogonal sep.

Intrinsic characterization for H-J Eqn.

Theorem: Necessary and sufficient conditions for the existence of an *orthogonal* separable coordinate system $\{x^i\}$ for the Hamilton-Jacobi equation $\mathcal{H}^1 = E$ on an N-dimensional pseudo-Riemannian manifold are that there exist N quadratic forms $\mathcal{H}^k = \sum_{i,j=1}^{N} H_{ij}^{(k)} p_i p_j$ on the manifold such that:

- $\{\mathcal{H}^k, \mathcal{H}^\ell\} = 0, \quad 1 \leq k, i \leq N$,
- The set $\{\mathcal{H}^k\}$ is linearly independent (as N quadratic forms).
- There is a basis $\{\omega_{(j)} : 1 \leq j \leq N\}$ of simultaneous eigenforms for the $\{\mathcal{H}^k\}$. If conditions (1)-(3) are satisfied then there exist functions $g^i(x)$ such that:

$$\omega_{(j)} = g^j dx^j, \quad j = 1, \cdots, N.$$

Intrinsic char. for Helmholtz Eqn. -1

Theorem: Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system $\{x^i\}$ for the Helmholtz equation $\Delta_N \Psi = E\Psi$ on an N-dimensional pseudo-Riemannian manifold are that there exists a linearly independent set $\{A_1 = \Delta_N, A_2, \cdots, A_N\}$ of second-order differential operators on the manifold such that:

- $[A_k, A_\ell] = 0, \quad 1 \leq k, \ell \leq N$,
- Each A_k is in self-adjoint form,
- There is a basis $\{\omega_{(j)} : 1 \leq j \leq N\}$ of simultaneous eigenforms for the $\{A_k\}$.

If conditions (1)-(3) are satisfied then there exist functions $g^i(x)$ such that: $\omega_{(j)} = g^j dx^j, \quad j = 1, \cdots, N$.

Intrinsic characterization

The main point of the theorems is that, under the required hypotheses the eigenforms ω^ℓ of the quadratic forms a^{ij} are normalizable, i.e., that up to multiplication by a nonzero function, ω^ℓ is the differential of a coordinate. This fact permits us to compute the coordinates directly from a knowledge of the symmetry operators.

Example 7. -1

Consider the Hamilton-Jacobi equation for two dimensional Minkowski space. In Cartesian coordinates this equation is $H \equiv u_x^2 - u_t^2 = E$. The vector space of all symmetries of the form $\mathcal{L} = a(x,t)u_x + b(x,t)u_y$ is closed under the bracket $\{\cdot,\cdot\}$; hence the symmetries form a Lie algebra. Furthermore $\{H,\mathcal{L}\} \equiv 0$ for each linear symmetry. The Lie algebra is three dimensional, with basis

$$
\begin{aligned}
\mathcal{L}_1 &= u_x,\ \mathcal{L}_2 = u_t,\ \mathcal{L}_3 = tu_x + xu_t, \\
\{\mathcal{L}_1,\mathcal{L}_2\} &= 0,\ \{\mathcal{L}_3,\mathcal{L}_1\} = \mathcal{L}_2,\ \{\mathcal{L}_3,\mathcal{L}_2\} = \mathcal{L}_1.
\end{aligned}
$$

Every symmetry quadratic in the first derivatives of u is a polynomial in the linear symmetries $\mathcal{L}_i$. All candidates for variable separation can be built from the basis symmetries

Example 7. -2

Consider the quadratic symmetry $\mathcal{A}^2 = 2\mathcal{L}_3\mathcal{L}_1$. With respect to Cartesian coordinates, the corresponding symmetric quadratic forms are

$$\mathcal{A}^2 \sim \begin{pmatrix} 2t & x \\ x & 0 \end{pmatrix}, \quad \mathcal{A}^1 = \mathcal{H} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly, $\mathcal{A}^2$ has roots $\rho = t \pm \sqrt{t^2 - x^2}$ (assuming $t > |x|$) with a basis of eigenforms $\omega_1 = (t + \sqrt{t^2 - x^2})dx - xdt$, $\omega_2 = (t - \sqrt{t^2 - x^2})dx - xdt$. By the Theorem, $\mathcal{A}^2$ does define a separable coordinate system $\{\xi^1, \xi^2\}$ for (5.2) and there exist functions f_i such that $d\xi^i = f_i\omega_i$. We find $f_1 = [\xi^2((\xi^2)^2 - (\xi^1)^2)]^{-1}$, $f_2 = -[\xi^1((\xi^2)^2 - \xi^1)^2)]^{-1}$, $t = \frac{1}{2}((\xi^1)^2 + (\xi^2)^2), \qquad x = \xi^1\xi^2$.

Example 7. -3

On the other hand the symmetry

$$\mathcal{A} = 2\mathcal{L}_3(\mathcal{L}_1 - \mathcal{L}_2)$$

has two equal roots and only one eigenform. Thus $\mathcal{A}$ cannot determine a separable coordinate system.

For manifolds of dimension $n \geq 3$ there is a second way that a system of $n-1$ commuting symmetries may fail to determine separable coordinates: although each quadratic symmetry determines a basis of eigenforms, there is no basis of eigenforms for all symmetries simultaneously.

Construction of separable coordinates -1

A complete construction of separable coordinate systems on the N-sphere and on Euclidean N-space, and a graphical method for constructing these systems has been worked out by Kalnins and Miller. Here we mention some of the main ideas.
The basic elliptic coordinate system on the N-sphere is denoted

$$[e_0|e_1|\cdots|e_N].$$

All separable coordinate systems on the N-sphere can be obtained by nesting these basic coordinates for the k-spheres for $k \leq N$.

Separable coordinates on N-sphere -2

Start with a basic elliptic coordinate system on the $(N-k)$-sphere and embedding in it a k-sphere. The k-sphere Cartesian coordinates $(V_0, \cdots, V_k)$ can be attached to any one of the $N-k+1$ Cartesian coordinates $(U_0, \cdots, U_{N-k})$ of the $(N-k)$-sphere. Let us attach it to the first coordinate. We have

$$(X_0, \cdots, X_N) = (U_0V_0, \cdots, U_0V_k, U_1, \cdots, U_{N-k}), \qquad \sum_{\ell=0}^{k} V_\ell^2 = 1,$$

$$V_\ell^2 = \frac{\prod_{i=1}^{k}(v_i - f_\ell)}{\prod_{i \neq \ell}(f_i - f_\ell)}, \quad U_0^2 = \frac{\prod_{t=1}^{N-k}(u_t - e_0)}{\prod_{i \neq 0}(e_i - e_0)},$$

$$ds^2 = ds_1^2 + U_0^2 ds_2^2, \quad ds_1^2 = \sum_{h=0}^{N-k} dU_h^2, \quad ds_2^2 = \sum_{\ell=0}^{k} dV_\ell^2.$$

Construction of separable coordinates -3

The resulting system denoted graphically by

$$
\begin{array}{ccccccccc}
 & & & [\; e_0 & | & e_1 & | & \cdots & | \; e_{N-k} \;] \\
 & & & \downarrow & & & & & \\
[\; f_0 & | & \cdots & | & f_k \;] & & & &
\end{array}
$$

Here is another possibility:

$$
\begin{array}{ccccccccccc}
 & & & [\; e_0 & | & e_1 & | & \cdots & | & e_{N-k-\ell-m} \;] & \\
 & & & \downarrow & & & & & & & \searrow \\
[\; f_0 & | & f_1 & | & \cdots & | & f_k \;] & & [g_0 & | \quad \cdots & | \; g_\ell] \\
 & & \downarrow & & & & & & & & \\
[\; h_0 & | & \cdots & | & h_m \;] & & & & & &
\end{array}
$$

Each separable system so arises.

Separable Euclidean systems -4

For Euclidean space the results are a bit more complicated. The basic ellipsoidal coordinate system on N-space is denoted

$$< e_0|e_1|\cdots|e_{N-1} >,$$

and the parabolic coordinate system is

$$(e_1|\cdots|e_{N-1}).$$

The graphs need no longer be trees; they can have several connected components. Each connected component is a tree with a root node that is either of the two basic forms. Just as above, spheres can be embedded in the root coordinates or to each other.

Construction of separable coordinates -5

Here are two examples: $1)$ Cartesian coordinates in two-space,

$$< e_0 > \qquad < e_0' >,$$

and $2)$ oblate spheroidal coordinates in three-space.

$$\begin{array}{c} < \quad e_0 \quad | \quad e_1 \quad > \\ \downarrow \\ [\quad a_1 \quad | \quad a_2 \quad] \end{array}$$

Symmetry adapted solutions -1

$$(5) \qquad H(x^i, u, u_i, u_{ij}, u_{ijk}, \cdots) = E \qquad 1 \leq i, j, k, \cdots \leq N$$

Lie derivative:

$$Z = \sum_{i=1}^{N} \xi_i(x)\partial_{x^i} + \phi(x,u)\partial_u$$

Extend by prolongation to get operator:

$$\hat{Z} = Z + \sum_{|K|>0} \phi^K(x, u^J)\partial_{u^K}$$

Z is a *Lie Symmetry* of (5) if $\hat{Z}H = 0$ whenever $H = E$.

Symmetry adapted solutions -2

If Z is a Lie symmetry, then by Lie's theorem there are new coordinates $s, y^2, \cdots, y^N, v$ such that

$$x = A(s, y), \qquad u = B(s, y, v), \qquad Z = \partial_s.$$

Thus (5) becomes

$$H(y^j, v, v^K) = E.$$

We can find solutions v such that

$$v = \lambda s + V(y^j).$$

Example 8. KdV

$$\partial_t u - \partial_{xxx} u - u\partial_x u = 0.$$

Symmetry:

$$Z = t\partial_x + c\partial_t - \partial_u, \quad c \neq 0.$$

New coordinates:

$$t = cs, \quad y = \frac{1}{2}cs^2 + x, \quad v = -s + u$$

New equation:

$$\partial_s v - c\partial_{yyy} v - cv\partial_y v - 1 = 0$$

THIS METHOD DOESN'T INCLUDE THE GENERAL STÄCKEL CONSTRUCTION.

Group Methods: Tensor Harmonics

For systems of equations admitting nontrivial Lie symmetry groups harmonic analysis can provide an effective tool for determining separable solutions in certain (subgroup) coordinate systems which are well adapted to the symmetries.

SYSTEMS OF EQUATIONS

- NO AGREED UPON DEFINITION OF VARIABLE SEPARATION.
- NO GENERAL MECHANISM FOR SEPARATION KNOWN.
- SOME INSIGHT FOR 1ST ORDER SEPARATION OF DIRAC TYPE EQUATIONS

Example 9. -1

$$\mathbf{H}\psi = E\psi, \qquad E,$$

where

$$\mathbf{H} = \sum_{i=1}^{n} H^i(x)\partial_i + V(x),$$

H^i, V are $N \times N$ matrices and ψ is an N-component spinor. We require H^i, $1 \le i \le n$, are nonsingular matrices. We define a γ*-integrable system* for $\mathbf{H}\psi = E\psi$ as a set of equations

$$\partial_i\psi = (\sum_{j=1}^{n} C_{ij}(x)\lambda^j - C_i(x))\psi \qquad i = 1, \ldots, n.$$

where the $C_{ij}(x)$, $C_i(x)$ are $N \times N$ matrices such that $\det(C_{ij}) \neq 0$, the λ^j are independent parameters with $\lambda^1 = E$,

Example 9. -2

The integrability conditions

$$\partial_i(\partial_j\psi) = \partial_j(\partial_i\psi), \qquad i \neq j,$$

imply

$$a) \qquad C_{jk}C_{i\ell} + C_{j\ell}C_{ik} = C_{ik}C_{j\ell} + C_{i\ell}C_{jk},$$

$$b) \qquad \partial_j C_{ik} - C_{ik}C_j - C_iC_{jk} = \partial_i C_{jk} - C_{jk}C_i - C_jC_{ik},$$

$$c) \qquad \partial_i C_j + C_iC_j = \partial_j C_i + C_jC_i.$$

Example 9. -3

Let the $nN \times nN$ bordered matrix $A = (A^{ij})$ be the inverse of (C_{jk}) :

$$\sum_{j=1}^{n} A^{ij} C_{jk} = \sum_{j=1}^{n} C_{kj} A^{ji} = \delta_k^i \mathcal{I}.$$

It follows that the solutions ψ satisfy the eigenvalue equations

$$A^k \psi \equiv \sum_{i=1}^{n} A^{ki}(x) \partial_i \psi + B^k(x) \psi = \lambda^k \psi, \quad 1 \leq k \leq n.$$

Example 9. -4

Theorem: The integrability conditions for the separation equations are satisfied identically iff there exist $N \times N$ matrices $A^{ki}(x), B^k(x), \quad 1 \leq k, i \leq n$ such that:

- The operators $\mathbf{A}^k = \sum_{i=1}^N A^{ki}\partial_{x^i} + B^k, \qquad k = 1, \cdots, N$ commute, i.e., $\mathbf{A}^k \mathbf{A}^\ell = \mathbf{A}^\ell \mathbf{A}^k$,
- $\mathbf{H} = \mathbf{A}^1$,
- $\sum_{i=1}^N A^{ki} C_{ij} = \mathcal{I}\delta_j^k$, where $\mathcal{I}$ is the $N \times N$ identity matrix.

SIMILARLY, THE STÄCKEL FORM CONDITIONS FOR SCALAR EQUATIONS CAN BE GENERALIZED TO THIS CASE.

γ-INTEGRABLE SYSTEMS -1

A γ-integrable system is *separable* if (by a change of frame $\psi = R\psi'$ if necessary) the factorization equations take the form

$$\partial_i \psi' = (\sum_{j=1}^{n} C_{ij}(x^i)\lambda^j - C_i(x^i))\psi' \qquad i = 1, \ldots, n.$$

in a *particular* coordinate system $\{x^1, \ldots, x^n\}$, i.e., $\partial_\ell C_{ij} = \partial_\ell C_i = 0$ if $\ell \neq i$.

γ-INTEGRABLE SYSTEMS -2

Theorem; Suppose the above is a separable system for $\mathbf{H}\psi = E\psi$ in coordinates $\{x^\ell\}$ and let $\mathbf{x}_0$ be a fixed vector. Then there are solutions of the form

$$\psi(\mathbf{x}) = R(\mathbf{x})\Theta^{(1)}(x^1)\ldots\Theta^{(n)}(x^n)\xi$$

where the Θ^ℓ are $N \times N$ matrices such that

$$\Theta^{(i)}(x^i)\Theta^{(j)}(x^j) = \Theta^{(j)}(x^j)\Theta^{(i)}(x^i)$$

for all i, j and $\Theta^{(i)}(x_0^i) = \mathcal{I}$.

MATRIX STÄCKEL FORM

Theorem Necessary and sufficient conditions that nonsingular matrices H^k satisfy the conditions

$$\sum_{i=1}^{n} H^i C_{ij}(x^i) = \delta_j^1 \mathcal{I}, \qquad 1 \leq j \leq n,$$

where

$$\det(C_{ij}) \neq 0$$

i.e., C is a Stäckel form matrix in the coordinates $\{x^i\}$, are

$$\partial_{jk} H^i = \partial_j H^k (H^k)^{-1} \partial_k H^i + \partial_k H^j (H^j)^{-1} \partial_j H^i, \quad j \neq k.$$

(Note that for $N = 1$ these equations agree with the Levi-Civita conditions where $H^i \equiv H_i^{-2}$.)

2nd order superintegrability (classical)

Classical superintegrable system on an n-dimensional local Riemannian manifold:

$$\mathcal{H} = \sum_{ij} g^{ij}\ p_i p_j + V(\mathbf{x}).$$

Require that Hamiltonian admits $2n-1$ functionally independent 2nd-order symmetries $\mathcal{S}_k = \sum a^{ij}_{(k)}(\mathbf{x}) p_i p_j + W_{(k)}(\mathbf{x})$, That is, $\{\mathcal{H}, \mathcal{S}_k\} = 0$ where $\{f, g\} = \sum_{j=1}^{n} (\partial_{x_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{x_j} g)$ is the Poisson bracket. Note that $2n-1$ is the maximum possible number of functionally independent symmetries.

Significance

Generically, every trajectory $\mathbf{p}(t), \mathbf{x}(t)$, i.e., solution of the Hamilton equations of motion, is characterized (and parametrized) as a common intersection of the (constants of the motion) hypersurfaces

$$\mathcal{S}_k(\mathbf{p}, \mathbf{x}) = c_k, \quad k = 0, \cdots, 2n-2.$$

The trajectories can be obtained without solving the equations of motion. This is better than integrability.

2D system

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{\lambda(\mathbf{x})} + V(\mathbf{x}).$$

Require that Hamiltonian admits 3 functionally independent 2nd-order symmetries $\mathcal{S}_k = \sum a^{ij}_{(k)}(\mathbf{x}) p_i p_j + W_{(k)}(\mathbf{x})$. Assume that V is at least a 2-parameter potential, i.e., we can prescribe $\partial_{x_1} V$ and $\partial_{x_2} V$ (as well as V) arbitrarily at each nonsingular point $(x, y) = (x_1, x_2)$.

Functional linear independence

The 3 functionally independent symmetries are **functionally linearly independent** if at each regular point (x_0, y_0) the 3 matrices $a^{ij}_{(1)}(x_0, y_0), a^{ij}_{(1)}(x_0, y_0), a^{ij}_{(1)}(x_0, y_0)$ are linearly independent.
There is essentially only one functionally linearly dependent superintegrable system in 2D:

$$\mathcal{H} = p_z p_{\bar{z}} + V(z),$$

where $V(z)$ is an arbitrary function of z alone. This system separates in only one set of coordinates $z, \bar{z}$.

Spaces of polynomial constants-4

THEOREM: Let $\mathcal{K}$ be a third order constant of the motion for a functionally linearly independent superintegrable system with 2-parameter potential V:

$$\mathcal{K} = \sum_{k,j,i=1}^{2} a^{kji}(x,y)p_k p_j p_i + \sum_{\ell=1}^{2} b^{\ell}(x,y)p_{\ell}.$$

Then $b^{\ell}(x,y) = \sum_{j=1}^{2} f^{\ell,j}(x,y)\frac{\partial V}{\partial x_j}(x,y)$ with $f^{\ell,j} + f^{j,\ell} = 0, \quad 1 \leq \ell, j \leq 2$. The a^{ijk}, b^{ℓ} are uniquely determined by the number $f^{1,2}(x_0,y_0)$ at some regular point (x_0,y_0) of V.

Structure theory -1

- Let

$$\mathcal{S}_1 = \sum a_{(1)}^{kj} p_k p_j + W_{(1)}, \quad \mathcal{S}_2 = \sum a_{(2)}^{kj} p_k p_j + W_{(2)}$$

be second order constants of the the motion and let $\mathcal{A}_{(i)}(x,y) = \{a_{(i)}^{kj}(x,y)\}$, $i = 1,2$ be 2×2 matrix functions. Then the Poisson bracket of these symmetries is given by

$$\{\mathcal{S}_1, \mathcal{S}_2\} = \sum_{k,j,i=1}^{2} a^{kji}(x,y) p_k p_j p_i + b^{\ell}(x,y) p_\ell$$

where

$$f^{k,\ell} = 2\lambda \sum_j (a_{(2)}^{kj} a_{(1)}^{j\ell} - a_{(1)}^{kj} a_{(2)}^{j\ell}).$$

Structure theory -2

- Thus $\{\mathcal{S}_1, \mathcal{S}_2\}$ is uniquely determined by the skew-symmetric matrix

$$[\mathcal{A}_{(2)}, \mathcal{A}_{(1)}] \equiv \mathcal{A}_{(2)}\mathcal{A}_{(1)} - \mathcal{A}_{(1)}\mathcal{A}_{(2)},$$

hence by the constant matrix $[\mathcal{A}_{(2)}(x_0, y_0), \mathcal{A}_{(1)}(x_0, y_0)]$ evaluated at a regular point.

2D multiseparability

COROLLARY: Let H be the Hamiltonian for a functionally linearly independent superintegrable system with nontrivial potential V and L be a second order constant of the motion with matrix function $\mathcal{A}(\mathbf{x})$. If at some regular point $\mathbf{x}_0$ the matrix $\mathcal{A}(\mathbf{x}_0)$ has 2 distinct eigenvalues, then H, L characterize an orthogonal separable coordinate system.

Note: Since a generic 2×2 symmetric matrix has distinct roots, it follows that any superintegrable nondegenerate potential is multiseparable.

3D structure and multiseparability

- $$\mathcal{H} = \frac{p_1^2 + p_2^2 + p_3^2}{\lambda(\mathbf{x})} + V(\mathbf{x}).$$
- Require Hamiltonian to admit 5 functionally independent aand functionally liniearly independent 2nd-order symmetries
- Assume V is at least a 3-parameter potential
- Then
$$\{\mathcal{S}_1, \mathcal{S}_2\} = 0 \Longleftrightarrow [\mathcal{A}_{(2)}, \mathcal{A}_{(1)}] = 0.$$
- Can use this to show that the system is multiseparable.

CHALLENGES

- Classify orthogonal separable systems on spaces that are NOT conformally flat (in particular, not of constant curvature).
- Classify nonorthogonal separable systems on constant curvature spaces.
- Develop a satisfactory theory of variable separation for spinor equations (e.g., the Dirac equation).
- Find the structure and classify spaces and potentials that are multiseparable (superintegrable systems).