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LIE THEORY AND THE WAVE EQUATION IN SPACE-TIME. 2. THE GROUP $SO(4, \mathbb{C})$*

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Abstract. Homogeneous solutions of the Laplace or wave equation in four complex variables correspond to eigenfunctions of the Laplace–Beltrami operator on the complex sphere $S_{3c}: \sum_{i=1}^{4} z_i^2 = 1$. It is shown explicitly that variables separate in this eigenvalue equation for exactly 21 orthogonal coordinate systems, each system characterized by a pair of commuting symmetry operators in the enveloping algebra of $so(4, \mathbb{C})$. Standard group-theoretic methods are applied to derive generating functions and integral representations for the separated solutions. Henrici's theory of expansions in products of Legendre functions is incorporated into this more general scheme.

1. Introduction. In [1] we studied the relation between symmetry and separation of variables for the differential equation in 3 real variables satisfied by solutions of the wave equation $\partial_{tt}\Phi - \Delta_3\Phi = 0$ which are homogeneous of degree σ in x, y, z, t. The appropriate symmetry group was $SO(3, 1)$. Here we examine this relationship in the case where all variables are complex. Instead of the Hilbert space theory for expansions of solutions of the differential equation in terms of separable solutions as developed in [1] we here construct a theory of analytic expansions in terms of separable solutions.

We begin with the complex Laplace equation

$$\Delta_4\Phi(y) = 0, \qquad \Delta_4 = \sum_{j=1}^{4} \partial_{y_jy_j},$$
$$\mathbf{y} = (y_1, y_2, y_3, y_4), \qquad y_j \in \mathbb{C}. \tag{1.1}$$

Clearly (1.1) is equivalent to the complex wave equation, (set $y_1 = x$, $y_2 = y$, $y_3 = z$, $y_4 = it$). We are interested in the solutions of (1.1) which are homogeneous of fixed degree $\sigma \in \mathbb{C}$: $\Phi(r\mathbf{y}) = r^\sigma\Phi(\mathbf{y})$. Introducing coordinates r, z_j such that $y_j = rz_j$, $\sum_{j=1}^{4} z_j^2 = 1$ we see that these homogeneous functions are uniquely determined by their values on the complex unit sphere $S_{3c}: z_2^1 + z_2^2 + z_3^2 + z_4^2 = 1$. Indeed $\Phi(\mathbf{y}) = r^\sigma\Phi(\mathbf{z})$. The group $SO(4, \mathbb{C}) \equiv SO(4)$ has a natural action on S_{3c} which is determined by the Lie derivatives

$$I_{jk} = z_j\partial_{z_k} - z_k\partial_{z_j}, \qquad 1 \leqq j, k \leqq 4, \quad j \neq k.$$

(Since this paper deals with *local* Lie theory we are concerned only with the behavior of analytic functions in small neighborhoods of a given point. Thus $f(r) = r^\sigma$ can be defined precisely in a neighborhood of $r_0 \neq 0$ by choosing any branch of the global analytic function, e.g., if $r_0 = R_0 e^{i\varphi_0}$, $R_0 > 0$, $-\pi < \varphi_0 < \pi$ we can define $f(r)$ for $r = Re^{i\varphi}$ in a small neighborhood of r_0 by $f(r) = \exp(\sigma \ln R)\, e^{i\sigma\varphi}$. The branch chosen makes no difference in the computations to follow. However, in § 4 it is necessary to be more careful about domains of definition in order to determine precisely the regions of validity of our identities.

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In that section we use the above definition of r^σ for $r=R>0$ and extend by analytic continuation.)

It is straightforward to show from (1.1) that the restriction ψ of the homogeneous function Φ to S_{3c} satisfies the eigenvalue equation for the Laplace operator on S_{3c}:

$$(1.2)\qquad (I_{12}^2+I_{13}^2+I_{14}^2+I_{23}^2+I_{24}^2+I_{34}^2)\psi(\mathbf{z})=-\sigma(\sigma+2)\psi(\mathbf{z}).$$

Moreover, the symmetry algebra of (1.2) is $so(4)$, the Lie algebra of $SO(4)$. In other publications we have developed a method which relates the symmetry group of a linear partial differential equation to the possible coordinate systems in which the equation admits solutions via separation of variables, e.g., [2], [3]. Here the method is applied to (1.2).

In § 2 we apply results of Eisenhart [4] to construct all complex orthogonal coordinate systems in which (1.2) admits separation. We show that there are exactly twenty-one such systems. In § 3 we show that each system is characterized by a pair of commuting second-order operators $\mathscr{L}_1$, $\mathscr{L}_2$ in the enveloping algebra of $so(4)$ in the sense that the corresponding separable solutions are common eigenfunctions of these operators with the separation constants as eigenvalues. We also discuss the relationship between the subalgebras $so(3)$, $so(3)\times so(3)$ and $\mathscr{E}(2)$ of $so(4)$ and some of the simpler coordinate systems.

In § 4 it is shown how the Lie algebraic characterization of the separable solutions of (1.2) can be used to derive generating functions and addition theorems for these special functions. Since the basic theory of such expansions has been discussed elsewhere, [5], [6], we merely present a few of the most interesting cases.

Among the results is a new group theoretic proof of the addition theorem for Gegenbauer polynomials $C_n^\lambda(x)$. The standard group-theoretic proofs of this result, [7, Chap. 11], use global representations of the family of groups $SO(m)$ and are valid only for half-integer values of λ. The proof given here is much simpler, uses local representations of $SO(4)$ and is valid for general complex λ. In [8], Henrici gave simple, elegant proofs of this addition theorem and many other generating functions for products of Gegenbauer functions by employing complex variable techniques on the partial differential equation (4.17) below, an equation which is distinct from (1.2). We will show, that (4.17) is actually equivalent to (1.2) under the action of the conformal symmetry group $SO(6)$ of (1.1) and point out the underlying group structure of Henrici's technique. A related proof of the addition theorem which implicitly employs separation of variables can be found in a recent note by Koornwinder [9].

Finally, in § 5 we show how to construct integral representations for each of the twenty-one classes of separated solutions of (1.2) by transferring the action of $SO(4)$ from S_{3c} to S_{2c}.

We are ultimately concerned with the classification of all separable and R-separable complex coordinate systems for (1.1) and the study of all special functions which arise from the equation via separation of variables. The determination of all homogeneous orthogonal separable systems given here is a first step toward realization of this program.

Note that by characterizing each separable system in terms of Lie algebra generators we have to a considerable extent reduced problems concerning the expansion of one set of separable solutions in other sets to a problem in the representation theory of the symmetry algebra. In [1] we studied unitary representations and obtained Hilbert space expansions whereas here we study local representations and obtain analytic expansions.

2. Separation of variables for the Laplace operator on S_{3c}. Here we consider the problem of separation of variables for the equation $\Delta\psi = \sigma(\sigma+2)\psi$ where Δ is the Laplace operator on the complex sphere S_{3c}. This is not equivalent to the corresponding problem on the real sphere S_3 studied by Olevskii [10] and Eisenhart [4] since we allow the coordinates to be complex quantities and ignore the ranges of variations of the coordinates. We do, however, restrict ourselves to orthogonal coordinate systems. The method we use for evaluating the systems is an adaption of that used by Eisenhart for a space of constant curvature. Here we look for all complex solutions for the metric coefficients rather than for all real solutions as did Eisenhart.

Let $\{x_1, x_2, x_3\}$ be a complex analytic coordinate system on S_{3c}. If the system is orthogonal then the metric takes the form

$$ds^2 = H_1^2\,dx_1^2 + H_2^2\,dx_2^2 + H_3^2\,dx_3^2 \tag{2.1}$$

and the equation $\Delta\psi = \sigma(\sigma+2)\psi$ in these coordinates reads

$$\frac{1}{H_1H_2H_3}\left[\left(\partial_{x_1}\left(\frac{H_2H_3}{H_1}\,\partial_{x_1}\psi\right) + \partial_{x_2}\left(\frac{H_1H_3}{H_2}\,\partial_{x_2}\psi\right) + \partial_{x_3}\left(\frac{H_1H_2}{H_3}\,\partial_{x_3}\psi\right)\right]\right. \\ = -\sigma(\sigma+2)\psi. \tag{2.2}$$

Eisenhart has shown that if (2.2) separates in the variables $\{x_1, x_2, x_3\}$ then the metric coefficients must have one of the forms

1. $H_1 = 1, \quad H_2 = \phi(x_1), \quad H_3 = \theta(x_1),$
2. $H_1 = 1, \quad H_2 = \phi(x_1), \quad H_3 = \phi(x_1)\theta(x_2),$
3. $H_1 = 1, \quad H_2^2 = (x_2 - x_3)X_2(x_2)\sigma_1^2(x_1), \quad H_3^2 = (x_2 - x_3)X_3(x_3)\sigma_1^2(x_1),$
4. $H_1^2 = \sigma_1(x_1) + e\sigma_3(x_3), \quad H_2^2 = \sigma_1(x_1)\sigma_3(x_3),$
   $H_3^2 = \sigma_1(x_1) + e\sigma_3(x_3), \quad e = \pm 1$
5. $H_i^2 = (x_i - x_j)(x_i - x_k)X_i(x_i), \quad i \neq j \neq k \neq i.$

In addition to having one of these forms the metric coefficients H_i^2 must satisfy the requirement that the space have constant unit curvature. This condition is

$$\begin{aligned}&\frac{1}{H_j^2}\left(2\frac{\partial^2}{\partial x_j^2}\log H_i^2 + \frac{\partial}{\partial x_j}\log H_i^2\,\frac{\partial}{\partial x_j}\log\frac{H_i^2}{H_j^2}\right)\\ &\quad+\frac{1}{H_i^2}\left(2\frac{\partial^2}{\partial x_i^2}\log H_j^2 + \frac{\partial}{\partial x_i}\log H_j^2\,\frac{\partial}{\partial x_i}\log\frac{H_j^2}{H_i^2}\right)\\ &\qquad\qquad+\frac{1}{H_k^2}\frac{\partial}{\partial x_k}\log H_i^2\,\frac{\partial}{\partial x_k}\log H_j^2 = -4.\end{aligned} \tag{2.3}$$

where i, j, k are distinct. We now compute the differential forms associated with the four types of metric and subject to constraints (2.3).

1. For metrics of type 1 we find from (2.3) for $i=1, j=2$ and $i=1, j=3$ that ϕ and θ satisfy the equation $d^2\psi/dx_1^2+\psi=0$, and for $i=2, j=3$ in (2.3) we have the constraint $(d\phi/dx_1)(d\theta/dx_1)+\phi\theta=0$. There are two distinct solutions:

$$\text{(i)}\quad \phi=\sin x_1, \quad \theta=\cos x_1,$$

$$\text{(ii)}\quad \phi=e^{ix_1}, \quad \theta=e^{ix_1}.$$

The corresponding metrics are

$$(1)\quad ds^2=dx_1^2+\sin^2 x_1\, dx_2^2+\cos^2 x_1\, dx_3^2,$$

$$(2)\quad ds^2=dx_1^2+e^{2ix_1}(dx_2^2+dx_3^2),$$

2. For metrics of type 2 we find from (2.3) with $i=1, j=2$ and $i=1, j=3$ that $\phi''+\phi=0$. For $i=2, j=3$ we find $\theta''+(\phi^2+\phi'^2)\theta=0$. The possible solutions to these equations are

$$\text{(i)}\quad \phi=\sin x_1, \quad \theta=\sin x_2,$$

$$\text{(ii)}\quad \phi=\sin x_1, \quad \theta=e^{ix_2},$$

$$\text{(iii)}\quad \phi=e^{ix_1}, \quad \theta=x_2.$$

The corresponding metrics are

$$(3)\quad ds^2=dx_1^2+\sin^2 x_1(dx_2^2+\sin^2 x_2\, dx_3^2),$$

$$(4)\quad ds^2=dx_1^2+\sin^2 x_1(dx_2^2+e^{2ix_2}\, dx_3^2),$$

$$(5)\quad ds^2=dx_1^2+e^{2ix_1}(dx_2^2+x_2^2\, dx_3^2).$$

3. For metrics of type 3 we find from (2.3) with $i=1, j=2$ and $i=1, j=3$ that $\sigma_1''+\sigma_1=0$. If $\sigma_1=\sin x_1$ then $H_1^2=1$, $H_2^2=(x_2-x_3)X_2\sin^2 x_1$ and $H_3^2=(x_2-x_3)X_3\sin^2 x_1$. For $i=2, j=3$ in (2.3) we obtain

$$2\left(\frac{1}{X_2}+\frac{1}{X_3}\right)+(x_2-x_3)\left[\left(\frac{1}{X_3}\right)'-\left(\frac{1}{X_2}\right)'\right]-4(x_2-x_3)^3=0.$$

Differentiation of this equation twice with respect to x_2 implies $(1/X_2)'''=-24$ so

$$1/X_2=-4x_2^3+bx_2^2+cx_2+d=f(x_2).$$

Similarly $X_3=-1/f(x_3)$. There are only three distinct systems of this type:

$$(6)\quad ds^2=dx_1^2+\sin^2 x_1(\mathrm{sn}^2(x_2,k)-\mathrm{sn}^2(x_3,k))(dx_2^2-dx_3^2),$$

$$(7)\quad ds^2=dx_1^2+\left(\frac{1}{\mathrm{ch}^2 x_2}-\frac{1}{\mathrm{ch}^2 x_3}\right)\sin^2 x_1(dx_2^2-dx_3^2),$$

$$(8)\quad ds^2=dx_1^2+\left(\frac{1}{x_3^2}-\frac{1}{x_2^2}\right)\sin^2 x_1(dx_2^2-dx_3^2).$$

Here, $\mathrm{sn}(x,k)$ is a Jacobi elliptic function and we adopt the notation $\mathrm{sh}\, x$, $\mathrm{ch}\, x$, $\mathrm{th}\, x$ for hyperbolic functions.

In these equations we have introduced new variables $\tilde{x}_j = \tilde{x}_j(x_j)$, $j = 2, 3$. In general, we do not distinguish between coordinate systems $\{x_1, x_2, x_3\}$ and $\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$ where $\tilde{x}_j = \tilde{x}_j(x_j)$, $j = 1, 2, 3$.

If $\sigma_1 = e^{ix_1}$ and $i = 2$, $j = 3$, then (2.3) reduces to

$$2\left(\frac{1}{X_2}+\frac{1}{X_3}\right)+(x_2-x_3)\left[\left(\frac{1}{X_3}\right)'-\left(\frac{1}{X_2}\right)'\right]=0.$$

Differentiating this equation twice with respect to x_2 we find $(1/X_2)''' = 0$ or $1/X_2 = ax_2^2 + bx_2 + c = h(x_2)$. Similarly $1/X_3 = -h(x_3)$. There are four distinct systems of this type:

$$(9)\quad ds^2 = dx_1^2 + e^{2ix_1}(\operatorname{ch}^2 x_2 - \operatorname{ch}^2 x_3)(dx_2^2 - dx_3^2),$$

$$(10)\quad ds^2 = dx_1^2 + e^{2ix_1}(e^{2x_2} + e^{2x_3})(dx_2^2 - dx_3^2),$$

$$(11)\quad ds^2 = dx_1^2 + e^{2ix_1}(x_2^2 + x_3^2)(dx_2^2 + dx_3^2),$$

$$(12)\quad ds^2 = dx_1^2 + e^{2ix_1}(4x_2 - 4x_3)(dx_2^2 - dx_3^2).$$

4. For metrics of type 4, equation (2.3) with $i = 1$, $j = 2$ yields the constraint

$$2\left(\sigma_1'' - \frac{\sigma_1'^2}{\sigma_1}\right) + \sigma_3\left(2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2}\right) + \frac{\sigma_3'^2}{\sigma_3} = -4(\sigma_1 + \sigma_3)^2.$$

Differentiating with respect to x_3 we obtain

$$\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} + \left(\frac{\sigma_3'^2}{\sigma_3}\right)' \frac{1}{\sigma_3'} = -8(\sigma_1 + \sigma_3).$$

We can separate variables in this equation according to the scheme

$$2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} + 8\sigma_1 = 4c,$$

$$\left(\frac{\sigma_3'^2}{\sigma_3}\right)' \frac{1}{\sigma_3'} + 8\sigma_3 = -4c$$

where c is a separation constant. First integrals of these equations are

$$\sigma_1'^2 = 4\sigma_1(f + c\sigma_1 - \sigma_1^2),$$

$$\sigma_3'^2 = 4\sigma_3(f - c\sigma_3 - \sigma_3^2),$$

f is a constant. Choosing new variables $\hat{x}_1 = \sigma_1$, $\hat{x}_3 = -\sigma_3$ we obtain the metric

$$ds^2 = \frac{1}{4}(\hat{x}_1 - \hat{x}_3)\left[\frac{d\hat{x}_1^2}{\hat{x}_1(a - \hat{x}_1)(b - \hat{x}_1)} - \frac{d\hat{x}_3^2}{\hat{x}_3(a - \hat{x}_3)(b - \hat{x}_3)}\right] + \hat{x}_1\hat{x}_3\, dx_2^2,$$

where $ab = -f$, $a + b = c$. There are four distinct cases:

If $a \neq b$, $|a|, |b| > 0$, the metric can be reduced to

$$(13)\quad ds^2 = -k^2(\operatorname{sn}^2(x_1, k) - \operatorname{sn}^2(x_3, k))(dx_1^2 - dx_3^2)$$

$$+\frac{k^2}{k'^2}\operatorname{cn}^2(x_1, k)\operatorname{cn}^2(x_3, k))\, dx_2^2, \qquad k' = \sqrt{1 - k^2},$$

If $a=b\neq 0$ we find

$$(14)\quad ds^2=(\mathrm{th}^2 x_1-\mathrm{th}^2 x_3)(dx_1^2-dx_3^2)+\mathrm{th}^2 x_1\,\mathrm{th}^2 x_3\,dx_2^2,$$

while if $a=0$, $b\neq 0$, we obtain

$$(15)\quad ds^2=\left(\frac{1}{\mathrm{ch}^2 x_1}-\frac{1}{\mathrm{ch}^2 x_3}\right)(dx_1^2-dx_3^2)+\frac{1}{\mathrm{ch}^2 x_1\,\mathrm{ch}^2 x_3}dx_2^2.$$

Finally, if $a=b=0$ the metric becomes

$$(16)\quad ds^2=\left(\frac{1}{x_1^2}+\frac{1}{x_3^2}\right)(dx_1^2+dx_3^2)+\frac{1}{x_1^2x_3^2}dx_2^2.$$

5. For metrics of type 5, equation (2.3) with $i=1$, $j=2$ becomes

$$\frac{1}{X_3}+\frac{1}{(x_1-x_2)^2}\left\{(x_3-x_2)^2\left[(x_1-x_3)\left(\frac{1}{X_1}\right)'-\left(\frac{2(x_3-x_1)}{x_2-x_1}+1\right)\frac{1}{X_1}\right]\right.$$
$$+(x_3-x_1)^2\left[(x_2-x_3)\left(\frac{1}{X_2}\right)'\right.$$
$$\left.\left.-\left(\frac{2(x_3-x_2)}{x_1-x_2}+1\right)\frac{1}{X_2}\right]\right\}+4(x_3-x_1)^2(x_3-x_2)^2=0.$$

Differentiating this equation twice with respect to x_2 we obtain a polynomial of order three in x_3. The coefficient $g(x_1,x_2)$ of x_3^3 must be identically zero. Thus

$$\frac{\partial^2 g}{\partial x_2^2}=\left(\frac{1}{X_2}\right)^{(4)}+96=0$$

and $1/X_2=-4x_2^4+ax_2^3+bx_2^2+cx_2+d=f(x_2)$. Similarly $1/X_1=f(x_1)$ and $1/X_3=f(x_3)$. Five coordinate systems of this type can be distinguished. In each case the metric assumes the form

$$ds^2=\frac{(x_1-x_2)(x_1-x_3)}{f(x_1)}dx_1^2+\frac{(x_2-x_3)(x_2-x_1)}{f(x_2)}dx_2^2+\frac{(x_3-x_1)(x_3-x_2)}{f(x_3)}dx_3^2$$

and the systems are distinguished by the multiplicities of the zeros of $f(x)$. The distinct possibilities are

$$(17)\quad f(x)=-4(x-a)(x-b)(x-1)x,\quad a\neq b,$$

$$(18)\quad f(x)=-4(x-2)(x-1)x^2,$$

$$(19)\quad f(x)=-4(x-1)^2x^2,$$

$$(20)\quad f(x)=-4(x-1)x^3,$$

$$(21)\quad f(x)=-4x^4.$$

This completes the list of orthogonal coordinate systems on the complex sphere S_{3c} which permit separation of variables for the equation $\Delta\psi=\sigma(\sigma+2)\psi$. There are exactly 21 such systems.

3. Lie algebra characteristics of the separable systems. The three-dimensional complex sphere S_{3c} consists of those points (z_1, z_2, z_3, z_4) in complex four-dimensional Euclidean space such that $z_1^2+z_2^2+z_3^2+z_4^2=1$. The connected Lie subgroup of the complex Euclidean group which leaves this manifold invariant is $SO(4, \mathbb{C})$, the complex rotation group. A basis for the six-dimensional Lie algebra $so(4, \mathbb{C})$ of $SO(4, \mathbb{C})$ is

$$I_{kl} = z_k \partial_l - z_l \partial_k, \quad k, l = 1, 2, 3, 4, \quad k \neq l, \quad I_{kl} = -I_{lk}. \tag{3.1}$$

These basis elements satisfy the commutation relations

$$[I_{kl}, I_{st}] = \delta_{ls} I_{kt} - \delta_{ks} I_{lt} - \delta_{lt} I_{ks} + \delta_{kt} I_{ls}. \tag{3.2}$$

Further, if we put

$$\begin{aligned} J_1 &= \tfrac{1}{2}(I_{23} - I_{14}), \quad J_2 = \tfrac{1}{2}(I_{13} + I_{24}), \quad J_3 = \tfrac{1}{2}(I_{12} - I_{34}), \\ L_1 &= \tfrac{1}{2}(I_{23} + I_{14}), \quad L_2 = \tfrac{1}{2}(I_{13} - I_{24}), \quad L_3 = \tfrac{1}{2}(I_{12} + I_{34}), \end{aligned} \tag{3.3}$$

it becomes evident that $so(4, \mathbb{C}) \cong so(3, \mathbb{C}) \oplus so(3, \mathbb{C})$. Indeed

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \quad [L_i, L_j] = \varepsilon_{ijk} L_k, \quad [J_i, L_j] = 0. \tag{3.4}$$

It can be verified by tedious computations that each of the 21 separable coordinate systems constructed in § 2 is characterized by a pair of commuting symmetric second-order operations $\mathcal{L}_1, \mathcal{L}_2$ in the enveloping algebra of $so(4, \mathbb{C})$. That is, the separable solutions $\psi = \psi_1(x_1)\psi_2(x_2)\psi_3(x_3)$ corresponding to such a system are characterized by the equations

$$\Delta \psi = \sigma(\sigma+2)\psi, \qquad \mathcal{L}_1 \psi = \lambda_1 \psi, \qquad \mathcal{L}_2 \psi = \lambda_2 \psi. \tag{3.5}$$

The eigenvalues λ_1, λ_2 are the separation constants. Expressed in terms of the generators of $so(4, \mathbb{C})$ the Laplace operator is

$$-\Delta = I_{12}^2 + I_{13}^2 + I_{14}^2 + I_{23}^2 + I_{24}^2 + I_{34}^2; \tag{3.6}$$

i.e., Δ is the Casimir operator for $so(4, \mathbb{C})$.

We now present the explicit coordinates and the corresponding operators $\mathcal{L}_1$, $\mathcal{L}_2$ for each of the 21 separable coordinate systems on S_{3c}.

(1) $z_1 = \sin x_1 \cos x_2, \quad z_2 = \cos x_1 \cos x_3,$

$z_3 = \cos x_1 \sin x_3, \quad z_4 = \sin x_1 \sin x_2,$

$\mathcal{L}_1 = I_{23}^2, \quad \mathcal{L}_2 = I_{14}^2;$

(2) $z_1 = \tfrac{1}{2}[e^{-ix_1} + (1 + x_2^2 + x_3^2)\, e^{ix_1}], \quad z_2 = i x_2\, e^{ix_1},$

$z_3 = i x_3\, e^{ix_1}, \quad z_4 = \dfrac{i}{2}[e^{-ix_1} + (-1 + x_2^2 + x_3^2)\, e^{ix_1}],$

$\mathcal{L}_1 = (I_{42} + i I_{21})^2, \quad \mathcal{L}_2 = (I_{34} + i I_{13})^2;$

(3) $z_1 = \sin x_1 \cos x_2, \quad z_2 = \sin x_1 \sin x_2 \cos x_3,$

$z_3 = \sin x_1 \sin x_2 \sin x_3, \quad z_4 = \cos x_1,$

$\mathcal{L}_1 = I_{12}^2 + I_{13}^2 + I_{23}^2, \quad \mathcal{L}_2 = I_{23}^2;$

(4) $z_1=\frac{1}{2}\sin x_1[e^{-ix_2}+(1-x_3^2)\,e^{ix_2}],\quad z_2=x_3\,e^{ix_2}\sin x_1,$

$z_3=\frac{-i}{2}\sin x_1[e^{-ix_2}-(1+x_3^2)\,e^{ix_2}],\quad z_4=\cos x_1,$

$\mathcal{L}_1=I_{12}^2+I_{13}^2+I_{23}^2,\quad \mathcal{L}_2=(I_{32}+iI_{21})^2;$

(5) $z_1=\frac{1}{2}[e^{-ix_1}+(1+x_2^2)\,e^{ix_1}],\quad z_2=i\,e^{ix_1}x_2\cos x_3,$

$z_3=i\,e^{ix_1}x_2\sin x_3,\quad z_4=(i/2)[e^{-ix_1}-(1-x_2^2)\,e^{ix_1}],$

$\mathcal{L}_1=(I_{42}+iI_{21})^2+(I_{34}+iI_{13})^2,\quad \mathcal{L}_2=I_{23}^2;$

(6) $z_1=\frac{1}{k'}\sin x_1\,\mathrm{dn}\,(x_2,k)\,\mathrm{dn}\,(x_3,k)\quad z_2=\frac{ik}{k'}\sin x_1\,\mathrm{cn}\,(x_2,k)\,\mathrm{cn}\,(x_3,k),$

$z_3=k\sin x_1\,\mathrm{sn}\,(x_1,k)\,\mathrm{sn}\,(x_3,k),\quad z_4=\cos x_1,$

$\mathcal{L}_1=I_{12}^2+I_{13}^2+I_{23}^2,\quad \mathcal{L}_2=I_{23}^2+k^2I_{13}^2;$

(7) $z_1=\frac{1}{2}\sin x_1\left(\frac{\mathrm{ch}\,x_3}{\mathrm{ch}\,x_2}+\frac{\mathrm{ch}\,x_2}{\mathrm{ch}\,x_3}\right),\quad z_2=\sin x_1\,\mathrm{th}\,x_2\,\mathrm{th}\,x_3,$

$z_3=i\sin x_1\left[\frac{1}{\mathrm{ch}\,x_2\,\mathrm{ch}\,x_3}-\frac{1}{2}\left(\frac{\mathrm{ch}\,x_3}{\mathrm{ch}\,x_2}+\frac{\mathrm{ch}\,x_2}{\mathrm{ch}\,x_3}\right)\right],\quad z_4=\cos x_1,$

$\mathcal{L}_1=I_{12}^2+I_{13}^2+I_{23}^2,\quad \mathcal{L}_2=-I_{12}^2-I_{13}^2+I_{23}^2+i\{I_{31},I_{32}\};$

(8) $z_1=\frac{-i\sin x_1}{8x_2x_3}[(x_3^2-x_2^2)^2+4],\quad z_2=\frac{\sin x_1}{2x_2x_3}[x_3^2+x_2^2],$

$z_3=\frac{\sin x_1}{8x_2x_3}[-(x_3^2-x_2^2)^2+4],\quad z_4=\cos x_1-\{I_{12},I_{13}\}+i\{I_{12},I_{23}\},$

$\mathcal{L}_1=I_{12}^2+I_{13}^2+I_{23}^2,\quad \mathcal{L}_2=-\{I_{12},I_{13}\}+i\{I_{12},I_{23}\};$

(9) $z_1=\frac{1}{2}(e^{-ix_1}+[1+\mathrm{ch}^2\,x_2+\mathrm{sh}^2\,x_3]\,e^{ix_1}),\quad z_2=i\,\mathrm{ch}\,x_2\,\mathrm{ch}\,x_3\,e^{ix_1},$

$z_3=\mathrm{sh}\,x_2\,\mathrm{sh}\,x_3\,e^{ix_1},\quad z_4=\frac{i}{2}(e^{-ix_1}+[-1+\mathrm{ch}^2\,x_2+\mathrm{sh}^2\,x_3]\,e^{ix_1}),$

$\mathcal{L}_1=(I_{42}+iI_{21})^2+(I_{34}+iI_{13})^2,\quad \mathcal{L}_2=I_{23}^2+(I_{34}+iI_{13})^2;$

(10) $z_1=\frac{1}{2}(e^{-ix_1}+[1+e^{2x_2}-e^{2x_3}]\,e^{ix_1}),\quad z_2=\frac{i}{\sqrt{2}}(\mathrm{sh}\,(x_2-x_3)+e^{x_2+x_3})\,e^{ix_1},$

$z_3=\frac{1}{\sqrt{2}}(\mathrm{sh}\,(x_2-x_3)-e^{x_2+x_3})\,e^{ix_1},$

$z_4=\frac{i}{2}(e^{-ix_1}+[-1+e^{2x_2}-e^{2x_3}]\,e^{ix_1},$

$\mathcal{L}_1=(I_{42}+iI_{21})^2+(I_{34}+iI_{13})^2,\quad \mathcal{L}_2=I_{23}^2-(I_{42}+I_{31}+i(I_{12}+I_{34}))^2;$

(11) $z_1=\frac{1}{2}(e^{-ix_1}+[1+\frac{1}{4}(x_2^2+x_3^2)^2]e^{ix_1}, \quad z_2=(i/2)(x_2^2-x_3^2)e^{ix_1},$

$z_3=ix_2x_3e^{ix_1}, \quad z_4=(i/2)(e^{-ix_1}+[-1+\frac{1}{4}(x_2^2+x_3^2)^2]e^{ix_1}),$

$\mathscr{L}_1=(I_{42}+iI_{21})^2+(I_{34}+iI_{13})^2, \quad \mathscr{L}_2=\{I_{23}, I_{42}+iI_{21}\};$

(12) $z_1=\frac{1}{2}(e^{-ix_1}+[1+2(x_2-x_3)^2(x_2+x_3)]e^{ix_1},$

$z_2=i[\frac{1}{2}(x_2-x_3)^2+(x_2+x_3)]e^{ix_1}, \quad z_3=[\frac{1}{2}(x_2-x_3)^2-(x_2+x_3)]e^{ix_1},$

$z_4=\frac{1}{2}(e^{-ix_1}+[-1+2(x_2-x_3)^2(x_2+x_3)]e^{ix_1}),$

$\mathscr{L}_1=(I_{42}+iI_{21})^2+(I_{34}+iI_{13})^2,$

$\mathscr{L}_2=\{I_{23}, I_{42}+I_{31}+iI_{21}+iI_{34}\}-i(I_{42}-I_{31}+i(I_{21}-I_{34}))^2;$

(13) $z_1=k\operatorname{sn}(x_1,k)\operatorname{sn}(x_3,k), \quad z_2=-i\frac{k}{k'}\operatorname{cn}(x_1,k)\operatorname{cn}(x_3,k)\cos x_2,$

$z_3=-i\frac{k}{k'}\operatorname{cn}(x_1,k)\operatorname{cn}(x_3,k)\sin x_2, \quad z_4=\frac{1}{k'}\operatorname{dn}(x_1,k)\operatorname{dn}(x_3,k),$

$\mathscr{L}_1=I_{23}^2, \quad \mathscr{L}_2=I_{12}^2+I_{13}^2+k^2I_{14}^2;$

(14) $z_1=\frac{1}{2}\left(\frac{\operatorname{ch}x_1}{\operatorname{ch}x_3}+\frac{\operatorname{ch}x_3}{\operatorname{ch}x_1}\right), \quad z_2=\operatorname{th}x_1\operatorname{th}x_3\operatorname{ch}x_2,$

$z_3=-i\operatorname{th}x_1\operatorname{th}x_3\operatorname{sh}x_2, \quad z_4=\frac{-i}{\operatorname{ch}x_1\operatorname{ch}x_3}+\frac{i}{2}\left(\frac{\operatorname{ch}x_1}{\operatorname{ch}x_3}+\frac{\operatorname{ch}x_3}{\operatorname{ch}x_1}\right),$

$\mathscr{L}_1=I_{23}^2, \quad \mathscr{L}_2=I_{24}^2+I_{34}^2-I_{12}^2-I_{13}^2-I_{14}^2-i\{I_{12}, I_{42}\}-i\{I_{13}, I_{43}\};$

(15) $z_1=\frac{-1}{2}\left(\frac{\operatorname{ch}x_3}{\operatorname{ch}x_1}+\frac{\operatorname{ch}x_1}{\operatorname{ch}x_3}\right)-\frac{x_2^2}{2\operatorname{ch}x_1\operatorname{ch}x_3}, \quad z_2=\frac{ix_2}{\operatorname{ch}x_1\operatorname{ch}x_3},$

$z_3=\operatorname{th}x_1\operatorname{th}x_3, \quad z_4=i\left[\frac{2-x_2^2}{2\operatorname{ch}x_1\operatorname{ch}x_3}-\frac{1}{2}\left(\frac{\operatorname{ch}x_1}{\operatorname{ch}x_3}+\frac{\operatorname{ch}x_3}{\operatorname{ch}x_1}\right)\right],$

$\mathscr{L}_1=(I_{42}+iI_{21})^2, \quad \mathscr{L}_2=2I_{12}^2+I_{13}^2+I_{14}^2-I_{34}^2+i(\{I_{12}, I_{42}\}+\{I_{13}, I_{43}\});$

(16) $z_1=\left[\frac{(x_1^2+x_3^2)^2+4}{8x_1x_3}+\frac{x_2^2}{2x_1x_3}\right], \quad z_2=\frac{-ix_2}{x_1x_3},$

$z_3=\frac{-i}{2}\left(\frac{x_1}{x_3}-\frac{x_3}{x_1}\right), \quad z_4=\frac{i(x_1^2+x_3^2)^2-4i}{8x_1x_3}+\frac{ix_2^2}{2x_1x_3},$

$\mathscr{L}_1=(I_{42}+iI_{21})^2, \quad \mathscr{L}_2=\{I_{32}, I_{42}+iI_{21}\}-\{I_{14}, iI_{34}-I_{13}\},$

(17) $z_1^2=\frac{-x_1x_2x_3}{ab}, \quad z_2^2=\frac{(x_1-1)(x_2-1)(x_3-1)}{(a-1)(b-1)},$

$z_3^2=\frac{-(x_1-b)(x_2-b)(x_3-b)}{(a-b)(b-1)b}, \quad z_4^2=\frac{(x_1-a)(x_2-a)(x_3-a)}{(a-b)(a-1)a},$

$\mathscr{L}_1=abI_{12}^2+aI_{13}^2+bI_{14}^2,$

$\mathscr{L}_2=(a+b)I_{12}^2+(a+1)I_{13}^2+(b+1)I_{14}^2+aI_{32}^2+bI_{42}^2+I_{43}^2;$

$$(18)\quad (iz_1+z_2)^2=\frac{x_1x_2x_3}{a},$$

$$z_1^2+z_2^2=\frac{1}{a^2}[(a+1)x_1x_2x_3-a(x_1x_2+x_1x_3+x_2x_3)],$$

$$z_3^2=\frac{-(x_1-1)(x_2-1)(x_3-1)}{a-1},\quad z_4^2=\frac{(x_1-a)(x_2-a)(x_3-a)}{a^2(a-1)},$$

$$\mathscr{L}_1=(I_{42}-iI_{14})^2-a(I_{32}+iI_{13})^2-aI_{12}^2,$$

$$\mathscr{L}_2=(a+1)I_{12}^2+I_{14}^2+I_{42}^2-a(I_{13}^2+I_{32}^2)+(I_{42}+iI_{14})^2+(I_{32}+iI_{13})^2;$$

$$(19)\quad (z_1+iz_2)^2=-(x_1-1)(x_2-1)(x_3-1),$$

$$z_1^2+z_2^2=2x_1x_2x_3-(x_1x_3+x_2x_3+x_1x_2)+1,\quad (z_3+iz_4)^2=-x_1x_2x_3,$$

$$z_3^2+z_4^2=x_1x_3+x_2x_3+x_1x_4-2x_1x_2x_3,$$

$$\mathscr{L}_1=2(I_{31}+iI_{32})^2+\{I_{31}+iI_{32},I_{24}+iI_{41}\}+I_{12}^2,$$

$$\mathscr{L}_2=2(I_{31}+iI_{32})^2+\{I_{31}+iI_{32},I_{24}+I_{41}\}-I_{34}^2;$$

$$(20)\quad (z_2-iz_1)^2=+x_1x_2x_3,\quad -2z_3(z_2-iz_1)=x_1x_2+x_1x_3+x_2x_3-x_1x_2x_3,$$

$$z_1^2+z_2^2+z_3^2=+x_1x_2x_3-x_1x_2-x_1x_3-x_2x_3+x_1+x_2+x_3,$$

$$z_4^2=-(x_1-1)(x_2-1)(x_3-1),$$

$$\mathscr{L}_1=(I_{41}+iI_{42})^2+\{I_{32}-iI_{13},I_{12}\},$$

$$\mathscr{L}_2=I_{41}^2+I_{42}^2-I_{34}^2-(I_{41}+iI_{42})^2+\{I_{41}+iI_{42},I_{34}\};$$

$$(21)\quad (z_1+iz_2)^2=2x_1x_2x_3,\quad (z_1+iz_2)(z_3+iz_4)=-(x_1x_2+x_2x_3+x_1x_3),$$

$$-(z_1+iz_2)(z_3-iz_4)+\tfrac{1}{2}(z_3+iz_4)^2=x_1+x_2+x_3,$$

$$\mathscr{L}_1=\tfrac{1}{2}\{I_{21},I_{14}+I_{23}+i(I_{31}+I_{24})\}-\tfrac{1}{4}[I_{13}+I_{24}+i(I_{23}+I_{41})]^2,$$

$$\mathscr{L}_2=\tfrac{1}{2}\{I_{21}+I_{43},I_{32}+I_{14}+i(I_{13}+I_{24})\}$$
$$+\tfrac{1}{2}\{I_{14}+I_{23}+i(I_{31}+I_{24}),I_{43}\}+\tfrac{1}{2}(I_{42}+iI_{23})^2-\tfrac{1}{2}(I_{13}+iI_{14})^2.$$

Here, $\{A,B\}=AB+BA$.

To understand the significance of these systems it is useful to examine some of the subalgebras of $so(4,\mathbb{C})$. As shown in (3.3) and (3.4) this algebra can be decomposed into $so(3,\mathbb{C})\oplus so(3,\mathbb{C})$, and it is easy to see that system (1) corresponds to this decomposition. Another $so(3,\mathbb{C})$ subalgebra of $so(4,\mathbb{C})$ has basis $\{I_{12},I_{13},I_{23}\}$ with commutation relations

$$[I_{12},I_{13}]=-I_{23},\ [I_{12},I_{23}]=I_{13},$$
$$[I_{13},I_{23}]=-I_{12}$$

and Casimir operator

$$I_{12}^2+I_{13}^2+I_{23}^2.$$

It is easily seen that the systems (3), (4), (6), (7), and (8) correspond to this Lie algebra reduction $so(4, \mathbb{C}) \supset so(3, \mathbb{C})$ and to coordinates on the sphere S_{2c}: $z_1^2+z_2^2+z_3^2=\text{const}$. Indeed as indicated in [11] there are exactly five such systems corresponding to the $so(3, \mathbb{C})$ subalgebra.

The operators

$$(3.7) \qquad E_1 = I_{42}+iI_{21}, \quad E_2 = I_{43}+iI_{31}, \quad E_3 = I_{23}$$

with commutation relations

$$(3.8) \qquad [E_1, E_2]=0, \qquad [E_1, E_3]=E_2, \qquad [E_2, E_3]=-E_1$$

form a basis for the Euclidean subalgebra $\mathscr{E}(2, \mathbb{C})$ with invariant operator

$$(3.9) \qquad E_1^2+E_2^2.$$

The systems (2), (5), (9), (10), (11) and (12) correspond to the reduction $so(4, \mathbb{C}) \supset \mathscr{E}(2, \mathbb{C})$. Indeed, as shown in [12, Chap. 1], the complex Helmholtz equation with symmetry algebra $\mathscr{E}(2, \mathbb{C})$ separates in exactly six coordinate systems. The remaining nine of our twenty-one systems are not obviously related to subalgebra reductions. (However, systems (13), (14) involve the diagonalization of I_{23} and systems (15), (16) involve the diagonalization of E_1.)

Our separable systems can be understood from another viewpoint. In [13] we presented a group-theoretic analysis of the six separable systems for the Laplace operator on the real sphere S_3: $y_1^2+y_2^2+y_3^2+y_4^2=1$. Here the symmetry algebra is $so(4, R)$. It is evident that each such real system can be analytically continued to a separable system on S_{3c}. Indeed the complexifications of these six systems correspond to our five complex systems (1), (3), (6), (13) and (17). (Elliptic cylindrical coordinates of types I and II complexify to the same system (13).) In [1] we analyzed the thirty-four separable systems for the Laplace operator on the hyperboloid $y_1^2-y_2^2-y_3^2-y_4^2=1$ (symmetry algebra $so(3, 1)$). Complexification of the thirty-four systems yields all complex systems classified here with the exception of the systems (10), (12), (16) and the nonsubgroup systems (19), (21). However, it is evident by inspection that these five remaining cases arise by complexification of separable coordinates for the Laplace operator on the real hyperboloid $y_1^2-y_2^2+y_3^2-y_4^2=1$, symmetry algebra $so(2, 2) \cong sl(2, R) \oplus sl(2, R)$. Thus all our complex separable coordinates are complexifications of real separable coordinates on the sphere S_3 and the hyperboloids $y_1^2-y_2^2-y_3^2-y_4^2=1$, $y_1^2-y_2^2+y_3^2-y_4^2=1$. Similarly the separated solutions are analytic continuations of the separated solutions for the real forms.

To be more specific, note that the coordinates corresponding to the subalgebra reduction $so(4, \mathbb{C}) \supset so(3, \mathbb{C})$ all have the form

$$(3.10) \quad z_1 = w_1 \sin x_1, \qquad z_2 = w_2 \sin x_1, \qquad z_3 = w_3 \sin x_1, \qquad z_4 = \cos x_1,$$

where $w_1^2+w_2^2+w_3^2=1$ and $w_j = w_j(x_2, x_3)$. The separated solutions are of the form

$$(3.11) \qquad f(x_1, x_2, x_3) = \sin^l x_1 \mathscr{C}_{\sigma-l}^{l+1}(\cos x_1) h(x_2, x_3)$$

where $\mathscr{C}_\alpha^\lambda(s)$ is a solution of the Gegenbauer equation

$$(3.12) \qquad (1-s^2)\mathscr{C}_\alpha^{\lambda\prime\prime}+(2\lambda-3)s\mathscr{C}_\alpha^{\lambda\prime}+\alpha(\alpha+2\lambda)\mathscr{C}_\alpha^\lambda=0$$

and

$$(I_{12}^2+I_{13}^2+I_{23}^2)h=-l(l+1)h. \tag{3.13}$$

Similarly the coordinates corresponding to the reduction $so(4, \mathbb{C}) \supset \mathscr{E}(2, \mathbb{C})$ all have the form

$$z_1=\tfrac{1}{2}(e^{-ix_1}+[1+w_2^2+w_3^2]e^{ix_1}), \qquad z_2=iw_2 e^{ix_1}, \tag{3.14}$$
$$z_3=iw_3 e^{ix_1}, \qquad z_4=\frac{i}{2}(e^{-ix_2}+[-1+w_2^2+w_3^2]e^{ix_1})$$

where $w_j=w_j(x_2, x_3)$, $j=2, 3$. The separated solutions are of the form

$$f(x_1, x_2, x_3)=e^{-ix_1}Z_{\pm(\sigma+1)}(i\omega e^{-ix_1})h(w_2, w_3), \tag{3.15}$$

where the cylindrical function $Z_\nu(s)$ is a solution of Bessel's equation

$$s^2Z_\nu''+sZ_\nu'+(s^2-\nu^2)Z_\nu=0$$

and h is a solution of the complex Helmholtz equation

$$(\partial_{w_2w_2}+\partial_{w_3w_3}+\omega^2)h(w_2, w_3)=0. \tag{3.16}$$

It follows from the above remarks that, except for the rather intractable systems (19) and (21), the separated solutions for all coordinate systems can be easily obtained by analytic continuation of results found in [1], [12] and [13].

4. Generating functions for the separated solutions. Here we are concerned with the analytic expansion of a particular separated solution of (2.2) in terms of a set of separated solutions. For the most part we shall confine our attention to expansions in terms of separated solutions corresponding to systems (1) and (3).

For system (1) with

$$\tau=\sin x_1 e^{ix_2}, \qquad \xi=\cos x_1 e^{ix_3}, \qquad w=\cos 2x_1, \tag{4.1}$$

one can easily verify that the functions

$$F_{\mu,m}^{(1)}(\tau, \xi, w)={}_2F_1\left(\begin{matrix}\dfrac{m+\mu-\sigma}{2}, \dfrac{m+\mu+\sigma}{2}+1\\ 1+\mu\end{matrix}\,\middle|\,\frac{1-w}{2}\right)\tau^\mu\xi^m \tag{4.2}$$

are solutions of (2.2) with

$$I_{14}F=i\mu F, \qquad I_{23}F=\mathrm{im}\,F.$$

(For $(\sigma-\mu-m)/2=n=0, 1, 2, \cdots$ this solution is proportional to $P_n^{(\mu,m)}(w)\tau^m\xi^m$ where

$$P_n^{(\alpha,\beta)}(x)=\binom{n+\alpha}{n}\,{}_2F_1\left(\begin{matrix}-n, \alpha+\beta+n+1\\ \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right) \tag{4.3}$$

is a Jacobi polynomial.) An independent solution is

$$(4.4)\qquad G^{(1)}_{\mu,m}(\tau,\xi,w)={}_2F_1\left(\begin{matrix}\frac{m+\mu-\sigma}{2},\ \frac{m+\mu+\sigma}{2}+1\\ 1+m\end{matrix}\ \middle|\ \frac{1+w}{2}\right)\tau^\mu\xi^m$$

if $(m+\mu\pm\sigma)/2$, m, μ are all noninteger.

For system (3) with

$$(4.5)\qquad \eta=-e^{ix_3}\sin x_2,\qquad \rho=-\cos x_2,\qquad q=\cos x_1,$$

it follows that the functions

$$(4.6)\qquad F^{(3)}_{l,m}(\eta,\rho,q)=\eta^m(1-q^2)^{l/2}C^{m+1/2}_{l-m}(\rho)C^{l+1}_{\sigma-l}(q)$$

satisfy (2.2) and

$$I_{23}F=\text{im}\,F,\qquad (I^2_{12}+I^2_{13}+I^2_{23})F=-l(l+1)F.$$

Here

$$C^\nu_\alpha(z)=\frac{\Gamma(\alpha+2\nu)}{\Gamma(\alpha+1)\Gamma(2\nu)}\,{}_2F_1\left(\begin{matrix}\alpha+2\nu,-\alpha\\ \nu+1/2\end{matrix}\ \middle|\ \frac{1-z}{2}\right)$$

ia a Gegenbauer function, a polynomial if $\alpha=0,1,2,\cdots$. An independent set of solutions is

$$(4.7)\qquad G^{(3)}_{l,m}(\eta,\rho,q)=\eta^m(1-q^2)^{l/2}C^{m+1/2}_{l-m}(\rho)D^{l+1}_{\sigma-l}(q),$$

where

$$D^\nu_\alpha(z)=e^{i\pi\nu}\frac{\Gamma(\alpha+2\nu)}{\Gamma(\nu)\Gamma(\alpha+\nu+1)}(2z)^{-\alpha-2}{}_2F_1\left(\begin{matrix}\nu+\alpha/2,\nu+\alpha/2+1/2\\ \nu+\alpha+1\end{matrix}\ \middle|\ z^{-2}\right).$$

The functions $C^\nu_\alpha(z)$, $D^\nu_\alpha(z)$ are analytic in the complex plane cut from -1 to $-\infty$ and from $+1$ to $-\infty$, respectively, along the real axis.

Now suppose $H(\tau,\xi,w)$, variables (4.1), is a solution of (2.2) which can be expanded in a convergent Laurent series in τ, ξ and is analytic in $1+w$ in a neighborhood of $w=-1$. Then it follows by Wiesner's principle, [5], [14] that

$$H(\tau,\xi,w)=\sum_{\mu,m}C_{\mu,m}G^{(1)}_{\mu,m}(\tau,\xi,w).$$

This is a generating function for the $\{G^{(1)}_{\mu,m}\}$. We can evaluate the constants $C_{\mu,m}$ by choosing special values for the variables. Similarly, if H is analytic in $1-w$ in a neighborhood of $w=1$ we can expand in terms of the basis $\{F^{(1)}_{\mu,m}\}$. Also, by making use of known expansion theorems for Gegenbauer polynomials, e.g., [15, p. 238], we can expand solutions of (2.2) in series of functions $\{F^{(3)}_{l,m}\}$ or $\{G^{(3)}_{l,m}\}$ where $l=\sigma-n$, $n=0,1,2,\cdots$.

A convenient way of constructing these generating functions H is to choose them to be separated solutions of (2.2) corresponding to one of our twenty-one coordinate systems. In this manner one can derive a wide variety of generating functions. However, the generating functions will usually lead to double sums. Here we limit ourselves to single-sum generating functions for the bases (1) and

(3) by restricting the generating functions to be eigenfunctions of I_{23} with eigenvalue im. Thus the sum over m can be omitted.

For example, expressing the solution (4.6) in terms of the variables (4.1) in the case $\sigma-l=n=0,1,2,\cdots, l-m=k=0,1,2,\cdots,\sigma\in\mathbb{C}$ and expanding in terms of the basis $\{G^{(1)}_{\mu,m}\}$ we find

$$\begin{aligned}&\tau^n[16\tau^2+(2\tau^2+w-1)^2]^{k/2}C_k^{\sigma-k-n+1/2}\left(\frac{w-1-2\tau^2}{\sqrt{16\tau^2+(2\tau^2+w-1)^2}}\right)\\ (4.8)\qquad & C_n^{\sigma-n+1}\left(\frac{2\tau^2+w-1}{4\tau i}\right)=\sum_{s=0}^{n+k}b_s\tau^{2s}{}_2F_1\left(\begin{matrix}s-n-k,\sigma+s-n-k+1\\ \sigma-n-k+1\end{matrix}\,\middle|\,\frac{1+w}{2}\right).\end{aligned}$$

Setting $w=-1$ in this expression we obtain the generating function

$$\tau^n(-2\tau^2-2)^k\frac{(2\sigma-2k-2n)_k}{k!}C_n^{\sigma-n+1}\left(\frac{\tau^2-1}{2\tau i}\right)=\sum_{s=0}^{n+k}b_s\tau^{2s}$$

for the coefficients b_s. Similar but more complicated expansions can be obtained for $\sigma-l$, $l-m$ noninteger. Conversely, for μ and $(\sigma-m-\mu)/2=n$ nonnegative integers we can expand the basis functions (4.2) in terms of the basis (4.6) to obtain

$$\begin{aligned}(4.9)\qquad &(q-\rho\sqrt{q^2-1})^\mu P_n^{(\mu,m)}(-1+2(1-q^2)(1-\rho^2))\\ &=\sum_{s=0}^{2n+\mu}a_s(q^2-1)^{s/2}C_s^{m+1/2}(\rho)C_{2n+\mu-s}^{s+m+1}(q).\end{aligned}$$

Replacing ρ by $\xi(q^2-1)^{-1/2}$ and letting $q\to1$ we find in the limit

$$(1-\xi)^\mu P_n^{(\mu,m)}(-1+2\xi^2)=\sum_{s=0}^{2n+\mu}a_s\binom{s+m-\frac12}{s}\binom{2n+2m+\mu+s+1}{2n+\mu-s}\xi^s,$$

a simple generating function for the coefficients a_s. More generally, expanding a function (4.4) in terms of the basis functions (4.6) we find

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}\dfrac{m+\mu-\sigma}{2},\dfrac{m+\mu+\sigma}{2}+1\\ 1+m\end{matrix}\,\middle|\,(\rho^2-1)(q^2-1)\right)(q-\rho\sqrt{q^2-1})^\mu\\ (4.10)\qquad &=\sum_{s=0}^{\infty}a_s(q^2-1)^{s/2}C_{\sigma-m-s}^{m+s+1}(q)C_s^{m+1/2}(\rho)\end{aligned}$$

valid for all ρ, q such that $|\rho\pm\sqrt{\rho^2-1}|>|(q-1)/(q+1)|^{1/2}$ and q is not pure imaginary. To compute the coefficients we set $\rho=\xi(q^2-1)^{-1/2}$ in (4.10) and let $q\to1$:

$$\begin{aligned}&(1-\xi)^\mu{}_2F_1\left(\begin{matrix}\dfrac{m+\mu-\sigma}{2},\dfrac{m+\mu+\sigma}{2}+1\\ 1+m\end{matrix}\,\middle|\,\xi^2\right)\\ &=\sum_{s=0}^{\infty}a_s\binom{\sigma+m+s+1}{\sigma-m-s}\binom{m+s-\frac12}{s}(2\xi)^s,\qquad |\xi|<1.\end{aligned}$$

Since $[I_{14}, I_{23}]=0$ it follows that the function $\exp(\alpha I_{14})F^{(3)}_{l,m}$ is an eigenfunction of I_{23} with eigenvalue im. Thus one can expand this function in terms of the $\{F^{(3)}\}$ basis with only a single sum. Consider the case $m\in\mathbb{C}$, $l-m=n$, $\sigma-l=k$, n, $k=0,1,2,\cdots$. A straightforward computation yields

$$\exp(\alpha I_{14})F^{(3)}_{l,m}(\eta,\rho,q)=\eta^m C^m_n\left(\left[\frac{(1-q^2)(\rho^2-1)+1-h^2(\alpha)}{1-h^2(\alpha)}\right]^{1/2}\right)C^{m+n+1}_k(h(\alpha))$$
$$\cdot(1-h^2(\alpha))^{n/2}(1-q^2)^{m/2}$$

where

$$h(\alpha)=q\sqrt{1-\alpha^2}-\rho\alpha\sqrt{1-q^2}.$$

Thus,

$$\text{(4.11)}\quad C^{m+1/2}_n\left(\left[\frac{(1-q^2)(\rho^2-1)+1-h^2(\alpha)}{1-h^2(\alpha)}\right]^{1/2}\right)C^{m+n+1}_k(h(\alpha))(1-h^2(\alpha))^{n/2}$$
$$=\sum_{s=0}^{n+k}a_s(\alpha)(1-q^2)^{s/2}C^{m+1/2}_s(\rho)C^{m+s+1}_{n+k-s}(q).$$

To obtain a simpler expression for the coefficients $a_s(\alpha)$ we set $\rho=\xi(1-q^2)^{-1/2}$ in (4.11) and let $q\to 1$:

$$C^{m+1/2}_n\left(\frac{\alpha+\xi\sqrt{1-\alpha^2}}{\sqrt{\alpha^2(1-\xi^2)+2\alpha\xi\sqrt{1-\alpha^2}}}\right)C^{m+n+1}_k(\sqrt{1-\alpha^2}-\alpha\xi)$$
$$\cdot(\alpha^2(1-\xi^2)+2\alpha\xi\sqrt{1-\alpha^2})^{n/2}$$
$$=\sum_{s=0}^{n+k}a_s(\alpha)\binom{m-\frac{1}{2}}{s}\binom{2m+n+k+s+1}{n+k-s}(2\xi)^s.$$

These expressions become much more tractable in the special case $n=0$. For that case and $t=\sqrt{1-\alpha^2}$ we see that the left-hand side of (4.11) is symmetric in q and t. Thus

$$a_s(t)=b_s(1-t^2)^{s/2}\,C^{m+s+1}_{k-s}(t)$$

and it is easy to check that

$$C^{m+1}_k(qt+\rho\sqrt{(1-q^2)(1-t^2)})=\frac{\Gamma(2m+1)}{[\Gamma(m+1)]^2}$$
$$\text{(4.12)}\qquad\cdot\sum_{s=0}^{k}\frac{2^{2s}(k-s)![\Gamma(m+s+1)]^2}{\Gamma(k+2m+s+2)}(2m+2s+1)[(1-q^2)(1-t^2)]^{s/2}$$
$$\cdot C^{m+s+1}_{k-s}(q)C^{m+s+1}_{k-s}(t)C^{m+1/2}_s(\rho),\qquad m\in\mathbb{C}\quad k=0,1,2,\cdots$$

This is the addition theorem for Gegenbauer polynomials, [7, p. 178]. For $\sigma-l$ an arbitrary complex number one can obtain similar expansions for the bases $\{F^{(3)}\}$, $\{G^{(3)}\}$, [8].

Note that from the group-theoretic point of view, our last computation amounts to the determination of the matrix elements of the operator $\exp(\alpha I_{14})$ with respect to the basis $\{F^{(3)}_{l,m}\}$. Similarly one can compute the matrix elements of group operators $\exp(\sum_{i<j}\alpha_{ij}I_{ij})$ with respect to the $\{F^{(1)}\}$ and $\{G^{(1)}\}$ bases. Since these results are essentially contained in [12] and [16], we shall not reproduce them here.

For system (5) with

$$\tau = e^{ix_1}, \qquad r = x_2, \qquad \theta = x_3, \tag{4.13}$$

the functions

$$F^{(5)}_{\omega,m}(\tau, r, \theta) = \tau^{-1} e^{im\theta} J_{\sigma-1}(i\omega\tau^{-1}) J_m(r\omega) \tag{4.14}$$

satisfy (2.2) and

$$I_{23}F = imF, \qquad \mathcal{L}_1 F = -\omega^2 F.$$

Expanding $F^{(5)}_{\omega,m}$ in terms of the basis $\{G^{(1)}_{\mu,m}\}$ we obtain the identity ($\tau = t^{-1}$, $\beta = i\omega$, $\nu = -\sigma - 1$):

$$\begin{aligned} &t^{-\nu-m} J_\nu(\beta t) J_m\left(\beta t\sqrt{\frac{1+w}{2}}\right)\left(\frac{1+w}{2}\right)^{-m/2} \\ &\quad = \sum_{s=0}^{\infty} a_s t^{2s} {}_2F_1\left(\begin{matrix}-s, -\nu-s \\ m+1\end{matrix}\,\middle|\, \frac{1+w}{2}\right). \end{aligned} \tag{4.15}$$

To evaluate the coefficients a_s it is enough to set $w = -1$:

$$t^{-\nu} J_\nu(\beta t) \frac{(\beta/2)^m}{\Gamma(m+1)} = \sum_{s=0}^{\infty} a_s t^{2s}.$$

We see that (4.15) is equivalent to the well-known power series expansion for a product of Bessel functions [7, p. 11].

Expanding $F^{(5)}_{\omega,m}$ in terms of the basis $\{F^{(3)}_{l,m}\}$ we find

$$\begin{aligned} &(q-\rho\sqrt{q^2-1})^{-1} J_\nu\left(\frac{\omega}{q-\rho\sqrt{q^2-1}}\right) J_m\left(\frac{\sqrt{(q^2-1)(\rho^2-1)}}{q-\rho\sqrt{q^2-1}}\right) \\ &\quad \cdot (\rho^2-1)^{-m/2}(q^2-1)^{-m/2} \\ &\quad = \sum_{s=0}^{\infty} b_s (q^2-1)^{s/2} C^{m+s+1}_{-\nu-m-s-1}(q) C^{m+1/2}_s(\rho) \end{aligned} \tag{4.16}$$

convergent for the same values of ρ, q as (4.10). As usual, a simpler generating function for the b_s can be obtained by setting $\rho = i\xi(1-q^2)^{-1/2}$ and letting $q \to 1$. A more complicated identity results when one expands $\exp(\alpha I_{14})F^{(5)}_{\omega,m}$ in terms of $\{F^{(3)}\}$ basis functions.

The expansions in terms of the $\{F^{(3)}\}$ basis listed above and various generalizations of these expansions are all treated in a beautiful paper by Henrici [8]. He studied the equation

$$\left(\partial_{xx} - \frac{(2\sigma+1)}{x}\partial_x + \partial_{yy} + \frac{(2m+1)}{y}\partial_y\right)\Phi(x, y) = 0 \tag{4.17}$$

which can be obtained from the complex Laplace equation $\Delta_4\psi = 0$ by separating off two variables, and showed that this equation admits R-separable solutions

$$(4.18)\qquad (\xi-\eta)^{m-\sigma}(\xi^2-1)^{(\sigma-l)/2}C_{\sigma-l}^{l+1}\left(\frac{\xi}{\sqrt{\xi^2-1}}\right)C_{l-m}^{m+1/2}(\eta),$$

$$\xi=\frac{1+ww^*}{2\sqrt{ww^*}},\qquad \eta=\frac{w+w^*}{2\sqrt{ww^*}},\qquad w=\frac{x+iy-c}{x+iy+c},\qquad w^*=\frac{x-iy-c}{x-iy+c},\quad c \text{ const.}$$

He then developed an ingenious theory of expansions of analytic solutions of (4.17) in terms of the basis (4.18). Furthermore he observed that (4.17) permits separable solutions in coordinate systems analogous to (1) and (5) as well as (3) and derived generating functions for Gegenbauer functions by expanding each of these separated solutions as series in the basis (4.18).

Note that equation (4.17) and equation (2.2) with $I_{23}^2\Psi = -m^2\Psi$ each arise from the complex Laplace equation by separating off two variables. Moreover, in the next paper in this series we shall show that these two reduced equations are equivalent under the action of the local symmetry group $O(6, \mathbb{C})$ of the Laplace equation. Thus, every separable system for (2.2) is mapped to an R-separable system for (4.17) and conversely.

It follows that Henrici's analysis of (4.17) carries over to

$$(4.19)\qquad \Delta\Psi=\sigma(\sigma+2)\Psi,\qquad I_{23}^2\Psi=-m^2\Psi.$$

The local symmetry group of (4.19) consists of the operators $\exp(\alpha I_{14})$, $\alpha\in\mathbb{C}$, i.e., these operators map solutions into solutions. Thus if Ψ is a known analytic solution of (4.19) we can discuss the expansion of $\exp(\alpha I_{14})\Psi$ in terms of the bases $\{F^{(1)}\}$ and $\{F^{(3)}\}$. In Henrici's work, which concerns only expansions in the $\{F^{(3)}\}$ basis, this freedom is expressed by choosing a family of coordinate systems parametrized by a complex variable c. Systems corresponding to distinct values of c are equivalent under an appropriate symmetry operator $\exp(\alpha I_{14})$.

By inspection we see that (4.19) separates in five coordinate systems: (1), (3), (5), (13), (14). In his work on (4.17) Henrici employs R-separation in systems (1), (3) and (5), but he fails to note the R-separation in analogies of (13) and (14). (System (13) yields products of associated Lamé functions and will not be treated here. See, however, [1].)

For system (14) the functions

$$\begin{aligned}&F_{\alpha,m}^{(14)}(u_1,x_2,u_3)\\ (4.20)\qquad &=e^{mx_2}(1-u_1)^{\alpha-m/2}(1-u_3)^{\alpha-m/2}(u_1u_3)^{m/2}\\ &\cdot{}_2F_1\left(\begin{matrix}\alpha+\sigma/2+1,\alpha-\sigma/2\\ m+1\end{matrix}\,\middle|\,u_1\right){}_2F_1\left(\begin{matrix}\alpha+\sigma/2+1,\alpha-\sigma/2\\ m+1\end{matrix}\,\middle|\,u_3\right),\\ &\qquad u_1=\operatorname{th}^2x_1,\qquad u_3=\operatorname{th}^2x_3\end{aligned}$$

satisfy (2.2) and

$$I_{23}F=imF,\qquad \mathscr{L}_2F=4(\alpha-m/2)^2F.$$

Expanding $F^{(14)}_{\alpha,m}$ in terms of the basis $\{G^{(1)}_{\mu,m}\}$ for $\alpha-\sigma/2=-n$, $n=0,1,2,\cdots$, we find

$$(4.21)\qquad \begin{aligned} &{}_2F_1\left(\begin{matrix}\sigma-n+1,-n\\ m+1\end{matrix}\,\middle|\,u_1\right){}_2F_1\left(\begin{matrix}\sigma-n+1,-n\\ m+1\end{matrix}\,\middle|\,u_3\right)\\ &\quad=\sum_{s=0}^{n} b_s\tau^{2s}{}_2F_1\left(\begin{matrix}s-n,\sigma+s-n+1\\ m+1\end{matrix}\,\middle|\,\frac{1+w}{2}\right),\\ &u_{\substack{1\\3}}=\frac{w+3}{4}-\frac{\tau^2}{2}\mp\frac{1}{2}\left[\left(\frac{w+3}{2}-\tau^2\right)^2-(2w+2)\right]^{1/2}.\end{aligned}$$

Setting $w=-1$ we find

$$\begin{aligned}{}_2F_1\left(\begin{matrix}\sigma-n+1,-n\\ m+1\end{matrix}\,\middle|\,1-\tau^2\right)&=\frac{\Gamma(m+1)\Gamma(m+2n-\sigma)}{\Gamma(m+n+1)\Gamma(m+n-\sigma)}{}_2F_1\left(\begin{matrix}\sigma-n+1,-n\\ \sigma-2n-m+1\end{matrix}\,\middle|\,\tau^2\right)\\ &=\sum_{s=0}^{n} b_s\tau^{2s}.\end{aligned}$$

Similar but more complicated expressions can be obtained for $n\neq 0,1,2,\cdots$.

Expanding $F^{(14)}_{\alpha,m}$ in terms of the basis $\{F^{(3)}_{l,m}\}$ for $\alpha-\sigma/2=-n$, $2\alpha-m=k$, k, $n=0,1,2,\cdots$, we obtain

$$(4.22)\qquad \begin{aligned}&[q-\rho\sqrt{q^2-1}]^k\,{}_2F_1\left(\begin{matrix}k+m+n+1,-n\\ m+1\end{matrix}\,\middle|\,u_1\right){}_2F_1\left(\begin{matrix}k+m+n+1,-n\\ m+1\end{matrix}\,\middle|\,u_3\right)\\ &\quad=\sum_{s=0}^{k+2n} a_s(q^2-1)^{s/2}C^{m+s+1}_{k+2n-s}(q)C^{m+1/2}_s(\rho),\\ &u_{\substack{1\\3}}=(1-\rho^2+\rho^2q^2+\rho q\sqrt{q^2-1})\\ &\qquad\pm[(1-\rho^2+\rho^2q^2+\rho q\sqrt{q^2-1})^2-(q^2-1)(\rho^2-1)]^{1/2}.\end{aligned}$$

A simpler generating function for the coefficients a_s can be found by setting $\rho=\xi(q^2-1)^{-1/2}$ and letting $q\to 1$.

For our final example we consider system (16) with basis functions

$$(4.23)\qquad \begin{aligned}&F^{(16)}_{\lambda,n}(x_1,x_2,x_3)\\ &\quad=\exp[i\lambda x_2+\sqrt{\lambda}(x_3^2-x_1^2)/2](x_1x_3)^{\sigma+2}L_n^{(\sigma+1)}(\sqrt{\lambda}x_1^2)L_n^{(\sigma+1)}(\sqrt{\lambda}x_3^2)\end{aligned}$$

where $L^{(\alpha)}_\nu(x)$ is a generalized Laguerre function, a polynomial if $n=0,1,2,\cdots$, [17, p. 268]. These functions satisfy the operator equations

$$(I_{42}+iI_{21})F=i\lambda F,\qquad \mathscr{L}_2F=-2\sqrt{\lambda}(2n+\sigma+2)F.$$

Note that the operator $K=I_{34}+iI_{13}=(x_1^2+x_3^2)^{-1}(x_3\partial_{x_3}-x_1\partial_{x_1})$ commutes with $I_{42}+iI_{21}$. Thus the function $\exp(\alpha K)F^{(16)}_{n,k}(x_1,x_2,x_3)$, $k=0,1,2,\cdots$, can be expanded in a series of functions (4.23) with λ fixed and n running over the nonnegative integers. The result is

$$L_k^{(\sigma+1)}\left(\frac{x_1^2}{2}-\frac{x_3^2}{2}-\alpha+\mathcal{R}\right)L_k^{(\sigma+1)}\left(\frac{x_3^2}{2}-\frac{x_1^2}{2}+\alpha+\mathcal{R}\right)$$

$$= \sum_{s=0}^{k} a_s L_s^{(\sigma+1)}(x_1^2) L_s^{(\sigma+1)}(x_3^2) \tag{4.24}$$

$$2\mathcal{R} = [(x_1^2 - x_3^2 - 2\alpha)^2 + 4x_1^2 x_3^2]^{1/2}.$$

(We choose the square root so that $2\mathcal{R} = x_1^2 + x_3^2$ when $\alpha = 0$.) For evaluation of the coefficients a_s it is enough to set $x_3 = 0$:

$$\binom{k+\sigma+1}{k} L_k^{(\sigma+1)}(x_1^2 - 2\alpha) = \sum_{s=0}^{k} a_s \binom{s+\sigma+1}{s} L_s^{(\sigma+1)}(x_1^2),$$

$$a_s = \binom{k+\sigma+1}{k}\binom{s+\sigma+1}{s}^{-1} L_{k-s}^{(-1)}(-2\alpha).$$

5. Integral representations for separated solutions. In analogy with a construction in [1] we can represent solutions of (2.2) as analytic functions on the complex sphere S_{2c}. Indeed, let $f(\mathbf{w})$ be analytic on $S_{2c}: w_1^2 + w_2^2 + w_3^2 = 1$, $w_1 = (1 - w_2^2 - w_3^2)^{1/2}$ and let $F(\mathbf{z})$ be a function on S_{3c} defined by

$$F(\mathbf{z}) = \mathcal{I}[f] = \iint_{\mathcal{D}} [w_1 z_1 + w_2 z_2 + w_3 z_3 + i z_4]^\sigma f(\mathbf{w}) \frac{dw_2\, dw_3}{w_1} \tag{5.1}$$

were $\mathcal{D}$ is a complex two-dimensional Riemann surface over w_2-w_3 space. We assume that the integration surface $\mathcal{D}$ and the analytic function f are chosen such that $\mathcal{I}[f]$ converges absolutely and arbitrary differentiation with respect to $z_1, \cdots, z_4$ is permitted under the integral sign. It follows that $F(\mathbf{z})$ is a solution of (2.2). (In fact F is a solution of the Laplace equation $\Delta_4 F = 0$ which is homogeneous of degree σ in $\mathbf{z}$.) Integrating by parts, we find that the operators I_{jk}, (3.1), acting on the solution space of (2.2) correspond to the operators

$$\begin{gathered} I_{12} = w_1 \partial_{w_2} - w_2 \partial_{w_1}, \qquad I_{13} = -w_3 \partial_{w_1}, \qquad I_{23} = -w_3 \partial_{w_2}, \\ I_{41} = -i(\sigma+2) w_1 + i(1 - w_1^2)\partial_{w_1} - i w_1 w_2 \partial_{w_2}, \\ I_{42} = -i(\sigma+2) w_2 - i w_2 w_1 \partial_{w_1} + i(1 - w_2^2)\partial_{w_2}, \\ I_{43} = -i(\sigma+2) w_3 - i w_3 w_1 \partial_{w_1} - i w_3 w_2 \partial_{w_2} \end{gathered} \tag{5.2}$$

acting on the analytic functions $f(\mathbf{w})$, provided $\mathcal{D}$ and f are chosen such that the boundary terms vanish:

$$I_{jk} F = \mathcal{I}(I_{jk} f).$$

The point of this construction is that we can use the operators (5.2) to compute an eigenfunction $f_{\lambda\mu}$:

$$\mathcal{L}_1 f = \lambda f, \qquad \mathcal{L}_2 f = \mu f,$$

where $\mathcal{L}_1$, $\mathcal{L}_2$ are the operators characterizing one of the separable systems

(1)–(21). It follows that the integral $F_{\lambda\mu} = \mathcal{I}(f_{\lambda\mu})$ is a solution of (2.2) which satisfies

$$\mathcal{L}_1 F = \lambda F, \qquad \mathcal{L}_2 F = \mu F,$$

where now $\mathcal{L}_1, \mathcal{L}_2$ are expressed in terms of the operators (3.1). Thus F must be a separable solution of (2.2) in the coordinates to which $\mathcal{L}_1, \mathcal{L}_2$ correspond. This fact enables us to evaluate the integral to within a few normalization constants which are determined by inspection. Thus, this procedure leads to integral representations for the separable solutions of (2.2).

We illustrate the method with a single example treated in detail. We adopt complex coordinates α, η on S_{2c} such that

$$(w_1, w_2, w_3) = (\cos\alpha, \sin\alpha\cos\eta, \sin\alpha\sin\eta),$$
$$\frac{dw_2\, dw_3}{w_1} = \sin\alpha\, d\alpha\, d\eta. \tag{5.3}$$

These coordinates will prove useful in the construction of integral representations for separable systems in which the operator $I_{23} = \partial_\eta$ is diagonalized. If $f(\alpha, \eta)$ satisfies $I_{23} f = imf$ then $f = h(\alpha)t^m$ where $t = e^{i\eta}$. We choose the integration surface in the form $\mathcal{D} = C_1 \times C_2$ where C_1 is the interval $[0, \pi]$ in the α-plane and C_2 is a simple closed curve surrounding the origin in the t-plane. Performing the t-integration and making use of the standard generating function for Gegenbauer polynomials [17, p. 175], we find

$$\begin{aligned} F_m(z) = \mathcal{I}[f] &= -i \int_0^\pi h(\alpha) \oint \left[iz_4 + z_1\cos\alpha + \frac{z}{2}\sin\alpha\left(\frac{u}{t} + \frac{t}{u}\right)\right]^\sigma t^{m-1}\, dt\, d\alpha \\ &= 2\pi\left(\frac{z}{2}\right)^\sigma u^m \int_0^\pi (\sin\alpha)^{\sigma+1} C_{\sigma-m}^{-\sigma}\left[\frac{-z_1\cos\alpha - iz_4}{z\sin\alpha}\right] h(\alpha)\, d\alpha \end{aligned} \tag{5.4}$$

is a solution of (2.2) such that $I_{23}F = imF$. Here

$$z_2 = z\left(\frac{u+u^{-1}}{2}\right), \qquad z_3 = z\left(\frac{u-u^{-1}}{2i}\right), \qquad z_1^2 + z_4^2 + z^2 = 1$$

and we assume that σ, m are complex numbers such that $\sigma - m = n = 0, 1, 2, \cdots$.

The requirement that $f(\alpha, \eta) = h(\alpha)t^m$ satisfy the system (3) eigenvalue equations

$$I_{23} f = imf, \qquad (I_{12}^2 + I_{13}^2 + I_{23}^2) f = -l(l+1) f$$

leads to a family of solutions

$$h(\alpha) = (\sin\alpha)^m C_{l-m}^{m+1/2}(\cos\alpha). \tag{5.5}$$

Substituting this expression into (5.4) under the assumptions

$$\sigma - l = k, \qquad l - m = n, \qquad \sigma \in \mathbb{C}, \qquad k, n = 0, 1, 2, \cdots,$$
$$\operatorname{Re} m > 0, \qquad \operatorname{Re}(m + \sigma) > 0$$

and using the fact that variables must separate in the resulting integral if coordinates (3) are employed, we obtain the identity

$$
\begin{aligned}
&AC_k^{\sigma-k+1}(\cos x_1)C_n^{\sigma-k-n+1/2}(\cos x_2)\\
(5.6)\quad &= (\sin x_1)^k(\sin x_2)^{k+n}\\
&\quad\cdot\int_0^\pi (\sin\alpha)^{2\sigma-k-n+1}C_{k+n}^{-\sigma}(i\cot x_1\csc x_2\csc\alpha\\
&\qquad\qquad +\cot x_2\cot\alpha)C_n^{\sigma-k-n+1/2}(\cos\alpha)\,d\alpha,
\end{aligned}
$$

where A is a constant to be determined. To evaluate A we first let $x_2\to 0$ and obtain

$$
\begin{aligned}
&AC_n^{\sigma-k-n+1/2}(1)C_k^{\sigma-k+1}(\cos x_1)\\
(5.7)\quad &=\frac{(\sin x_1)^k\Gamma(k+n-\sigma)}{\Gamma(-\sigma)(n+k)!}2^{n+k}\\
&\quad\cdot\int_0^\pi(\sin\alpha)^{2\sigma-2k-2n+1}C_n^{\sigma-k-n+1/2}(\cos\alpha)(i\cot x_1+\cos\alpha)^{k+n}\,d\alpha,
\end{aligned}
$$

an identity which is apparently due to Durand [18]. Finally, letting $x_1\to 0$ and using the orthogonality relations for Gegenbauer polynomials we obtain

$$A=(-1)^{k+n}(i)^k 2^{2\sigma-k+1}\frac{\Gamma(\sigma+1)\Gamma(\sigma-k)}{\Gamma(2\sigma-k+2)}.$$

By varying the eigenfunctions (5.5) and the integration surface $\mathscr{D}$ one can find a variety of such identities. In each case the integral must separate in coordinates (3) and this permits easy evaluation. Similar remarks hold for each of the twenty-one separable systems.

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