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The Wave Equation and Separation of Variables on the Complex Sphere S_4

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It is shown that every orthogonal separable coordinate system for the Helmholtz equation on S_4 leads to an R-separable system for the complex wave equation. All orthogonal separable systems on S_4 are classified and each is characterized by a commuting triplet of operators from the enveloping algebra of $o(5)$. A consequence of the classification is that the most general cyclidic coordinates for the wave equation arise from ellipsoidal coordinates on S_4.

1. INTRODUCTION

In this paper we determine all orthogonal separable coordinate systems for the Helmholtz equation

$$\Delta \Psi = \lambda \Psi, \qquad \lambda \neq 0, \tag{1.1}$$

where Δ is the Laplace–Beltrami operator on the four-dimensional complex sphere

$$S_4\colon \sum_{l=1}^{5} (z^l)^2 = 1.$$

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As is well known [1], the symmetry algebra of S_4 and (1.1) is $o(5)$ with basis

$$I_{jk} = -I_{kj} = z^j\partial_{z^k} - z^k\partial_{z^j} \qquad 1 \leqslant j < k \leqslant 5 \tag{1.2}$$

and commutation relations

$$[I_{jk}, I_{\ell m}] = \delta_{k\ell} I_{jm} - \delta_{j\ell} I_{km} - \delta_{km} I_{j\ell} + \delta_{jm} I_{k\ell}. \tag{1.3}$$

Here,

$$\Delta = \sum_{1 \leqslant j < k \leqslant 5} I_{jk}^2. \tag{1.4}$$

While the separation problem for (1.1) is of some intrinsic interest, our primary motivation for studying it is its relation to the separation problem for the physically important wave equation

$$\square\Phi = 0, \qquad \square = \partial_{tt} - \partial_{xx} - \partial_{yy} - \partial_{zz}. \tag{1.5}$$

Now the real equation (1.5) and the real Laplace equation in four-dimensional Euclidean space are real forms of the complex Laplace equation

$$\Delta_4 \Phi(\mathbf{y}) = 0, \qquad \Delta_4 = \sum_{j=1}^{4} \partial_{y^j y^j} \tag{1.6}$$

and the possible real R-separable coordinates for these equations can be obtained from a knowledge of the complex R-separable coordinates for (1.6). In Ref. [2] we showed that every R-separable orthogonal coordinate system for (1.6) corresponds to a separable coordinate system for the Helmholtz equation on one of the local Riemann spaces E_4, $S_1 \times S_3$, $S_2 \times S_2$ and S_4, where E_4 is 4-dimensional complex Euclidean space and S_j is the complex j-dimensional unit sphere. To clarify this statement we review some facts concerning the symmetry algebra $o(6)$ of (1.6). A basis for this 15-dimensional Lie algebra is given by the generators

$$\begin{aligned} P_j &= \partial_{y^j}, \qquad j = 1,\ldots, 4, \\ M_{jk} &= -M_{kj} = y^j\partial_{y^k} - y^k\partial_{y^j}, \qquad 1 \leqslant j < k \leqslant 4, \\ D &= -\left(1 + \sum_{\ell=1}^{4} y^\ell \partial_{y^\ell}\right), \end{aligned} \tag{1.7}$$

$$K_j = \left((y^j)^2 - \sum_{\ell \neq j} (y^\ell)^2\right)\partial_{y^j} + 2y^j \sum_{\ell \neq j} y^\ell \partial_{y^\ell} + 2y^j.$$

Another basis for $o(6)$ is given by generators $\Gamma_{pq} = -\Gamma_{qp}$, $1 \leqslant p < q \leqslant 6$, with commutation relations

$$[\Gamma_{pq}, \Gamma_{rs}] = \delta_{qr}\Gamma_{ps} - \delta_{pr}\Gamma_{qs} - \delta_{qs}\Gamma_{pr} + \delta_{ps}\Gamma_{qr}, \tag{1.8}$$

and these two bases can be related by

$$\begin{aligned} P_j &= \Gamma_{1,j+1} - i\Gamma_{j+1,6}, & K_j &= \Gamma_{1,j+1} + i\Gamma_{j+1,6}, & 1 &\leqslant j \leqslant 4, \\ D &= -i\Gamma_{16}, & M_{jk} &= \Gamma_{j+1,k+1}, & 1 &\leqslant j < k \leqslant 3, \\ M_{4\ell} &= \Gamma_{\ell+1,5}, & & & \ell &= 1, 2, 3. \end{aligned} \tag{1.9}$$

Note that

$$\Delta_4 = \sum_{k=1}^{4} P_k^2$$

and that every separable coordinate system for the Helmholtz equation on E_4,

$$\left(\sum_{k=1}^{4} P_k^2\right)\Theta = \lambda\Theta, \qquad \lambda \neq 0 \tag{1.10}$$

determines a separable system for (1.1) (although the separated solutions are not the same). The symmetry algebra of (1.10) is the 10-dimensional Euclidean algebra $\mathscr{E}(4)$ with basis, $\{P_j, M_{k\ell},\ 1 \leqslant j \leqslant 4,\ 1 \leqslant k < \ell \leqslant 4\}$, a subalgebra of $o(6)$.

Now consider the subalgebra $o(5)$ of $o(6)$ with basis $\{\Gamma_{jk},\ 1 \leqslant j < k \leqslant 5\}$, and Casimir operator $\Delta' = \sum_{1 \leqslant j < k \leqslant 5} \Gamma_{jk}^2$. It is not difficult to show that (1.6) is equivalent to

$$\Delta'\Phi = 2\Phi. \tag{1.11}$$

As shown above, $o(5)$ can be realized as the symmetry algebra for the space S_4, (1.2), with action $\{I_{pq}\}$. This realization can be achieved by the multiplier transformation $\Gamma = MIM^{-1}$, where

$$M\Theta(\mathbf{y}) = (1 + z_5)\,\Theta(\mathbf{y}) \tag{1.12}$$

and the $\mathbf{z}$-coordinates on S_4 are related to the $\mathbf{y}$-coordinates by

$$y^j = z_j/(1 + z_5), \qquad j = 1,\dots, 4. \tag{1.13}$$

In particular, $\Gamma_{j+1,k+1} \leftrightarrow I_{jk}$, $1 \leqslant j < k \leqslant 3$, $\Gamma_{12} \leftrightarrow I_{15}$, $\Gamma_{25} \leftrightarrow I_{41}$ and the remaining operators can be generated from these by taking commutators.

Thus the solutions of (1.6) can be represented in the form $\Phi = (1 + z_5)\,\Psi(\mathbf{z})$, where

$$\Delta\Psi = 2\Psi \tag{1.14}$$

and Δ is given by (1.4). It follows that every separable coordinate system for the Helmholtz equation (1.1) yields an R-separable system for (1.6).

We see from the above and related constructions that R-separable coordinates for the flat-space Laplace equation can be obtained from separable coordinates for the Helmholtz equations on the manifolds E_4, S_2, S_3 and S_4. Moreover, in Ref. [2] the authors showed that *all* orthogonal R-separable systems for the Laplace equation arise in this manner. Indeed the following facts are easy consequences of the results of [2]:

Let M be a (local) complex four-dimensional conformally flat Riemannian manifold with metric ds^2 and let Δ_M be the Laplace–Beltrami operator on M. If $\{x^j\}$ is an R-separable orthogonal coordinate system for the Laplace equation $\Delta_M\Phi = 0$ then $ds^2 = \rho(x^j)\, d\hat{s}^2(x^j)$, where $d\hat{s}^2$ is the metric on one of the manifolds $M' = E_4$, $S_2 \times S_2$, $S_3 \times S_1$, S_4 and $\{x^j\}$ is a separable coordinate system for the Helmholtz equation $\Delta_{M'}\Psi = \lambda\Psi$. The same conclusion follows if $\{x^j\}$ is a separable coordinate system for the Helmholtz equation $\Delta_M\Phi = \lambda\Phi$.

Thus the study of variable separation for the Laplace and Helmholtz equations on any conformally flat manifold inevitably leads to the problem of classifying the separable systems for the Helmholtz equations on E_4, S_2, S_3 and S_4.

Detailed classifications of orthogonal separable coordinate systems for the Riemannian manifolds E_4, S_2, S_3 have been given by the authors in Refs. [4, 5], together with characterizations of these systems in terms of the symmetry groups of the corresponding manifolds. Here we present a similar classification of the orthogonal separable systems for S_4. The results are of special interest for the Laplace equation since the most general cyclidic systems for this equation originate as ellipsoidal systems on S_4.

In Section 2 we shall adapt a method due to Eisenhart [6] and classify the metrics of all orthogonal separable coordinates $\{x^j\}$ on S_4. Although these results are new they are straightforward to obtain. In Section 3 we shall explicitly relate the separable coordinates $\{x^j\}$ to the "standard" coordinates $\{z^k\}$ on S_4 and determine the corresponding separated solutions

$$\Psi = \prod_{j=1}^{4} E_j(x^j) \tag{1.15}$$

of (1.1). Given the separable metric $ds^2(x^j)$ for S_4 it is easy to see that there must exist a coordinate transformation relating $\{x^j\}$ to the standard coordinates and that this transformation is unique up to an action of the symmetry group $O(5)$ on S_4. However, the problem of explicitly constructing the transformation is often formidable. This portion of our paper required a major computational effort drawing on geometrical techniques related to the projective representation of cyclides [12] and on the experience and intuition of the authors. Fortunately, now that these coordinate transformations have been constructed, their validity is straightforward to check.

In Section 3 we also compute a triplet $\{\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3\}$ of commuting second-order symmetric operators in the enveloping algebra of $o(5)$ for each separable system, such that the separated solutions are characterized as eigenfunctions

$$\mathscr{L}_k \Psi = \ell_k \Psi, \qquad k = 1, 2, 3. \tag{1.16}$$

Here ℓ_1, ℓ_2, ℓ_3 are the separation constants. The significance of Eqs. (1.16) is that they provide a direct relationship between separable systems on S_4 and the representation theory of $o(5)$. This relationship permits one to use simple models of $o(5)$ representations to derive addition theorems and expansion formulas relating the various separated solutions of (1.1). Examples of such derivations are given in Refs. [5, 13, 16].

Computation of the operators $\{\mathscr{L}_k\}$ is straightforward but extremely tedious, as the reader can verify by trying to check any but the simplest examples. However, we believe these explicit relations between the enveloping algebra of $o(5)$ and separable coordinates on S_4 to be among the most important contributions of this paper. In particular, many of these operators turn out to be Casimir operators for subalgebras of $o(5)$, thus clearly pointing out the nature of the corresponding coordinates.

2. Classification of the Metrics

Here we use techniques developed by Eisenhart to find all orthogonal separable coordinate systems for Eq. (1.1). It follows from Ref. [6] that necessary and sufficient conditions for a set of orthogonal coordinates $\{x^1, x^2, x^3, x^4\}$ on S_4 to afford a separation of variables for (1.1) are that

(1) the coefficients of the metric

$$ds^2 = \sum_{i=1}^{5} (dz^i)^2 = \sum_{i=1}^{4} H_i^2 (dx^i)^2 \tag{2.1}$$

are in Stäckel form with respect to $\{x^j\}$, i.e.,

$$\frac{\partial^2}{\partial x^j\,\partial x^k}\ln H_i^2-\frac{\partial}{\partial x^j}\ln H_i^2\,\frac{\partial}{\partial x^k}\ln H_i^2 \tag{2.2}$$

$$+\frac{\partial}{\partial x^j}\ln H_i^2\,\frac{\partial}{\partial x^k}\ln H_j^2+\frac{\partial}{\partial x^k}\ln H_i^2\,\frac{\partial}{\partial x^j}\ln H_k^2=0, \qquad j\neq k.$$

and

(2) The Robertson condition is satisfied, i.e.,

$$R_{ij}=0, \qquad i\neq j, \tag{2.3}$$

where R_{ij} is the Ricci tensor. Now the manifold S_4 is characterized by the property that in any orthogonal coordinate system $\{x^j\}$ the Riemann curvature tensor takes the form [8]

$$R_{ijk\ell}=H_i^2H_j^2(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk}). \tag{2.4}$$

It follows from (2.4) that condition (2.3) is automatically satisfied. Furthermore, from (2.4) and the general expression for the curvature tensor in terms of the metric (Ref. [8, p. 44]), we find the condition

$$R_{jiik}=\frac{3}{4}H_i^2\frac{\partial^2}{\partial x^j\,\partial x^k}\ln H_i^2=0 \qquad (i,j,k\neq). \tag{2.5}$$

In Ref. [9] the authors computed all possible metric forms on four-dimensional Riemannian manifolds which satisfy conditions (2.2), (2.3) and (2.5). The possibilities, which were also listed in Refs. [2, 4], are of eight types.

To determine the separable systems for (1.1) we need only subject these metric types to the conditions (2.4). Due to (2.5) the only nontrivial conditions are

$$R_{ijji}=-H_i^2H_j^2, \qquad i\neq j$$

or

$$\frac{1}{H_j^2}\left[2\left(\frac{\partial}{\partial x^j}\right)^2\ln H_i^2+\frac{\partial}{\partial x^j}\ln\frac{H_i^2}{H_j^2}\right]$$

$$+\frac{1}{H_i^2}\left[2\left(\frac{\partial}{\partial x^i}\right)^2\ln H_j^2+\frac{\partial}{\partial x^i}\ln H_j^2\,\frac{\partial}{\partial x^i}\ln\frac{H_j^2}{H_i^2}\right] \tag{2.6}$$

$$+\frac{1}{H_k^2}\frac{\partial}{\partial x^k}\ln H_i^2\,\frac{\partial}{\partial x^k}\ln H_j^2+\frac{1}{H_\ell^2}\frac{\partial}{\partial x^\ell}\ln H_i^2\,\frac{\partial}{\partial x^\ell}\ln H_j^2=-4$$

with i, j, k, ℓ all distinct.

We omit the straightforward solution of Eqs. (2.6) for each metric type and merely list the results in terms of classes of metrics with similar properties.

Type [1]

There are no forms of type [1] because Eqs. (2.6) imply $R_{1331}=0$.

Type [2], *Class* I

$$ds^2 = \frac{(x^2-x^1)}{4}\left[\frac{(dx^1)^2}{(x^1-a)(x^1-1)\,x^1} - \frac{(dx^2)^2}{(x^2-a)(x^2-1)\,x^2}\right] + \frac{x^1x^2}{a}\,d\omega^2 \qquad a \neq 0, 1. \tag{2.7}$$

Here $d\omega^2$ is one of the five orthogonal separable metric forms on the complex two-sphere [5]:

$$S_2\colon w_1^2+w_2^2+w_3^2=1, \qquad d\omega^2 = dw_1^2+dw_2^2+dw_3^2. \tag{2.8}$$

Type [2], *Class* II

$$ds^2 = \frac{(x^2-x^1)}{4}\left[\frac{(dx^1)^2}{(x^1-1)(x^1)^2} - \frac{(dx^2)^2}{(x^2-1)(x^2)^2}\right] + (x^1-1)(x^2-1)\,d\omega^2. \tag{2.9}$$

Again $d\omega^2$ is one of the five orthogonal separable forms on S_2.

Type [2], *Class* III

$$ds^2 = \frac{(x^2-x^1)}{4}\left[\frac{(dx^1)^2}{(x^1)^2(x^1-1)} - \frac{(dx^2)^2}{(x^2)^2(x^2-1)}\right] + x^1x^2\,d\eta^2. \tag{2.10}$$

Here $d\eta^2$ is one of the six orthogonal separable metric forms on complex two-dimensional Euclidean space [5]:

$$E_2\colon (y_1, y_2), \qquad d\eta^2 = dy_1^2+dy_2^2. \tag{2.11}$$

Type [2], *Class* IV

$$ds^2 = \frac{(x^2-x^1)}{4}\left[\frac{(dx^1)^2}{(x^1)^3} - \frac{(dx^2)^2}{(x^2)^3}\right] + x^1x^2\,d\eta^2. \tag{2.12}$$

$d\eta^2$ is the same as for Class III.

TABLE I

The Roots a, b, c of f

Case	a	b	c	
1	a	b	1	$a, b, 1 \neq 0$, distinct
2	a	a	1	$a \neq 1, 0$
3	1	1	1	
4	a	1	0	$a \neq 1, 0$
5	1	1	0	
6	1	0	0	
7	0	0	0	

Type [3], *Class* V

$$ds^2 = x^2x^3x^4(dx^1)^2 - d\bar{s}^2,$$

$$d\bar{s}^2 = \frac{(x^2-x^3)(x^2-x^4)}{4f(x^2)}(dx^2)^2 + \frac{(x^3-x^2)(x^3-x^4)}{4f(x^3)}(dx^3)^2 + \frac{(x^4-x^2)(x^4-x^3)}{4f(x^4)}(dx^4)^2, \tag{2.13}$$

$$f(z) = (z-a)(z-b)(z-c)z.$$

Here $d\bar{s}^2$ is one of the seven metrics for completely elliptic coordinates on the sphere S_3 [5]. The possibilities for the roots of f are given in Table I.

Type [4], *Class* VI

$$ds^2 = \frac{(x^2-x^1)}{4}\left[\frac{(dx^1)^2}{(x^1-a)(x^1-1)\,x^1} - \frac{(dx^2)^2}{(x^2-a)(x^2-1)\,x^2}\right] + x^1x^2(dx^3)^2 + (x^1-1)(x^2-1)(dx^4)^2. \tag{2.14}$$

There are two separable systems here: (1) $a \neq 0, 1$ and (2) $a = 0$.

Type [5], *Class* VII

$$ds^2 = -\sum_{i=1}^{4} \frac{(x^i-x^j)(x^i-x^k)(x^i-x^\ell)(dx^i)^2}{4g(x^i)} \qquad (i, j, k, \ell \neq), \tag{2.15}$$

$$g(z) = (z-a)(z-b)(z-c)(z-d)z.$$

These are the most general ellipsoidal systems for S_4. In Table II we distinguish seven cases in terms of the roots of g.

TABLE II

The Roots a, b, c, d of g

Case	a	b	c	d	
1	a	b	c	1	$a, b, c \neq 1, 0$, distinct
2	a	b	1	0	$a, b \neq 1, 0$, distinct
3	a	1	0	0	$a \neq 1, 0$
4	1	0	0	0	
5	0	0	0	0	
6	a	1	1	0	$a \neq 1, 0$
7	1	1	1	0	

Type [6], *Class* VIII

$$ds^2 = (dx^1)^2 + (\sin x^1)^2\, d\Omega^2. \tag{2.16}$$

Here $d\Omega^2$ is one of the twenty-one orthogonal separable metrics on the sphere S_3 [5].

Type [6], *Class* IX

$$ds^2 = (dx^1)^2 + e^{2ix^1}\, d\Phi^2. \tag{2.17}$$

Here $d\Phi^2$ is any orthogonal separable metric in Euclidean three-space E_3.

Type [7], *Class* X

$$ds^2 = (dx^1)^2 + \cos^2 x^1 (dx^2)^2 + \sin^2 x^1\, d\omega^2. \tag{2.18}$$

The metric $d\omega^2$ is defined as in Class I.

Type [8]

There are no systems of type [8] which are not already included as special cases of other types.

3. Orthogonal Separable Coordinate Systems on S_4

Here we present the coordinate systems corresponding to the separable metrics of the preceding section, i.e., we express these systems in terms of the standard coordinates $\{z^j\}$ on S_4: $\sum_{j=1}^{5}(z^j)^2 = 1$. (Each metric determines an equivalence class of such coordinates under the action of the group $O(5)$. We simply choose a representative from each equivalence class.) In addition we give the triplet of commuting second-order symmetric operators $\{\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3\}$ which characterize the separation, and we give the separation equations.

Class I

The general coordinate system of this type is

$$z^j = (x^1x^2/a)^{1/2}\, w_j, \qquad j=1,2,3,$$
$$z^4 = [(x^1-1)(x^2-1)/(1-a)]^{1/2}, \tag{3.1}$$
$$z^5 = [(x^1-a)(x^2-a)/a(a-1)]^{1/2},$$

where $\sum_{j=1}^{3} w_j^2 = 1$ and the $\{w_j\}$ correspond to one of the five separable systems on S_2. The defining operators, whose eigenvalues are the separation constants, are

$$\mathscr{L}_1 = a(I_{14}^2 + I_{24}^2 + I_{34}^2) + I_{15}^2 + I_{25}^2 + I_{35}^2$$
$$+(a+1)(I_{12}^2 + I_{13}^2 + I_{23}^2), \tag{3.2}$$
$$\mathscr{L}_2 = I_{12}^2 + I_{13}^2 + I_{23}^2,\ \mathscr{L}_3.$$

Here $\mathscr{L}_2$ is the Casimir operator for the $o(3)$ subalgebra of $o(5)$ with basis $\{I_{12}, I_{13}, I_{23}\}$ and $\mathscr{L}_3$ is the second-order symmetric operator in the enveloping algebra of $o(3)$ which characterizes the corresponding separable system on S_2 [5].

The separation equations for the separated solution $\Psi = \prod_{i=1}^{4} E_i(x^i)$ of (1.1) are

$$\frac{4}{x^i}\,[(x^i-a)(x^i-1)\,x^i]^{1/2}\,\frac{d}{dx^i}\left(x^i[(x^i-a)(x^i-1)\,x^i]^{1/2}\,\frac{dE_i}{dx^i}\right)$$
$$+\left(-\frac{a\ell_2}{x^i} + \lambda x^i + \ell_1\right)E_i = 0, \qquad i=1,2, \tag{3.3}$$
$$(I_{12}^2 + I_{13}^2 + I_{23}^2)\,E_3E_4 = \ell_2 E_3E_4.$$

Class II

The general system of this type is

$$z^j = [(x^1-1)(x^2-1)]^{1/2}\, w_j, \qquad j=1,2,3,$$
$$z^4 + iz^5 = -[x^1x^2]^{1/2}, \tag{3.4}$$
$$z^4 - iz^5 = -[x^2/x^1]^{1/2} - [x^1/x^2]^{1/2} + [x^1x^2]^{1/2},$$

where, as for Class I, the $\{w_j\}$ correspond to one of the separable systems on S_2. The defining operators are

$$\mathscr{L}_1 = (I_{41} + iI_{51})^2 + (I_{42} + iI_{52})^2 + (I_{43} + iI_{53})^2$$
$$+ I_{45}^2 - (I_{12}^2 + I_{13}^2 + I_{23}^2), \tag{3.5}$$
$$\mathscr{L}_2 = I_{12}^2 + I_{13}^2 + I_{23}^2,\ \mathscr{L}_3.$$

Again $\mathscr{L}_2$ is the Casimir operator for the $o(3)$ subalgebra with basis $\{I_{12}, I_{13}, I_{23}\}$ and $\mathscr{L}_3$ is the second-order operator in the enveloping algebra of $o(3)$ which characterizes the separable system on S_2.

The separation equations for $\Psi = \prod_{i=1}^{4} E_i(x^i)$ are

$$\frac{4x^i}{(x^i-1)^{1/2}} \frac{d}{dx^i}\left[x^i(x^i-1)^{3/2}\frac{dE_i}{dx^i}\right] + \left(-\frac{\ell_2}{x^i-1} + \lambda x^i + \ell_1\right)E_1 = 0, \qquad i = 1, 2, \tag{3.6}$$

$$\mathscr{L}_2 E_3 E_4 = \ell_2 E_3 E_4.$$

Class III

The general coordinate system is

$$\begin{aligned} z^1 + iz^2 &= -(x^1x^2)^{1/2}, \\ z^1 - iz^2 &= -(x^1/x^2)^{1/2} - (x^2/x^1)^{1/2} + (x^1x^2)^{1/2}(1 + y_1^2 + y_2^2), \\ z^3 &= (x^1x^2)^{1/2} y_1, \\ z^4 &= (x^1x^2)^{1/2} y_2, \\ z^5 &= [(x^1-1)(x^2-1)]^{1/2}, \end{aligned} \tag{3.7}$$

where $d\eta^2 = dy_1^2 + dy_2^2$ and the $\{y_j(x^3, x^4)\}$ correspond to one of the six separable systems on E_2. The defining operators are

$$\begin{aligned} \mathscr{L}_1 &= (I_{15} + iI_{25})^2 + I_{13}^2 + I_{23}^2 + I_{34}^2 + I_{12}^2 + I_{14}^2 \\ &\quad + I_{24}^2 + (I_{13} + iI_{23})^2 + (I_{14} + iI_{24})^2, \\ \mathscr{L}_2 &= (I_{13} + iI_{23})^2 + (I_{14} + iI_{24})^2, \mathscr{L}_3. \end{aligned} \tag{3.8}$$

Here $\mathscr{L}_2$ is the Casimir operator for the $\mathscr{E}(2)$ subalgebra of $o(5)$ with basis $\{I_{13} + iI_{23}, I_{14} + iI_{24}, I_{34}\}$ and $\mathscr{L}_3$ is the second-order operator in the enveloping algebra of $\mathscr{E}(2)$ which characterizes the corresponding separable system on E_2 [5].

The separation equations for $\Psi = \prod_{i=1}^{4} E_i$ are

$$4(x^i-1)\frac{d}{dx^i}\left[(x^i)^2(x^i-1)^{1/2}\frac{dE_i}{dx^i}\right] + \left(\frac{\ell_2}{x^i} - \lambda x^i + \ell_1\right)E_i = 0, \qquad i = 1, 2, \tag{3.9}$$

$$\mathscr{L}_2 E_3 E_4 = \ell_2 E_3 E_4.$$

Class IV

In this case

$$\begin{aligned} z^1 + iz^2 &= (x^1x^2)^{1/2}, \\ z^1 - iz^2 &= -(x^1 - x^2)^2/4(x^1x^2)^{3/2} - (x^1x^2)^{1/2}[y_1^2 + y_2^2], \\ z^3 &= (x^1x^2)^{1/2}\, y_1, \; z^4 = (x^1x^2)^{1/2}\, y_2, \\ z^5 &= \tfrac{1}{2}((x^1/x^2)^{1/2} + (x^2/x^1)^{1/2}), \end{aligned} \tag{3.10}$$

where as for Class III the $\{y_j(x^3, x^4)\}$ correspond to one of the six separable systems on E_2.

The defining operators are

$$\begin{aligned} \mathscr{L}_1 &= i\{I_{12}, I_{15} + iI_{25}\} - \{I_{35}, I_{13} + I_{14} + i(I_{23} + I_{24})\}, \\ \mathscr{L}_2 &= (I_{13} + iI_{23})^2 + (I_{14} + iI_{24})^2, \mathscr{L}_3, \end{aligned} \tag{3.11}$$

where

$$\{A, B\} = AB + BA,$$

$\mathscr{L}_2$ is the Casimir operator for the $\mathscr{E}(2)$ subalgebra with basis $\{I_{13} + iI_{23}, I_{14} + iI_{24}, I_{34}\}$ and $\mathscr{L}_3$ is the $\mathscr{E}(2)$ symmetry operator which characterizes the separable coordinates on E_2. The separation equations for $\Psi = \prod_{i=1}^4 E_i$ are

$$\begin{aligned} &4(x^i)^{1/2} \frac{d}{dx^i}\left((x^i)^{3/2} \frac{dE_i}{dx^i}\right) \\ &\qquad + \left(\frac{\ell_2}{x^i} - \lambda x^i + \ell_1\right) E_i = 0, \qquad i = 1, 2, \\ &\qquad \mathscr{L}_2 E_3 E_4 = \ell_2 E_3 E_4. \end{aligned} \tag{3.12}$$

Class V

These systems correspond to the metric (2.13), where $d\bar{s}^2$ is the metric for ellipsoidal coordinates on S_3. In order to more readily compare our results with Ref. [5] we make the change of variable $z^j \to z^j - \alpha, j = 2, 3, 4$, where α is a fixed constant. For all cases the separation equations for $\Psi = \prod_{i=1}^4 E_i$ take the form

$$\begin{aligned} &-4[(x^j - \alpha)^{-1} f(x^j)]^{1/2} \frac{d}{dx^j}\left\{[(x^j - \alpha) f(x^j)]^{1/2} \frac{dE_j}{dx^j}\right\} \\ &\qquad + [-\lambda(x^j)^3 + \ell_1(x^j)^2 + \ell_2 x^j + \ell_3]\, E_j = 0, \qquad j = 2, 3, 4, \\ &\qquad \left(\frac{d}{dx^1}\right)^2 E_1 = \ell_3' E_1, \end{aligned} \tag{3.13}$$

where

$$-\lambda(x^j-\alpha)^3+\ell'_1(x^j-\alpha)^2+\ell'_2(x^j-\alpha)+\ell'_3$$
$$\equiv -\lambda(x^j)^3+\ell_1(x^j)^2+\ell_2 x^j+\ell_3.$$

We label the seven possible systems by the roots of $f(z)$ and convenient choices of α. To save space and since all the remaining systems are limiting cases of (i), we list the defining operators only for system (i).

(i) $\alpha=0,\ f(z)=(z-a)(z-b)(z-1)z$,

$$(z^1)^2=\frac{x^1x^2x^3}{ab}(\cos x^1)^2,\ (z^2)^2=\frac{x^1x^2x^3}{ab}(\sin x^1)^2,$$

$$(z^3)^2=-\frac{(x^1-1)(x^2-1)(x^3-1)}{(a-1)(b-1)},$$

$$(z^4)^2=-\frac{(x^1-a)(x^2-a)(x^3-a)}{(a-b)(a-1)a},$$

$$(z^5)^2=-\frac{(x^1-b)(x^2-b)(x^3-b)}{(b-a)(b-1)b}.$$

The defining operators are

$$\mathscr{L}_1=(a+b)(I_{31}^2+I_{32}^2)+(b+1)(I_{41}^2+I_{42}^2)$$
$$+(a+1)(I_{51}^2+I_{52}^2)+aI_{35}^2+bI_{34}^2+I_{45}^2$$
$$+(a+b+1)\,I_{12}^2,$$
$$\mathscr{L}_2=ab(I_{31}^2+I_{32}^2)+b(I_{41}^2+I_{42}^2)+a(I_{51}^2+I_{52}^2)$$
$$+(ab+a+b)\,I_{12}^2,$$
$$\mathscr{L}_3=I_{12}^2.$$

(ii) $\alpha=1,\ f(z)=(z-a)(z-1)\,z^2$,

$$(z^1)^2=\frac{(x^2-1)(x^3-1)(x^4-1)}{a-1}(\cos x^1)^2,$$

$$(z^2)^2=\frac{(x^2-1)(x^3-1)(x^4-1)}{a-1}(\sin x^1)^2,$$

$$(z^3)^2=-\frac{(x^2-a)(x^3-a)(x^4-a)}{a^2(a-1)},$$

$$(z^4+iz^5)=-x^2x^3x^4/a,$$

$$(z^4)^2+(z^5)^2=\frac{1}{a^2}\left[-(a+1)\,x^2x^3x^4+a(x^2x^3+x^2x^4+x^3x^4)\right].$$

(iii) $\alpha = 1, f(z) = (z-1)z^3$,

$$\begin{pmatrix}(z^1)^2\\(z^2)^2\end{pmatrix} = -(x^2-1)(x^3-1)(x^4-1)\begin{pmatrix}(\cos x^1)^2\\(\sin x^1)^2\end{pmatrix},$$

$$(z^3 - iz^4)^2 = x^2x^3x^4,$$

$$2z^5(z^3 - iz^4) = -(x^2x^3 + x^2x^4 + x^3x^4) + x^2x^3x^4,$$

$$(z^3)^2 + (z^4)^2 + (x^5)^2 = x^2x^3x^4 - (x^2x^3 + x^2x^4 + x^3x^4) + x^2 + x^3 + x^4.$$

(iv) $\alpha = 0, f(z) = (z-a)(z-1)z^2$,

$$(z^1 + iz^2)^2 = -x^2x^3x^4/a, \qquad (z^3)^2 = -x^2x^3x^4(x^1)^2/a,$$

$$(z^1)^2 + (z^2)^2 = a^{-2}[-(a+1)x^2x^3x^4 + a(x^2x^4 + x^3x^4 + x^2x^3)] + x^2x^3x^4(x^1)^2/a,$$

$$(z^4)^2 = \frac{(x^2-1)(x^3-1)(x^4-1)}{a-1},$$

$$(z^5)^2 = \frac{(x^2-a)(x^3-a)(x^4-a)}{a^2(1-a)}.$$

(v) $\alpha = 0, f(z) = (z-1)^2 z^2$,

$$(z^1 + iz^2)^2 = -x^2x^3x^4, \qquad (z^3)^2 = -x^2x^3x^4(x^1)^2,$$

$$(z^1)^2 + (z^2)^2 = -2x^2x^3x^4 + x^2x^4 + x^3x^4 + x^2x^3 + x^2x^3x^4(x^1)^2,$$

$$(z^4 + iz^5)^2 = -(x^2-1)(x^3-1)(x^4-1),$$

$$(z^4)^2 + (z^5)^2 = 2x^2x^3x^4 - (x^2x^4 + x^3x^4 + x^2x^3) + 1.$$

(vi) $\alpha = 0, f(z) = (z-1)z^3$,

$$(z^1 + iz^2)^2 = x^2x^3x^4, \qquad (z^5)^2 = (x^1)^2 x^2x^3x^4,$$

$$2z^3(z^1 + iz^2) = -(x^2x^3 + x^2x^4 + x^3x^4) + x^2x^3x^4,$$

$$(z^1)^2 + (z^2)^2 + (z^3)^2 = x^2 + x^3 + x^4 - (x^2x^3 + x^2x^4 + x^3x^4) + (1-(x^1)^2)x^2x^3x^4,$$

$$(z^4)^2 = -(x^2-1)(x^3-1)(x^4-1).$$

(vii) $\alpha = 0, f(z) = z^4$,

$$(z^1 + iz^2)^2 = -x^2x^3x^4, \qquad (z^5)^2 = x^2x^3x^4(x^1)^2,$$

$$2(z^1 + iz^2)(z^3 + iz^4) = x^2x^3 + x^2x^4 + x^3x^4,$$

$$(z^1 + iz^2)(z^3 - iz^4) + (z^3 + iz^4)^2 = x^2 + x^3 + x^4,$$

$$(z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = 1 - (x^1)^2 x^2x^3x^4.$$

Class VI

For the metric (2.14) with $a \neq 0, 1$ we have coordinates

$$\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = [x^1 x^2/a]^{1/2} \begin{pmatrix} \cos x^3 \\ \sin x^3 \end{pmatrix},$$
$$\begin{pmatrix} z^3 \\ z^4 \end{pmatrix} = [(x^1-1)(x^2-1)/(1-a)]^{1/2} \begin{pmatrix} \cos x^4 \\ \sin x^4 \end{pmatrix}, \tag{3.14}$$
$$z^5 = [(x^1-a)(x^2-a)/a(a-1)]^{1/2}.$$

The separation equations for $\Psi = \prod_{i=1}^4 E_i$ take the form

$$4(x^j-a)^{1/2} \frac{d}{dx^j} \left[x^j (x^j - 1)(x^j - a)^{1/2} \frac{dE_j}{dx^j} \right]$$
$$+ (\lambda (x^j)^3 + \ell_1 (x^j)^2 + \ell_2 x^j + \ell_3) E_j = 0, \qquad j = 1, 2, \tag{3.15}$$
$$a \left(\frac{d}{dx^3} \right)^2 E_3 = \ell_3 E_3, \qquad (a-1) \left(\frac{d}{dx^4} \right)^2 E_4 = (\ell_1 - \ell_2 + \ell_3 + \lambda) E_4$$

and the defining operators are

$$\mathcal{L}_2 = a(I_{14}^2 + I_{13}^2 + I_{24}^2 + I_{23}^2) + I_{15}^2 + I_{25}^2 + (2a+1) I_{12}^2,$$
$$\mathcal{L}_3 = a I_{12}^2, \qquad \mathcal{L}_1 = \mathcal{L}_2 - \mathcal{L}_3 + \Delta_4 + (a-1) I_{34}^2. \tag{3.16}$$

For the metric (2.14) with $a = 0$ we have

$$(z^1 + iz^2)^2 = -x^1 x^2, \qquad z^3 = (x^1 x^2)^{1/2} x^3,$$
$$(z^1)^2 + (z^2)^2 = x^1 + x^2 - x^1 x^2 (1 + (x^3)^2), \tag{3.17}$$
$$\begin{pmatrix} z^4 \\ z^5 \end{pmatrix} = [(x^1-1)(x^2-1)]^{1/2} \begin{pmatrix} \cos x^4 \\ \sin x^4 \end{pmatrix}.$$

The separation equations for $j = 1, 2$ are

$$4(x^j)^{1/2} \frac{d}{dx^j} \left[(x^j)^{3/2} (x^j - 1) \frac{dE_j}{dx^j} \right]$$
$$+ [\lambda (x^j)^3 + \ell_1 (x^j)^2 + \ell_2 x^j + \ell_3] E_j = 0 \tag{3.18}$$

whereas the equations for $j = 3, 4$ agree with (3.15). The defining operators are

$$\mathcal{L}_1 = -(I_{14} + iI_{24})^2 - 2(I_{13} + iI_{23})^2 - (I_{15} + iI_{25})^2 + I_{13}^2 + I_{23}^2,$$
$$\mathcal{L}_2 = -(I_{13} + iI_{23})^2, \tag{3.19}$$
$$\mathcal{L}_3 = \mathcal{L}_2 - \mathcal{L}_1 + \Delta_4 - I_{45}^2.$$

Class VII

The separation equations for $\Psi = \prod_{i=1}^{4} E_i$ assume the form

$$(g(x^j))^{1/2} \frac{d}{dx^j}\left[(g(x^j))^{1/2} \frac{dE_j}{dx^j}\right] + [\lambda(x^j)^3 + \ell_1(x^j)^2 + \ell_2 x^j + \ell_3]\, E_j = 0, \qquad j = 1,..., 4, \tag{3.20}$$

$$g(z) = (z-a)(z-b)(z-c)(z-d)z.$$

The coordinates and operators take various forms depending on the roots of g. To more easily describe the coordinates we define

$$S_{a,n} = \sum_{\alpha_1,...,\alpha_n} \frac{1}{n!}(x^{\alpha_1} - a) \cdots (x^{\alpha_n} - a),$$

where $1 \leqslant n \leqslant 4$ and the summation extends over all choices of $\alpha_1,..., \alpha_n$ from the set $\{1, 2, 3, 4\}$ with no repetitions, e.g.,

$$S_{a,4} = (x^1 - a)(x^2 - a)(x^3 - a)(x^4 - a).$$

We have computed the defining operators for the coordinate systems listed below but because of the length of these expressions we list the results only for system (i).

(i) $g(z) = (z-a)(z-b)(z-c)(z-1)z,$

$$(z^1)^2 = S_{1,4}/[(1-a)(1-b)(1-c)],$$

$$(z^2)^2 = S_{a,4}/[(a-b)(a-c)(a-1)a],$$

$$(z^3)^2 = S_{b,4}/[(b-a)(b-c)(b-1)b],$$

$$(z^4)^2 = S_{c,4}/[(c-a)(c-b)(c-1)c],$$

$$(z^5)^2 = S_{0,4}/[abc],$$

$$\begin{aligned}\mathscr{L}_1 = {} & (a+b+c)\, I_{15}^2 + (a+b)\, I_{14}^2 + (a+c)\, I_{13}^2 + (b+c)\, I_{12}^2 \\ & + (a+1)\, I_{34}^2 + (b+1)\, I_{24}^2 + (c+1)\, I_{23}^2 + (a+c-1)\, I_{35}^2 \\ & + (b+c+1)\, I_{25}^2 + (a+b+1)\, I_{45}^2,\end{aligned}$$

$$\begin{aligned}\mathscr{L}_2 = {} & (ab+ac+bc)\, I_{15}^2 + (ab+a+b)\, I_{45}^2 + (ac+a+c)\, I_{35}^2 \\ & + (bc+b+c)\, I_{25}^2 + abI_{14}^2 + acI_{13}^2 + bcI_{12}^2 + aI_{34}^2 \\ & + bI_{24}^2 + cI_{23}^2,\end{aligned}$$

$$\mathscr{L}_3 = abcI_{15}^2 + bcI_{25}^2 + acI_{35}^2 + abI_{45}^2.$$

(ii) $g(z) = (z-a)(z-b)(z-1)\,z^2$,

$$(z^1 + iz^2)^2 = -S_{0,4}/ab,$$

$$(z^1)^2 + (z^2)^2 = (S_{0,3}/ab) - [(a+b+ab)\,S_{0,4}/(ab)^2],$$

$$(z^3)^2 = S_{1,4}/[(1-a)(1-b)],$$

$$(z^4)^2 = S_{a,4}/[(a-b)(a-1)\,a^2],$$

$$(z^5)^2 = S_{b,4}/[(b-a)(b-1)\,b^2].$$

(iii) $g(z) = (z-a)(z-1)\,z^3$,

$$(z^1 + iz^2)^2 = S_{0,4}/a, \qquad (z^4)^2 = S_{1,4}/(1-a),$$

$$2z^3(z^1 + iz^2) = -S_{0,3}/a + (a+1)\,S_{0,4}/a^2,$$

$$(z^1)^2 + (z^2)^2 + (z^3)^2 = S_{0,2}/a - (a+1)\,S_{0,3}/a^2 + (a^2+a+1)\,S_{0,4}/a^3,$$

$$(z^5)^2 = S_{a,4}/[a^3(a-1)].$$

(iv) $g(z) = (z-1)\,z^4$,

$$(z^1 + iz^2)^2 = -S_{0,4}, \qquad (z^5)^2 = S_{1,4},$$

$$2(z^1 + iz^2)(z^3 + iz^4) = S_{0,3} - S_{0,4},$$

$$(z^1 + iz^2)(z^3 - iz^4) + (z^3 + iz^4)^2 = -S_{0,2} + S_{0,3} - S_{0,4},$$

$$(z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = S_{0,1} - S_{0,2} + S_{0,3} - S_{0,4}.$$

(v) $g(z) = z^5$,

$$(z^1 + iz^2)^2 = S_{0,4},$$

$$2(z^1 + iz^2)(z^3 + iz^4) = -S_{0,3},$$

$$2z^5(z^1 + iz^2) + (z^3 + iz^4)^2 = S_{0,2}, \qquad \sum_{j=1}^{5} (z^j)^2 = 1,$$

$$(z^1 + iz^2)(z^3 - iz^4) + 2z^5(z^3 + iz^4) = -S_{0,1}.$$

(vi) $g(z) = (z-a)(z-1)^2\,z^2$,

$$(z^1 + iz^2)^2 = -S_{0,4}/a, \qquad (z^3 + iz^4)^2 = -S_{1,4}/(a-1),$$

$$(z^1)^2 + (z^2)^2 = [S_{0,3} - 2S_{0,4}]/a - S_{0,4}/a^2,$$

$$(z^3)^2 + (z^4)^2 = [S_{1,3} + 2S_{1,4}]/(a-1) - S_{1,4}/(a-1)^2,$$

$$(z^5)^2 = S_{a,4}/[a(a-1)]^2.$$

(vii) $g(z) = (z-1)^2 z^3$,

$$(z^1 - iz^2)^2 = S_{0,4}, \qquad 2z^3(z^1 - iz^2) = -S_{0,3} + 2S_{0,4},$$
$$(z^4 + iz^5)^2 = S_{1,4}, \qquad (z^4)^2 + (z^5)^2 = -S_{1,3} - 3S_{1,4},$$
$$(z^1)^2 + (z^2)^2 + (z^3)^2 = S_{0,2} - 2S_{0,3} + 3S_{0,4}.$$

Class VIII

The general coordinate system of this type is

$$z^j = \xi_j \sin x^1, \qquad j = 1,\ldots, 4,\ z^5 = \cos x^1, \tag{3.21}$$

where the $\{\xi_j(x^2, x^3, x^4)\}$ are orthogonal separable coordinates on the sphere S_3: $\sum_{j=1}^4 (\xi^j)^2 = 1$. These coordinates correspond to the Lie algebra reduction $o(5) \supset o(4)$, where a basis for $o(4)$ is $\{I_{k\ell},\ 1 \leqslant k < \ell \leqslant 4\}$. The twenty-one possible systems and their corresponding operators are listed in Ref. [5].

Class IX

The general system of this type is

$$\begin{pmatrix} z^1 \\ z^5 \end{pmatrix} = \frac{1}{2}\left\{ e^{-ix^1} + \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + y_2^2 + y_3^2 + y_4^2 \right] e^{ix^1} \right\}, \qquad z^j = iy_j e^{ix^1}, \qquad j = 2, 3, 4, \tag{3.22}$$

where the $\{y_j(x^2, x^3, x^4)\}$ are orthogonal separable coordinates on Euclidean three-space E_3: $d\Phi^2 = dy_2^2 + dy_3^2 + dy_4^2$; see Ref. [10]. These coordinates correspond to the reduction $o(5) \supset \mathscr{E}(3)$ where a basis for $\mathscr{E}(3)$ is given by $\{I_{23}, I_{24}, I_{34}, I_{12} - iI_{52}, I_{13} - iI_{53}, I_{14} - iI_{54}\}$.

Class X

The general coordinates have the form

$$z^j = w_j \sin x^1, \qquad j = 1, 2, 3, \qquad z^4 = \cos x^1 \cos x^2, \qquad z^5 = \cos x^1 \sin x^2, \tag{3.23}$$

where $\{w_j\}$ corresponds to a separable system on S_2, as for class I. The defining operators are

$$\mathscr{L}_1 = I_{12}^2 + I_{13}^2 + I_{23}^2, \qquad \mathscr{L}_2 = I_{45}^2, \qquad \mathscr{L}_3, \tag{3.24}$$

where $\mathscr{L}_1$ is the Casimir operator for the $o(3)$ subalgebra with basis

$\{I_{12}, I_{13}, I_{23}\}$ and $\mathscr{L}_3$ is a second-order element in the enveloping algebra of $o(3)$ [5]. The separation equations are

$$[\cos x^1(\sin x^1)^2]^{-1} \frac{d}{dx^1}\left[\cos x^1(\sin x^1)^2 \frac{dE_1}{dx^1}\right]$$

$$+\left[-\frac{\ell_1}{(\sin x^1)^2}+\frac{\ell_2}{(\cos x^1)^2}-\lambda\right] E_1=0, \tag{3.25}$$

$$\left(\frac{d}{dx^2}\right)^2 E_2=\ell_2 E_2, \qquad \mathscr{L}_1 E_3 E_4=\ell_1 E_3 E_4.$$

4. Comments and Conclusions

All classes of coordinates except VII above are associated with Riemann manifolds of lower dimension than four. Indeed, the coordinates in these classes can all be built up from a knowledge of separable coordinates for E_2, S_2, S_3 and E_3 by adding new variables in analogy with the polyspherical and hyperspherical coordinates of Ref. [15]. It is only the class VII coordinates which cannot be constructed from separable systems on lower dimensional manifolds. Furthermore, it is this class which leads to the most general cyclidic coordinates for the Laplace equation (1.6), of which all other orthogonal R-separable systems are degenerate limits [12, 14].

We have included the degenerate cases for two reasons. First of all there is the matter of completeness of the results. Techniques such as those in Ref. [15] do not seem amenable to direct proofs that they give all possible separable systems of a given type. For S_4 we have supplied the completeness proof and determined the precise relations between S_4 and the lower dimensional manifolds. (An interesting problem here is the determination of the precise relationship between the subalgebra structure of $o(5)$ and these degenerate coordinates.) A second reason is that for many cases the separated solutions of (1.1) are not the same as the corresponding solutions on submanifolds, i.e., new special functions arise. We have determined the ordinary differential equations satisfied by these functions.

A separable coordinate system for (1.1) is called *split* if the defining operators are of the form $\mathscr{L}_j = I_j^2$, $j=1, 2, 3$, where $\{I_1, I_2, I_3\}$ is a basis for a three-dimensional abelian subalgebra of $o(5)$. (In the terminology of Refs. [10, 11] we say that such a system has three *ignorable* variables.) From our list of coordinates we see that (1.1) admits, up to equivalence, only one system of orthogonal split separable coordinates. This system belongs to class IX and corresponds to the operators $\{I_{12}-iI_{52}, I_{13}-iI_{53}, I_{14}-iI_{54}\}$. Moreover, by an extension of the results of Ref. [11] one can show that $o(5)$ contains only one three-dimensional abelian subalgebra up to conjugacy. It follows that there is only one split separable system for (1.1), orthogonal or not. (This is in strong contrast with the flat space Helmholtz equation in four variables where there are several split nonorthogonal systems.)

We remark that $o(5)$ is also associated with R-separation of variables for the Laplace and wave equations in three variables [7, 12–14].

With this paper the determination and group theoretic characterization of all R-separable orthogonal coordinate systems for the four-dimensional Laplace equation is complete. Although the final list of systems is new, we have shown that the coordinate surfaces for all R-separable systems are either families of orthogonal cyclides or their degenerate limits. Thus they could have been obtained using the geometric approach of Bôcher [12]. To prove completeness and to properly characterize the systems it was necessary for us to make use of results of Eisenhart [6] and techniques from Lie theory.

For nonorthogonal R-separable systems the results are entirely different. The approaches of Bôcher and Eisenhart no longer apply and we can find a number of new systems not obtainable by classical techniques. The relationship between separation and commuting sets of operators in the enveloping algebra of $o(6)$ still holds, however. We intend to report on these results in a forthcoming paper.

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