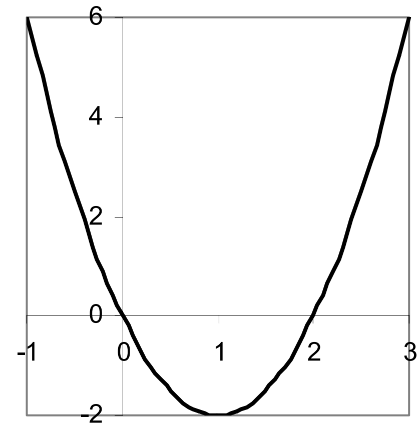
Solutions to First Midterm Exam

1. The graph of the function $f(x) = 2x^2 - 4x$ is a parabola with vertex $(x, y) = (1, -2)$, as shown in the graph at the right. Where is the vertex of the parabola $y = f(x+1) + 1$?

(a) (1,2)
(b) (2,1)
(c) (0,-1)
(d) (2,-1)
(e) None of the above.



The answer is (c). The graph of $y = f(x+1) + 1$ is simply the graph of $y = f(x)$ moved to the left one unit and up one unit. The vertex moves from $(x, y) = (1, -2)$ to $(x, y) = (0, -1)$.

2. When plotted on a log-log scale, the function $y = f(x)$ is a straight line whose slope is 1.7 and whose y-intercept is -2. More precisely, x and y are related by the equation

$$\log_{10} y = 1.7 \log_{10} x - 2 .$$

Which of the following expresses the functional relation between x and y?

(a) $y = -2 \cdot 10^{1.7x}$
(b) $y = \frac{1}{100} \cdot 10^{1.7x}$
(c) $y = -2 \cdot x^{1.7}$
(d) $y = \frac{1}{100} x^{1.7}$
(e) None of the above.

The answer is (d).

$$\begin{aligned} \log_{10} y &= 1.7 \log_{10} x - 2 \\ &= \log_{10} x^{1.7} - \log_{10} 100 \\ &= \log_{10}\left(\frac{x^{1.7}}{100}\right) \end{aligned}$$

Therefore, $y = \frac{1}{100} x^{1.7}$.

3. Compute the following limits.

a. $\lim\limits_{x\to 0}\dfrac{2x^2+3x}{x^2-x}$

$$\lim_{x\to 0}\frac{2x^2+3x}{x^2-x}=\lim_{x\to 0}\frac{2x(x+3/2)}{x(x-1)}=\lim_{x\to 0}\frac{2(x+3/2)}{x-1}=\frac{2(0+3/2)}{0-1}$$
$$=-3$$

b. $\lim\limits_{x\to 0}\dfrac{\sqrt{3x+1}-1}{2x}$

$$\lim_{x\to 0}\frac{\sqrt{3x+1}-1}{2x}=\lim_{x\to 0}\frac{\sqrt{3x+1}-1}{2x}\times\frac{\sqrt{3x+1}+1}{\sqrt{3x+1}+1}$$
$$=\lim_{x\to 0}\frac{3x+1-1}{2x(\sqrt{3x+1}+1)}=\lim_{x\to 0}\frac{3}{2(\sqrt{3x+1}+1)}$$
$$=\frac{3}{4}$$

c. $\lim\limits_{x\to 1}\dfrac{2-x-x^2}{1-x}$

$$\lim_{x\to 1}\frac{2-x-x^2}{1-x}=\lim_{x\to 1}\frac{(2+x)(1-x)}{1-x}=\lim_{x\to 1}(2+x)$$
$$=3$$

4. A population of bacteria is growing exponentially. At the beginning of the experiment, the mass of the population is 1 milligram (10^{-3} gram). Three hours later, the mass of the population is 1.5 milligrams. Assume that the same exponential growth continues indefinitely

a. What will be the mass of the population 5 days after the beginning of the experiment?

Let $M(t)$ be the mass (in grams) of the population at time t (in hours from the beginning of the experiment). Then $M(t) = M_0 a^t$, where $M_0 = 10^{-3}$, and where a is a constant to be determined. Since $M(3) = 1.5\times10^{-3}$, it follows that $10^{-3}a^3 = 1.5\times10^{-3}$, so $a = 1.5^{1/3}$. Five days is 120 hrs, so the mass of the population after 5 days is

$$M(120) = M_0 a^{120} = 10^{-3}(1.5)^{120/3} \approx 11{,}000$$

So the answer is about 11,000 grams, or 11 kilograms.

b. How long will it take for the mass of the population to reach 6,000,000,000,000,000,000,000 metric tons (the approximate mass of the earth)? (1 metric ton = 10^6 grams)

We must find the value of t so that $M(t) = M_0 a^t = 6\times10^{27}$. This means that $a^t = 1.5^{t/3} = 6\times10^{30}$, so

$$(t/3)\log_{10}1.5 = 30 + \log_{10}6,$$

$$t = \frac{90 + 3\log_{10}6}{\log_{10}1.5} \approx 524\,.$$

So the answer is about 524 hrs, or 21days, 20 hours.

5. Find the center and radius of the circle whose equation is $x^2 + y^2 - 8x + 6y = 0$.

We must complete the squares:

$$\begin{aligned} &x^2 + y^2 - 8x + 6y = 0 \\ &x^2 - 8x + y^2 + 6y = 0 \\ &x^2 - 8x + 16 + y^2 + 6y + 9 = 16 + 9 = 25 \\ &(x-4)^2 + (x+3)^2 = 5^2 \end{aligned}$$

So the center is at $(x, y) = (4,-3)$, and the radius is 5.

6. Let $f(x)=\dfrac{x-2}{x-1}$, and let $g(x)=\dfrac{x^2}{x^2+1}$.

a. Find $f\circ g$**.**

$$f\circ g(x)=\frac{g(x)-2}{g(x)-1}=\frac{\dfrac{x^2}{x^2+1}-2}{\dfrac{x^2}{x^2+1}-1}=\frac{x^2-2x^2-2}{x^2-x^2-1}=x^2+2$$

b. Find f^{-1}**.**

We must solve for x: $y=f(x)=\dfrac{x-2}{x-1}$.

$$\begin{aligned} y(x-1)&=x-2\\ xy-x&=y-2\\ x(y-1)&=y-2\\ x&=\frac{y-2}{y-1}\end{aligned}$$

So $f^{-1}(y)=\dfrac{y-2}{y-1}$, or $f^{-1}(x)=\dfrac{x-2}{x-1}$

7. Solve for x:

a. $e^x=2+\dfrac{3}{e^x}$

$$(e^x)^2-2e^x-3=0$$
$$(e^x-3)(e^x+1)=0$$

So $e^x=3$, or $e^x=-1$. Since the latter is impossible, we must have $e^x=3$, or $x=\ln 3$.

b. $\ln(2x-3)-\ln(x+1)=0$

$$\ln(2x-3)=\ln(x+1)$$
$$2x-3=x+1$$
$$x=4$$