SEPARATION OF VARIABLES METHODS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS IN MATHEMATICAL PHYSICS

E.G. KALNINS* AND W. MILLER, JR.†**

ABSTRACT. In these two expository talks we shall first review briefly the technique of separation of variables and its (intrinsic) group-theoretic significance for some fundamental scalar PDEs of mathematical physics: Hamilton-Jacobi, Helmholtz, Schrödinger, etc. The main emphasis, however, will be on recent progress in the analysis of these methods for systems of equations, where the proper definition of variable separation and its intrinsic significance are not yet clear. We report briefly on the work of Bagrov et al., Shapovalov et al., Kamran et al., Shishkin et al., and of the authors and Williams among others; and present examples including the Dirac equation (spin 1/2) and the Maxwell equations (spin 1) for the Minkowski, Robertson-Walker and Kerr metrics, and others.

1. Preliminary definitions of separability (scalar equations). These lectures constitute a presentation of some of the basic concepts in the theory of separation of variables for partial differential equations and comments on recent progress in this field. We will pay particular attention to variable separation for systems of equations, the part of the field which is of most active current interest.

The primary use of variable separation is for computation of explicit solutions of partial differential equations [12, 53]. The solutions can be calculated by solving ordinary differential equations (the separation equations). Many of the solutions obtained by this method prove so important that these functions are studied and tabulated in their own right: the special functions of mathematical physics. For Hamilton-Jacobi equations variable separation is used to obtain complete integrals which in turn lead to explicit solutions of the associated Hamiltonian system. For linear equations the Fourier method can be used to solve boundary value problems and represent a wide variety of functions as sums or integrals of separated solutions.

Intuitively, a (scalar) partial differential equation ($u_i = \partial_{x^i} u$)

$$H(x^i, u, u_i, u_{ij}, u_{ijk}, \cdots) = E \qquad 1 \le i, j, k, \cdots \le N$$

is said to be (additively) separable in the independent variables $x_1, \ldots, x_n$ if the equation admits a nontrivial solution of the form $u = \sum_{i=1}^{n} S^{(i)}(x_i)$. One can also talk about product separation $v = \prod_{i=1}^{n} T^{(i)}(x_i)$ or more complicated types of

*Mathematics Department, University of Waikato, Hamilton, New Zealand.
†Supported in part by the National Science Foundation under grant DMS 88–23054
**School of Mathematics, and Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, MN 55455

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separation such as $w = \tan[\sum_{i=1}^{n} S^{(i)}(x_i)]$. However a change of dependent variable reduces these other types to additive separation, e.g., $u = \ln v$ or $u = \arctan w$.

This intutive approach to separation does not, by itself, give one the practical tools to actually compute and classify separable solutions. A more technical (but nonintuitive) approach to separation is to postulate the (ordinary differential) separation equations. As an example we review the classical construction of orthogonal separable systems for the Hamilton-Jacobi equation [70, 71]:

$$\sum_{i=1}^{N} H_i^{-2} u_i^2 = E. \tag{1.1}$$

Here, $ds^2 = \sum_{i=1}^{N} H_i^2 (dx^i)^2$ is the metric corresponding to the orthogonal cooordinates x^i. We assume additive separation in these coordinates, so that $\partial_j u_i = \partial_j \partial_i u = 0$ for $i \neq j$. We postulate that u satisfies the separation equations:

$$u_i^2 + \sum_{j=1}^{N} s_{ij}(x^i)\lambda^j = 0, \qquad i = 1, \cdots, N \quad \lambda^1 = -E. \tag{1.2}$$

Here $\partial_k s_{ij}(x^i) = 0$ for $k \neq i$ and $\det(s_{ij}) \neq 0$. We say that $S = (s_{ij})$ is a *Stäckel matrix.*

Then (1.1) can be recovered from (1.2) provided $H_j^{-2} = (S^{-1})^{1j}$. Moreover, it is easy to verify that the quadratic forms $\mathcal{H}^\ell = \sum_{j=1}^{N} (S^{-1})^{\ell j} u_j^2$ satisfy $\mathcal{H}^\ell = -\lambda^\ell$ for a separable solution.

Furthermore, setting $u_i = p_i$, we have

$$\{\mathcal{H}^\ell, \mathcal{H}^j\} = 0, \qquad \ell \neq j$$

where

$$\{\mathcal{H}, \mathcal{K}\} = \sum_{i=1}^{N} (\partial_{x^i}\mathcal{H}\partial_{p_i}\mathcal{K} - \partial_{x^i}\mathcal{K}\partial_{p_i}\mathcal{H})$$

is the Poisson Bracket. Thus the $\mathcal{H}^\ell, \quad 2 \leq \ell \leq N$, are *constants of the motion* for the *Hamiltonian* $\mathcal{H}^1$. We see that the Stäckel construction is intimately associated with quadratic symmetries of the Hamilton-Jacobi equation (1.1) [57].

Similar constructions apply to 2nd-order linear PDEs and lead to 2nd-order symmetry operators, i.e., 2nd-order operators mapping solutions of these equations to solutions, [24, 25, 35].

2. Compromise approach to additive separation. The following approach to separation starts from the intuitive concept and leads naturally to the separation equations. We look for (additively separable solutions) of the equation

$$H(x^i, u, u_i, u_{ii}, u_{iii}, \cdots) = E, \qquad 1 \leq i \leq N \tag{2.1}$$

in the coordinates x^i. (Here, $u_i = \partial_{x^i} u$, $u_{ii} = \partial_{x^i x^i} u$, $\cdots$, and $u_{ij} \equiv 0$ for $i \neq j$.) We look for solutions of the form:

$$u = \sum_{i=1}^{N} S^{(i)}(x^i)$$

Introducing new notation, let

$$u_{i,1} \equiv u_1, \quad u_{i,j+1} \equiv \partial_{x^i} u_{i,j}, \qquad j = 1, 2, \cdots,$$

let m_i be the largest integer ℓ such that $\partial_{u_{i,\ell}} H = H_{u_{i,\ell}} \not\equiv 0$ and let D_i be the total differentiation operator

$$D_i = \partial_{x^i} + u_{i,1}\partial_u + u_{i,2}\partial_{u_{i,1}} + \cdots + u_{i,m_i+1}\partial_{u_{i,m_i}} + \cdots .$$

Then the equation $D_i H(x, u) = 0$ implies

$$u_{i,m_i+1} = -\frac{\tilde{D}_i H}{H_{u_{i,m_i}}}, \qquad i = 1, 2, \cdots, N, \tag{2.2}$$

where

$$\tilde{D}_i = \partial_{x^i} + u_{i,1}\partial_u + u_{i,2}\partial_{u_{i,1}} + \cdots + u_{i,m_i}\partial_{u_{i,m_i-1}}.$$

It follows that u satisfies the integrability conditions $D_j u_{i,m_i+1} = 0, \quad j \neq i$, or

$$\begin{aligned} H_{u_{i,m_i}} H_{u_{j,m_j}} (\tilde{D}_i \tilde{D}_j H) + H_{u_{i,m_i} u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H) \\ = H_{u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H_{u_{i,m_i}}) + H_{u_{i,m_i}} (\tilde{D}_j H)(\tilde{D}_i H_{u_{j,m_j}}). \end{aligned} \tag{2.3}$$

Conversely we have the result:

Theorem 1 [54, 55]. *If conditions (2.3) are satisfied identically (for $i \neq j$) in the dependent variables $u, u_{k,\ell}$, then the partial differential equation $H = E$ admits a $\sum_{i=1}^{N} m_i + 1$ parameter family of separable solutions.*

The following example, orthogonal R-separation for the Helmholtz equation, shows how the requirement that equations (2.3) are satisfied identically leads to the technical separation equations. Consider the Helmholtz (Schrödinger, Klein-Gordon) equation

$$\Delta_N \Psi(x) = E\Psi(x). \tag{2.4}$$

Here, Δ_N is the Laplacian on a Riemannian or pseudo-Riemannian manifold, expressed in an orthogonal coordinate system x^i:

$$\Delta_N = \frac{1}{\sqrt{H_1 \cdots H_N}} \sum_{i=1}^{N} \partial_{x^i}(H_1 \cdots H_N H_i^{-2} \partial_{x^i})$$

where $ds^2 = \sum_{i=1}^{N} H_i^2 (dx^i)^2$. We look for multiplicative R-separation of this equation:

$$\Psi = \exp R(x) \prod_{i=1}^{N} \Psi_{(i)}(x^i).$$

Here, $R(x)$ is a fixed function to be determined. (If $R = 0$ we have ordinary multiplicative separation.) Now set $u = R - \ln \Psi$ to get a standard PDE which we can test for additive separation:

$$\begin{aligned} \sum_{i=1}^{N} [H_i^{-2}(u_{ii} + u_i^2) + (2H_i^{-2}\partial_i R + s_i)u_i + H_i^{-2}(\partial_{ii} R + (\partial_i R)^2 + s_i \partial_i R)] \\ = E. \end{aligned} \tag{2.5}$$

Here,

$$s_i = \frac{1}{H_1 \cdots H_N} \partial_i (H_1 \cdots H_N H_i^{-2}).$$

Substituting (2.5) into the integrability conditions (2.3) and equating coefficients of the derivative terms we find:

(1) Coeff. of u_i^2: The H_i^{-2} are in Stäckel form. Indeed the conditions are

$$\partial_{jk} H_i^{-2} = \partial_j H_i^{-2} \partial_k \ln H_j^{-2} + \partial_k H_i^{-2} \partial_j \ln H_k^{-2}, \quad j \neq k, \tag{2.6}$$

the Levi-Civita separability conditions which are well known to be the necessary and sufficient conditions for Stäckel form [24].

(2) Coeff. of u_{ii}: Determines R.

(3) Coeff. of 1: Generalized Robertson conditions. These are the extra conditions that an additively separable orthogonal system for the Hamilton-Jacobi equation (1.1) must satisfy in order that it also be multiplicatively separable for (2.4), [35, 37]. If $R \equiv 0$ these conditions are simply that the space must be Ricci flat, [24, 25].

We see that all R-separable solutions of (2.4) follow from the Stäckel construction.

In general, we have the following results for scalar PDEs, [1, 13, 14, 17, 30-34, 36, 37, 39, 40, 59, 62, 74]:

Equation	Type of Separation
Hamilton-Jacobi	additive separation
Helmholtz or Klein-Gordon	multiplicative R-separation
Laplace or wave	multiplicative R-separation
heat/time-dependent Schrödinger	multiplicative R-separation

(1) All separation is determined via the Stäckel procedure.
(2) Separation can be characterized via the symmetry operators for the equation.
(3) All separable systems can (in principle) be classified.
(4) The theory applies to N-dimensional Riemannian and pseudo-Riemannian manifolds and to both orthogonal and non-orthogonal separation.

Variable separation for these scalar equations can be characterized intrinsically, i.e., there is a coordinate free characterization of coordinates, [32, 37, 61-64]. For example, for the Hamilton-Jacobi and Helmholtz equations the relevant theorems are:

Theorem 2. *Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system $\{x^i\}$ for the Hamilton-Jacobi equation $\mathcal{H}^1 = E$ on an N-dimensional pseudo-Riemannian manifold are that there exist N quadratic forms $\mathcal{H}^k = \sum_{i,j=1}^N H_{ij}^{(k)} p_i p_j$ on the manifold such that:*

(1) $\{\mathcal{H}^k, \mathcal{H}^\ell\} = 0, \quad 1 \le k, i \le N,$
(2) *The set $\{\mathcal{H}^k\}$ is linearly independent (as N quadratic forms).*
(3) *There is a basis $\{\omega_{(j)} : 1 \le j \le N\}$ of simultaneous eigenforms for the $\{\mathcal{H}^k\}$.*

If conditions (1)-(3) are satisfied then there exist functions $g^i(x)$ such that:

$$\omega_{(j)} = g^j dx^j, \quad j = 1, \cdots, N.$$

Theorem 3. *Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system $\{x^i\}$ for the Helmholtz equation $\Delta_N\Psi = E\Psi$ on an N-dimensional pseudo-Riemannian manifold are that there exists a linearly independent set $\{A_1 = \Delta_N, A_2, \cdots, A_N\}$ of 2nd-order differential operators on the manifold such that:*

(1) $[A_k, A_\ell] = 0, \quad 1 \le k, \ell \le N,$
(2) *Each A_k is in self-adjoint form,*
(3) *There is a basis $\{\omega_{(j)} : 1 \le j \le N\}$ of simultaneous eigenforms for the $\{A_k\}$.*

If conditions (1)-(3) are satisfied then there exist functions $g^i(x)$ such that:

$$\omega_{(j)} = g^j dx^j, \quad j = 1, \cdots, N.$$

We mention also that these methods are sufficiently powerful to allow us to derive lists of all separable coordinate systems for a given Riemannian or pseudo-Riemannian space, [25, 30, 31, 39, 40].

3. Separation for systems of equations. For systems of PDEs there are a number of interesting and important examples of variable separation but, as yet, no agreed upon general definition of variable separation. Moreover, the general mechanism for variable separation for systems is not yet known. For systems of Dirac type with 1st-order separation equations we can develop a theory analogous to the scalar theory, and we proceed to this study first. (One of the authors developed some of this material in the expository paper [55]. We have since learned that the principal results were known much earlier [67, 68].) Consider the system

$$\mathbf{H}\psi = E\psi, \qquad E \quad \text{constant}, \tag{3.1}$$

where $\mathbf{H} = \sum_{i=1}^n H^i(x)\partial_i + V(x)$; H^i, V are $N \times N$ matrices; and ψ is an N-component spinor. We require that H^i, $1 \le i \le n$, are nonsingular matrices.

We define a *γ-integrable system* for $\mathbf{H}\psi = E\psi$ as a set of equations

$$\partial_i\psi = (\sum_{j=1}^n C_{ij}(x)\lambda^j - C_i(x))\psi \qquad i = 1, \ldots, n.$$

where the $C_{ij}(x)$, $C_i(x)$ are $N \times N$ matrices such that $\det(C_{ij}) \neq 0$, the λ^j are independent parameters with $\lambda^1 = E$, and for every initial point x^0 and N-spinor ξ there is a solution $\psi(x)$ of $\mathbf{H}\psi = E\psi$ with the property $\psi(x^0) = \xi$. The integrability conditions $\partial_i(\partial_j\psi) = \partial_j(\partial_i\psi), \quad i \neq j$, imply

$$\begin{aligned} &a) \quad C_{jk}C_{i\ell} + C_{j\ell}C_{ik} = C_{ik}C_{j\ell} + C_{i\ell}C_{jk}, \\ &b) \quad \partial_j C_{ik} - C_{ik}C_j - C_iC_{jk} = \partial_i C_{jk} - C_{jk}C_i - C_jC_{ik}, \\ &c) \quad \partial_i C_j + C_iC_j = \partial_j C_i + C_jC_i. \end{aligned} \tag{3.2}$$

Let the $nN \times nN$ bordered matrix $A = (A^{ij})$ be the inverse of (C_{jk}) :

$$\sum_{j=1}^n A^{ij}C_{jk} = \sum_{j=1}^n C_{kj}A^{ji} = \delta^i_k \mathfrak{I}. \tag{3.3}$$

It follows that the solutions ψ satisfy the eigenvalue equations

$$A^k\psi \equiv \sum_{i=1}^n A^{ki}(x)\partial_i\psi + B^k(x)\psi = \lambda^k\psi, \quad 1 \le k \le n.$$

Theorem 4 [67]. *The integrability conditions for the separation equations are satisfied identically iff there exist $N \times N$ matrices $A^{ki}(x), B^k(x)$, $1 \le k, i \le n$ such that: (1) The operators $\mathbf{A}^k = \sum_{i=1}^N A^{ki}\partial_{x^i} + B^k$, $k = 1, \cdots, N$ commute, i.e.,*

$$\mathbf{A}^k \mathbf{A}^\ell = \mathbf{A}^\ell \mathbf{A}^k, \tag{3.4}$$

(2) $\mathbf{H} = \mathbf{A}^1$, and (3) $\sum_{i=1}^N A^{ki} C_{ij} = \mathcal{I}\delta_j^k$, where $\mathcal{I}$ is the $N \times N$ identity matrix.

The basic relation for 1st-order separation of $N \times N$ systems needed to prove the equivalence of (3.2) and (3.4) is:

$$\sum_\ell (A^{ij}A^{k\ell} + A^{i\ell}A^{kj})(C_{js}C_{\ell t} + C_{jt}C_{\ell s}) = 2(\delta_s^k \delta_t^i + \delta_t^k \delta_s^i)\mathcal{I} \tag{3.5}$$

where $\mathcal{I}$ is the $N \times N$ identity matrix. This relation can be easily verified from (3.2) and (3.3). We see that γ-integrable systems are associated with sets of commuting 1st-order matrix symmetry operators for (3.1), in a manner similar to that in the scalar theory.

A γ-integrable system is *separable* if (by a change of frame $\psi = R\psi'$ if necessary) the factorization equations take the form

$$\partial_i \psi' = (\sum_{j=1}^n C_{ij}(x^i)\lambda^j - C_i(x^i))\psi' \qquad i = 1, \ldots, n.$$

in a *particular* coordinate system $\{x^1, \ldots, x^n\}$, i.e., $\partial_\ell C_{ij} = \partial_\ell C_i = 0$ if $\ell \ne i$.

Theorem 5 [68]. *Suppose the above is a separable system for $\mathbf{H}\psi = E\psi$ in coordinates $\{x^\ell\}$, let $\mathbf{x}_0$ be a fixed n-vector and ξ a constant N-vector. Then there are solutions of the form*

$$\psi(\mathbf{x}) = R(\mathbf{x})\Theta^{(1)}(x^1)\ldots\Theta^{(n)}(x^n)\xi$$

where the Θ^ℓ are $N \times N$ matrices such that

$$\Theta^{(i)}(x^i)\Theta^{(j)}(x^j) = \Theta^{(j)}(x^j)\Theta^{(i)}(x^i)$$

for all i, j and $\Theta^{(i)}(x_0^i) = \mathcal{I}$.

Thus in the 1st-order theory the separation is via matrix multiplication. Similarly, the Stäckel form conditions (2.6) for scalar equations can be generalized to this case.

Theorem 6 [55]. *Necessary and sufficient conditions that nonsingular matrices H^k satisfy the conditions*

$$\sum_{i=1}^n H^i C_{ij}(x^i) = \delta_j^1 \mathcal{I}, \qquad 1 \le j \le n,$$

where $\det(C_{ij}) \ne 0$, i.e., C is a Stäckel form matrix in the coordinates $\{x^i\}$, are

$$\partial_{jk} H^i = \partial_j H^k (H^k)^{-1} \partial_k H^i + \partial_k H^j (H^j)^{-1} \partial_j H^i, \quad j \ne k.$$

(Note that for $N = 1$ these equations agree with the Levi-Civita conditions where $H^i \equiv H_i^{-2}$.)

We can also generalize the scalar Stäckel transform [16, 55]:

Theorem 7. *Suppose $\{x^i\}$ is an $(R-)$separable coordinate system for the equation*

$$\left(\sum_i H^i(x)\partial_i + (V+\mu W)\right)\psi = E\psi$$

where μ is a parameter, $H^i, V, W, \{x^i\}$ are independent of μ and W is nonsingular. Then $\{x^i\}$ is also an $(R-)$separable coordinate system for the equation

$$(\sum_i W^{-1}H^i\partial_i + W^{-1}V)\psi' = E\psi'.$$

Although the 1st-order separation theory for Dirac-like equations contains many familiar features from the scalar theory (the connection with commuting families of symmetry operators, separation equations, the Stäckel mechanism for separation, etc.) many problems remain:

(1) What is the relationship between the symmetry operators $\mathbf{A}^k$, the separable coordinates and the change of frame matrix $R(x)$?
(2) How do we classify the possible separable systems?
(3) Not all separation is 1st-order (as we shall show).

For the Dirac equation itself we can, without much difficulty, say a little more about item (3). Corresponding to arbitrary local coordinates $\{x^0, x^1, x^2, x^3\}$ on a four-manifold with signature $(-1,1,1,1)$ and contravariant metric tensor $(g^{\mu\nu})$ the Dirac equation takes the form

$$\left(\sum_{\nu=0}^{3}\gamma^\nu(x)(\partial_\nu - \Gamma_\nu(x)) + m\right)\psi = 0.$$

Here γ^ν, Γ_ν are 4×4 matrices, $m \neq 0$ is the rest mass of the electron and ψ is a 4-spinor. The γ^ν are determined up to similarity by the requirement

$$\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu = 2g^{\mu\nu}\mathcal{I}, \quad 0 \le \mu,\nu \le 3 \tag{3.6}$$

and the $\Gamma_\nu(x)$ are Christoffel matrices [11, 18, 27]. (In Minkowski space with cartesian coordinates $\{t,x,y,z\}$ and metric $(-1,1,1,1)$ we have $\Gamma_\nu \equiv 0$.) For a fixed orthogonal coordinate system $g^{\mu\nu} = H_\mu^{-2}\delta^{\mu\nu}$ is diagonal and $g^{\mu\mu} = H_\mu^{-2} \neq 0$ so each of the matrices $\gamma^\mu(x)$ is nonsingular.

We apply identity (3.5) to the Dirac equation. Setting $i = k = 1, N = 4$ in (3.5) and using (3.6) we find

$$\sum_\mu H_\mu^{-2}(C_{\mu\nu}C_{\mu\alpha} + C_{\mu\alpha}C_{\mu\nu}) = 2\delta_\nu^1\delta_\alpha^1\mathcal{I}.$$

We can solve these equations uniquely for H_μ^{-2} in terms of the $A^{\mu\nu}$ so the equations are of rank 4. Therefore H_μ^{-2} is in Stäckel form, i.e., it satisfies a system of scalar equations

$$\sum_{\mu=0}^{3} H_\mu^{-2}\mathcal{S}_{\mu\nu}(x^\mu) = \delta_\nu^0$$

where det $\mathcal{S} \neq 0$.

We conclude from this that a (1st-order) separable solution of the Dirac equation must also separate the (scalar) Hamilton-Jacobi equation (and in an Einstein space, also the Klein-Gordon equation). Thus for a given space the possible Dirac separable systems are a subset of the already classified Hamilton-Jacobi separable systems. The vector space of 1st-order matrix symmetry operators for the Dirac equation can be computed directly, [19, 43, 52, 60]. (Note that in general the space of 1st-order symmetry operators doesn't form a Lie algebra because the commutator of two 1st-order matrix symmetries may be 2nd-order. This phenomenon doesn't occur for scalar equations. For Minkowski spacetime the 10-dimensional Lie algebra of Poincaré symmetries is a proper subspace of this vector space.) Furthermore [45], for the Dirac equation, every triplet of commuting symmetry operators must contain at least 2 Poincaré symmetries (symmetries of the form $\mathbf{A} = \partial_{x^\beta} + K$ in specially adapted coordinates). Therefore, [15] the most general possible (1st-order) Dirac orthogonal separable metric is of the form

$$ds^2 = (f_1(x^1) + f_2(x^2))\left((dx^1)^2 + (dx^2)^2 + \frac{(dx^3)^2}{g_1(x^1) + g_2(x^2)} + \frac{(dx^4)^2}{h_1(x^1) + h_2(x^2)}\right).$$

This result can be generalized to nonorthogonal separation. It appears that detailed understanding of 1st-order Dirac separable systems should be practical in the near future. (Bagrov et al. have classified a large number of cases of separation of the Minkowski space Dirac equation with an added vector potential [2-10, 65]. For related work see [23].)

Now we switch our point of view. Instead of studying separation of a single system of equations in a variety of spaces, we consider a single physically interesting space, the Kerr metric, and a variety of spinor equations on that space. The Kerr metric is [22, 49]

$$\begin{aligned} ds^2 =& (1 - \frac{2Mr}{\rho^2})dt^2 - (\frac{\rho^2}{\Delta})dr^2 - \rho^2 d\theta^2 \\ &+ (\frac{2aMr\sin^2\theta}{\rho^2})dtd\theta - [(r^2+a^2) + \frac{2a^2Mr\sin^2\theta}{\rho^2}]\sin^2\theta d\phi^2, \end{aligned}$$

where $\Delta = r^2 + a^2 - 2Mr$, $\rho^2 = r^2 + a^2\cos^2\theta$. It is a solution of the free field Einstein equations which describes the geometry in the vicinity of a rotating black hole of mass M and angular momentum per unit mass a. When $a = 0$ the Kerr metric reduces to the Schwarzschild metric; when $M = 0$ it is the Minkowski (flat) space metric in oblate spheroidal coordinates.

The following equations "separate" in these coordinates [21, 22, 28, 44, 46-48]:

(1) The Hamilton-Jacobi and Klein-Gordon equations. *Standard scalar theory.*
(2) The Dirac equation (spin $\frac{1}{2}$ particles with mass). *Almost standard 1st-order theory.*
(3) Neutrino equation (spin $\frac{1}{2}$ particles with zero mass). *Almost standard 1st-order theory.*
(4) Maxwell's equations (spin 1 particles with zero mass). *Strange*, i.e., separation not well understood.
(5) Rarita-Schwinger equations (spin $\frac{3}{2}$ particles with zero mass). *Strange.*
(6) Gravitational perturbation equations (spin 2 particles, zero mass). *Strange.*

To examine several of these equations in some detail we adopt the Newman-Penrose spinor formalism [41, 56, 58, 73] and the Kinnersly nullcoframe:

$$\begin{aligned}
\ell_i dx^i &= \frac{1}{\Delta}(\Delta dt - \rho^2 dr - a\Delta \sin^2\theta d\phi), \quad n_i dx^i = \frac{1}{2\rho^2}(\Delta dt + \rho^2 dr - a\Delta\sin^2\theta d\phi) \\
m_i dx^i &= \frac{1}{\bar\rho\sqrt{2}}(ia\sin\theta dt - \rho^2 d\theta - i(r^2+a^2)\sin\theta d\phi) \\
\bar m_i dx^i &= \frac{1}{\bar\rho^*\sqrt{2}}(-ia\sin\theta dt - \rho^2 d\theta + i(r^2+a^2)\sin\theta d\phi)
\end{aligned}$$

The associated nullframe is:

$$\begin{aligned}
D = \ell^i\partial_{x^i} &= \frac{1}{\Delta}\left((r^2+a^2)\partial_t + \Delta\partial_r + a\partial_\phi\right), \ \tilde\Delta = n^i\partial_{x^i} = \frac{1}{2\rho^2}\left((r^2+a^2)\partial_t - \Delta\partial_r + a\partial_\phi\right) \\
\delta = m^i\partial_{x^i} &= \frac{1}{\sqrt{2}\bar\rho}\left(ia\sin\theta\partial_t + \partial_\theta + i\csc\theta\partial_\phi\right) \\
\bar\delta = \bar m^i\partial_{x^i} &= \frac{1}{\sqrt{2}\bar\rho^*}\left(-ia\sin\theta\partial_t + \partial_\theta - i\csc\theta\partial_\phi\right)
\end{aligned}$$

$$\Delta = r^2 + a^2 - 2Mr, \quad \rho^2 = r^2 + a^2\cos^2\theta, \quad \bar\rho = r + ia\cos\theta,$$

and the spin coefficients are:

$$\begin{aligned}
&\kappa = \sigma = \tilde\lambda = \nu = \epsilon = 0, \quad \tilde\rho = -\frac{1}{\bar\rho^*}, \quad \beta = \frac{\cot\theta}{\bar\rho 2\sqrt{2}}, \quad \pi = \frac{ia\sin\theta}{(\bar\rho^*)^2\sqrt{2}} \\
&\tau - \frac{ia\sin\theta}{\rho^2\sqrt{2}}, \quad \mu = -\frac{\Delta}{2\rho^2\bar\rho^*}, \quad \gamma = \mu + \frac{r-M}{2\rho^2}, \quad \alpha = \pi - \beta^*.
\end{aligned}$$

We will consider 4 equations in a Kerr spacetime and in the given coordinates: the Klein-Gordon, Dirac, Neutrino and Maxwell equations.

(1) Klein-Gordon Equation

$$\Box\Phi \equiv (\nabla_{AA'}\nabla^{AA'})\Phi = m^2\Phi$$

The separated solutions take the form: $\Phi = R_0(r)S_0(\theta)e^{im\phi + i\sigma t}$ where m and σ are separation constants. There are two 1st-order symmetries: ∂_ϕ, ∂_t where $\partial_\phi\Phi = im\Phi$ and $\partial_t\Phi = i\sigma\Phi$, and one 2nd-order symmetry: $U\Phi = \lambda\Phi$,

$$U = (K^{AA'BB'}\nabla_{BB'})(K_{AA'}{}^{CC'}\nabla_{CC'}) - \frac{1}{3}K^{AA'BB'}M_{BB'}\nabla_{AA'}$$

where $M_{AB'} = \nabla^{BA'}K_{BB'AA'}$. Here, $K_{AA'BB'}$ is a Killing Yano tensor, i.e.,

$$\nabla_{(CC'}K_{AA')BB'} = 0, \quad K_{AA'BB'} + K_{BB'AA'} = 0,$$

unique for the Kerr metric. Also, $\mathcal{K}^{AA'CC'} = K^{AA'}{}_{BB'}K^{BB'CC'}$ is a 2nd-order Killing tensor, i.e.,

$$\mathcal{K}^{AA'CC'} = \mathcal{K}^{CC'AA'}, \quad \nabla_{(AA'}\mathcal{K}_{BB'CC')} = 0$$

which corresponds to the 2nd-order terms in Theorem 3.

(2) Dirac Equation

In Newman-Penrose notation this reads

$$\nabla_{AX'}\chi^{X'} = \frac{im}{\sqrt{2}}\phi_A, \quad \nabla_{AX'}\phi^A = -\frac{im}{\sqrt{2}}\chi_{X'}.$$

The R-separated solutions are [21, 22, 41]

$$\begin{aligned} \phi_1 &= \bar\rho^* R_{-\frac12}(r)S_{-\frac12}(\theta)e^{i\sigma t+im\phi}, \quad \phi_0 = -R_{\frac12}(r)S_{\frac12}(\theta)e^{i\sigma t+im\phi} \\ \chi_{1'} &= -\bar\rho R_{-\frac12}(r)S_{\frac12}(\theta)e^{i\sigma t+im\phi}, \quad \chi_{0'} = -R_{\frac12}(r)S_{-\frac12}(\theta)e^{i\sigma t+im\phi}. \end{aligned}$$

In addition to the symmetry operators ∂_ϕ, ∂_t there is a nontrivial symmetry operator:

$$\begin{bmatrix} 0 & L_A{}^{A'} \\ N^A{}_{A'} & 0 \end{bmatrix} \begin{pmatrix} \phi_A \\ -\chi_{A'} \end{pmatrix} = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} \phi_A \\ -\chi_{A'} \end{pmatrix}$$

where

$$L_{AA'} = K_{AA'}{}^{BB'}\nabla_{BB'} + \frac13 M_{AA'}, \quad N_{AA'} = K_{AA'}{}^{BB'}\nabla_{BB'} - \frac13 M_{AA'}.$$

(3) Neutrino Equation

$$\nabla^{AA'}\phi_A = 0.$$

The nontrivial symmetry operator is:

$$L_B{}^{A'}N^A{}_{A'}\phi_A = -\frac{\lambda^2}{2}\phi_B.$$

(4) Maxwell's Equations (Spinor form)

$$\nabla^A{}_{A'}\Phi_{AB} = 0, \quad \Phi_{AB} = \Phi_{BA},$$

[20, 22, 38, 41]. The separated solutions for 2 of the 3 distinct components are:

$$\Phi_0 = R_1(r)S_1(\theta)e^{i\sigma t+im\phi}, \quad \Phi_2 = 2(\bar\rho^*)^2 R_{-1}(r)S_{-1}(\theta)e^{i\sigma t+im\phi}.$$

(In this new notation for the spinor $\Phi_{BA} = \Phi_I$, $I = A+B$.) Now $R_{\pm1}$, $S_{\pm1}$ are *Teukolsky functions*, i.e., they satisfy 2nd-order ordinary differential equations of the form

$$\begin{aligned} (\Delta\mathcal{D}_1\mathcal{D}_1^+ - 2i\sigma r - \lambda)R_1 = 0, &\quad (\mathcal{L}_0^+\mathcal{L}_1 + 2a\sigma\cos\theta + \lambda)S_1 = 0 \\ (\Delta\mathcal{D}_0^+\mathcal{D}_0 + 2i\sigma r + \lambda)R_{-1} = 0, &\quad (\mathcal{L}_0\mathcal{L}_1^+ - 2a\sigma\cos\theta - \lambda)S_{-1} = 0. \end{aligned}$$

Here,

$$\mathcal{D}_n = \partial_r + \frac{iK}{\Delta} + 2n\frac{r-M}{\Delta}, \quad \mathcal{D}_n^+ = \partial_r - \frac{iK}{\Delta} + 2n\frac{r-M}{\Delta}, \quad \mathcal{L}_n = \partial_\theta + Q + n\cot\theta,$$
$$\mathcal{L}_n^+ = \partial_\theta - Q + n\cot\theta, \quad K = (r^2+a^2)\sigma + am, \quad Q = a\sigma\sin\theta + m\csc\theta$$

From the *Teukolsky-Starobinsky identities*, [22, 42, 72], we can show that

$$\Delta \mathcal{D}_0 \mathcal{D}_0 R_{-1} = \mathcal{C} \Delta R_{+1}, \quad \Delta \mathcal{D}_0^{+} \mathcal{D}_0^{+} R_{+1} = \mathcal{C}^{*} \Delta R_{-1}$$

(and similar formulas for $S_{\pm 1}$ involving also the separation parameter λ), can work out the relative normalizations of $R_{\pm 1}$, $S_{\pm 1}$, and can obtain [22]

$$\Phi_1 = \frac{1}{\sqrt{2}\bar{\rho}^{*}\mathcal{C}} \left[\mathcal{D}_0 \mathcal{L}_1 - \frac{1}{\bar{\rho}^{*}} (\mathcal{L}_1 + ia \sin\theta \mathcal{D}_0) \right] R_{-1} S_1 .$$

(Note that Φ_1 is not separable in the same simple sense as Φ_0 and Φ_2.) The non-trivial separation constants $\mathcal{C}$, and λ are characterized by the symmetry operators ${\mathcal{C}^{A'B'}}_{AB}$, Λ where

$$\begin{aligned} {\mathcal{C}^{A'B'}}_{AB} \phi_{A'B'} &= \Big({K^{A}}_{(A'}{}^{CC'} {K^{B}}_{B')}{}^{DD'} \nabla_{CC'} \nabla_{DD'} \\ &\quad + \frac{4}{3} {M^{A}}_{(A'} {K^{B}}_{B')}{}^{DD'} \nabla_{DD'} + \frac{2}{9} {M^{A}}_{(A'} {M^{B}}_{B')} \Big) \phi_{AB} \\ &= \frac{1}{2} \mathcal{C} \phi_{AB}, \end{aligned}$$

$$\begin{aligned} \Lambda_{(A}{}^{B} \phi_{C)B} &= \left(K_{A}^{A'CC'} \Delta_{CC'} - \frac{1}{3} {M_{A}}^{A'} \right) \left({K^{B}}_{A'}{}^{DD'} \Delta_{DD'} + \frac{2}{3} {M^{B}}_{A'} \right) \phi_{BC} \\ &= \frac{\lambda}{2} \phi_{AC}, \end{aligned}$$

[20, 41, 73]. Note that the symmetry operator $\mathcal{C}$ is a 2nd-order differential operator. (The 1st-order theory for Dirac-like equations doesn't explain the separation mechanism here.)

Recently Williams and the authors [42, 73] have extended the Teukolsky-Starobinsky identities to arbitrary half integral or integral spins and have found a covariant system of equations which yields these general Teukolsky equations.

4. Hertz potentials. The method of Hertz potentials, [48], is another powerful method for generating explicit solutions of spinor equations that is closely related to variable separation. In our example we again adopt the Newman-Penrose spinor formalism, this time in Minkowski space.

The covariant zero-mass field equation for a free spin-s field is [48, 58]:

$$\nabla^{AX'} \phi_{AB\cdots K} = 0, \tag{4.1}$$

where $\phi_{AB\cdots K}$ is a totally symmetric spinor with $2s$ indices. The spinor d'Alembertian operator (a scalar) is: $\Box \equiv \nabla_{AX'} \nabla^{AX'}$. A *Hertz potential* for (4.1) is a totally symmetric $2s$-spinor $P_{AB\cdots K}$ satisfying

$$\Box \bar{P}^{X'N'\cdots W'} = 2 \nabla^{A(X'} G_{A}^{N'\cdots W')} \tag{4.2}$$

where $G_{AN'\cdots W'}$ is an arbitrary gauge spinor with one unprimed and $2s-1$ symmetrized primed indices.

Theorem 8 [48]. *Suppose P is a Hertz potential. Then the spinor*

$$\phi_{AB\cdots K} = \nabla_{AM'}\nabla_{BN'}\cdots\nabla_{KW'}\bar{P}^{M'N'\cdots W'} - \nabla_{(BN'\ldots}\nabla_{KW'}G_{A)}^{N'\cdots W'}$$

is a solution of the zero-mass field equation $\nabla^{AX'}\phi_{AB\cdots K} = 0$.

This result can be extended to curved spacetime (Kerr metric) for $s = \frac{1}{2}, 1, \frac{3}{2}, 2$, [48]. In these cases, G can be chosen such that the equation for the Hertz potential has only one nontrivial component and variables separate in a simple fashion. The Hertz potential thus yields a "separable" solution of the field equations. Unfortunately the method of Hertz potentials is not general; it works only for algebraically special spacetimes.

5. Other approaches and examples. In this section we review some other approaches to separation for spinor equations and present some intriguing examples which should (we hope) help point the way to a comprehensive theory of variable separation.

(1) The approach of Shiskin et al. to the Dirac equation [1, 69]

In this approach the problem is: given coordinates and frame (to within a constant change of frame), determine the separation equations. For example, consider the Dirac equation in Minkowski space and Cartesian coordinates, but with an added vector potential

$$H\Psi = \left(\sum_{|al=0}^{3}\gamma^\alpha(\partial_\alpha - iA_\alpha) + m\right)\Psi = 0,$$

$$A_\alpha(x) \quad \text{vector potential}, \qquad \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\mathfrak{I}\delta^{\alpha\beta}.$$

We set $\Phi = \Gamma^{-1}\Psi$, Γ a constant 4×4 matrix, and try to write $H\Psi = 0$ in the form

$$(K_0 + K_{123})\Phi = 0$$

where K_0 depends on only one of the variables, say x_0, K_{123} depends only on x_1, x_2, x_3 and

$$[K_0, K_{123}] = 0.$$

These requirements impose conditions on the A_α and Γ whose only possible solutions for Γ are

$$1)\ \Gamma = \gamma^0, \qquad 2)\ \Gamma = \gamma^1\gamma^2\gamma^3.$$

(The solutions for the A_α are easy to compute but divide into several cases.) We can continue the process to separate operators K_1, K_2, K_3, for x_1, x_2, x_3, one at a time. The separated solution is a simultaneous eigenfuntion of K_0, $K_0\phi = \lambda\phi$, and of K_1, K_2, K_3.

A similar approach yields separation equations for elliptic cylinder, parabolic cylinder and spheroidal coordinates, etc. This method is effective in many cases corresponding to 1st-order separation, but there are problems:

a) It is not completely systematic. One must postulate the coordinates and guess the right frame to within a constant frame change.

b) No definition of separation is provided.

(2) The metric $ds^2 = dt^2 - a(t)^2\left[dx^2 + b(x)^2(dy^2 + c(y)^2 dz^2)\right]$

Special cases of this metric are the Friedman-Robertson-Walker universes, [50, chapter 11]. These are the cases in which the spatial part of the metric (the part enclosed in square brackets) is a constant curvature Riemannian 3-manifold with curvature -1, +1, or 0. The three cases correspond to a Robertson-Walker expanding universe, finite universe or Minkowski universe, respectively.

To study variable separation of spinor equations for the general metric in these coordinates we again adopt Newman-Penrose notation [56, 58] and choose the null coframe:

$$\begin{aligned}
\ell_i dx^i &= \frac{1}{\sqrt{2}}(dt - a\ dx), \quad n_i dx^i = \frac{1}{\sqrt{2}}(dt + a\ dx)\\
m_i dx^i &= \frac{1}{\sqrt{2}}(ab\ dy + iabc\ dz), \quad \bar{m}_i dx^i = \frac{1}{\sqrt{2}}(ab\ dy - iabc\ dz)\\
(x^i) &= (t, x, y, z).
\end{aligned}$$

It follows from this that the null frame is

$$\begin{aligned}
D = \ell^i \partial_{x_i} &= \frac{1}{\sqrt{2}}(\partial_t + \frac{1}{a}\partial_x), \quad \Delta = n^i \partial_{x_i} = \frac{1}{\sqrt{2}}(\partial_t - \frac{1}{a}\partial_x)\\
\delta = m^i \partial_{x^i} &= \frac{1}{\sqrt{2}}(-\frac{1}{ab}\partial_y - \frac{i}{abc}\partial_z), \quad \bar{\delta} = \bar{m}^i \partial_{x^i} = \frac{1}{\sqrt{2}}(-\frac{1}{ab}\partial_y + \frac{i}{abc}\partial_z)
\end{aligned}$$

and the spin coefficients are

$$\kappa = \sigma = \lambda = \nu = \pi = \tau = 0, \quad \epsilon = -\gamma = \frac{a_t}{2\sqrt{2}a}, \quad \rho = -\frac{1}{\sqrt{2}a}(a_t + \frac{b_x}{b})$$

$$\mu = \frac{1}{\sqrt{2}a}(a_t - \frac{b_x}{b}), \quad \alpha = -\beta = \frac{c_y}{2\sqrt{2}abc}.$$

For the Dirac equation

$$\begin{pmatrix} 0 & 0 & D + \bar{\epsilon} - \bar{\rho} & \delta + \bar{\pi} - \bar{\alpha} \\ 0 & 0 & \bar{\delta} + \bar{\beta} - \bar{\tau} & \Delta + \bar{\mu} - \bar{\gamma} \\ \Delta + \mu - \gamma & -(\delta + \beta - \tau) & 0 & 0 \\ -(\bar{\delta} + \pi - \alpha) & D + \epsilon - \rho & 0 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ \bar{Q}^{\dot{0}} \\ \bar{Q}^{\dot{1}} \end{pmatrix} = im \begin{pmatrix} P_0 \\ P_1 \\ \bar{Q}^{\dot{0}} \\ \bar{Q}^{\dot{1}} \end{pmatrix}.$$

there are separable solutions of the form, [26]:

$$\begin{pmatrix} P_0 \\ P_1 \\ \bar{Q}^{\dot{0}} \\ \bar{Q}^{\dot{1}} \end{pmatrix} = e^{\lambda_3 z} \begin{pmatrix} T_1(t)X_1(x)Y_1(y) \\ T_1(t)X_2(x)Y_2(y) \\ T_2(t)X_1(x)Y_1(y) \\ T_2(t)X_2(x)Y_2(y) \end{pmatrix}.$$

There are 4 separation constants and three symmetry operators, one of which is 2nd-order. We give no more details about this example since it is covered in Niky Kamran's talk, elsewhere in this proceedings.

Maxwell's equation (spinor form) is $\nabla^{AA'}\phi_{AB} = 0$ which reads

$$(5.1)\qquad \begin{pmatrix} \bar{\delta}+\pi-2\alpha & -D+2\rho & 0 \\ 0 & \bar{\delta}+2\pi & -(D-\rho+2\epsilon) \\ (\Delta+\mu-2\gamma) & -(\delta-2\tau) & 0 \\ 0 & (\Delta+2\mu) & -(\delta+2\beta-\tau) \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

There are separable solutions of the form

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} A(t)h_0(x)g_0(y) \\ A(t)h_1(x)g_1(y) \\ A(t)h_2(x)g_2(y) \end{pmatrix} e^{-i\lambda z}.$$

To see how the separation works note that the 1st equation (5.1) is

$$-\frac{1}{\sqrt{2}ab}[\partial_y - \frac{i}{c}\partial_z + \frac{c_y}{c}]\phi_0 - \frac{1}{\sqrt{2}}[(\partial_t + \frac{2a_t}{a}) + \frac{1}{a}(\partial_x + \frac{2b_x}{b})]\phi_1 = 0.$$

It separates into

$$(\partial_y - \frac{\lambda}{c} + \frac{c_y}{c})g_0(y) = \lambda_4 g_1(y), \quad \frac{\lambda_4}{b}h_0(x) + (\mu + \partial_x + \frac{2b_x}{b})h_1(x) = 0,$$

$$A_t + \frac{2a_t}{a}A = \mu A.$$

where λ, λ_4 and μ are separation constants. The other 3 equations (5.1) separate in a similar manner. The miracle is that these 12 separation equations are *consistent* if one restricts the separation constants suitably.

The spin 1 (massive) equations are (in tensor notation)

$$\Box A_\mu - R_\mu{}^\nu A_\nu = m^2 A_\nu, \quad \nabla^\nu A_\nu = 0.$$

They do not separate in the given tetrad, but with the change of tetrad

$$\mathcal{A}_0 = A_0 + A_3, \quad \mathcal{A}_1 = A_1, \quad \mathcal{A}_3 = A_0 - A_3, \quad \mathcal{A}_2 = A_2$$

we find separated solutions of the general form

$$\mathcal{A}_0 = \chi_0(t)b_0(x)g_1(y)e^{-i\lambda z}, \quad \mathcal{A}_1 = \chi_1(t)b_+(x)g_0(y)e^{-i\lambda z}$$
$$\mathcal{A}_2 = -2\chi_1(t)b_-(x)g_2(y)e^{-i\lambda z}, \quad \mathcal{A}_3 = -2\chi_1(t)b_0(x)g_1(y)e^{-i\lambda z}.$$

For our last example, Wunsch's equations (spin 1, massive)

$$\nabla^{AA'}\phi_{AB} = m\psi_B{}^{A'}, \quad \nabla_{(AA'}\psi^{A'}{}_{B)} = -m\phi_{AB}$$

admit separable solutions of the form

$$\phi_0 = a_1(t)b_0(x)g_0(y)e^{-i\lambda z}, \quad \phi_1 = a_1(t)b_1(x)g_1(y)e^{-i\lambda z}, \quad \phi_2 = a_1(t)b_2(x)g_2(y)e^{-i\lambda z},$$
$$\psi_0{}^{0'} = a_2(t)b_1(x)g_1(y)e^{-i\lambda z}, \quad \psi_0{}^{1'} = -a_2(t)b_0(x)g_0(y)e^{-i\lambda z},$$
$$\psi_1{}^{0'} = a_2(t)b_2(x)g_2(y)e^{-i\lambda z}, \quad \psi_1{}^{1'} = -a_2(t)b_1(x)g_1(y)e^{-i\lambda z}.$$

Group Theoretic Methods: Tensor Harmonics

For systems of equations admitting nontrivial Lie symmetry groups, harmonic analysis can provide an effective tool for determining separable solutions in certain (subgroup) coordinate systems which are adapted to the symmetries. This well developed procedure provides the proper frame or tetrad for separation automatically and for the Dirac equation it yields examples of separation in which the separation equations are both 1st and 2nd order, [27a, 27b, 29, 51]. The principal drawback of this method is that it applies only to so-called subgroup coordinates such as, for example, spherical coordinates for the Dirac equation in Minkowski space. It does not apply to coordinates such as parabolic cylinder, elliptic cylinder or oblate spheroidal which also separate the Minkowski space Dirac equation with appropriate choice of frame.

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