# INTEGRABILITY, STÄCKEL SPACES & RATIONAL POTENTIALS

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ABSTRACT. For a variety of classical mechanical systems embeddable into flat space with cartesian coordinates $\{x_i\}$ and for which the Hamilton-Jacobi equation can be solved via separation of variables in a particular curvalinear system $\{u_j\}$, we answer the following question. When is the separable potential function $V$ expressible as a polynomial (or as a rational function) in the defining coordinates $\{x_i\}$? Many examples are given.

**PACS**: 02.20.-a, 03.65.Fd, 03.65.Ge, 02.30.Gp

**I. Introduction.** In recent years there has been renewed interest in the notion of integrability as it applies to the solution of mechanical systems. Many examples of such systems are known. The crucial requirement for the notion of integrability to be valid is the existence of a suitable number of constants of the motion. This

† Work supported in part by the National Science Foundation under grant DMS 94–00533

guarantees in principle that the corresponding mathematical problem can be solved. One of the classical ways of solving some integrable mechanical systems is using the method of separation of variables. Commonly quoted integrable systems are the so called Neumann system of a particle moving on a sphere under the influence of the anisotropic harmonic potential and also the problem of geodesic flow on an ellipsoid, [1-4]. The problem we consider here is, generally, how to classify systematically those classical mechanical systems that are of the form

$$H = \sum_{i,j=1}^{n} g^{ij} p_i p_j + V(x^1, ..., x^n) \tag{1.1}$$

via separation of variables techniques. Here the metric $g^{ij}$ corresponds to some Riemannian space that is embedded into flat space and the $x^i$ are cartesian coordinates in that space. Specifically, we will determine the possible separable potentials $V$ which are polynomials or rational functions of the cartesian coordinates. In particular we consider the case of spaces of constant curvature, i.e., Euclidean space, the positive definite hyperboloid and the sphere. Also considered here are complex versions of these spaces. For the real spaces we have mentioned, all the separable coordinate systems are known [5,6]. The requirement that the potential provide a separation of variables is also known, e.g., [7]. Indeed if the coordinates $y^i$ provide a separation of variables via some transformation $y^i = y^i(x^1, ...., x^n)$, $i = 1..n$, the only non zero elements of the contravariant tensor

$$\bar{g}^{ij} = \sum_{p,q=1}^{n} g^{pq} \frac{\partial y^i}{\partial x^p} \frac{\partial y^j}{\partial x^q}$$

are [8-15]

$$g^{ij} = \delta^{ij}[S^{i1}/S], \quad i,j = 1,\cdots,n_1$$

$$g^{r\alpha} = [S^{r1}/S]f_r(y^r), \quad r = n_1+1,\cdots,n_1+n_2+2, \quad \alpha = n_1+n_2+2,\cdots,n_1+n_2+n_3+2$$

$$g^{\alpha\beta} = [\sum_{k=1}^{n_1+n_2} h_k^{\alpha\beta}(y^k)(S^{k1}/S)], \quad \alpha,\beta = n_1+n_2+2,\cdots,n_1+n_2+n_3+2$$

Here $n_1+n_2+n_3 = n$, and there exists a Stäckel matrix $\bar{S} = (\bar{S}_{ij}(y^i))_{n\times n}$ such that $S = \det \bar{S}$ and $S^{ij}$ is the $ij$ cofactor of $\bar{S}$. The requirement that the potential separate in these coordinates is that [7]

$$V(y^1,...,y^n) = \sum_{k=1}^{n_1+n_2} v(y^k)\frac{S^{k1}}{S}. \tag{1.2}$$

When this is the case $V$ is said to be a *Stäckel multiplier* in the coordinate system $y^i$. Coordinates of the type $y^i, i = 1,\cdots,n_1$ are called *Stäckel coordinates.* Coordinates of type $y^r$, $r = n_1+1,\cdots,n_1+n_2+2$ are called *first order coordinates.* Coordinates of type $y^\alpha, \alpha = n_1+n_2+2,\cdots,n_1+n_2+n_3+2$ are called *ignorable variables.* We will restrict ourselves here to coordinate systems such that $\bar{g}^{ij}$ is a diagonal metric. This is both necessary and sufficient for the case of separation of variables for real spaces of constant curvature [6].

In the case of real Euclidean space let us consider the generic elliptical coordinates $\{u_i\}$ defined by

$$x_i^2 = c^2\Pi_{j=1}^n(u_j - e_i)/\Pi_{k\neq i}(e_k - e_i)), \quad i,j,k = 1,\cdots,n \tag{1.3}$$

and $e_i \neq e_k$ for $i \neq k$. For elliptical coordinates the new variables are subject to the restrictions $e_1 < u_1 < e_2 < u_2 < .... < e_n < u_n$. The general form for the Stäckel multiplier associated with these coordinates can be readily calculated. This may be deduced from the infinitesimal distance

$$ds^2 = \frac{c^2}{4}\sum_{k=1}^{n} \frac{\Pi_{m\neq i}(u_m - u_i)}{\Pi_{\ell=1}^{n}(u_k - e_\ell)} du_k^2.$$

Indeed the form of such a Stäckel multiplier must be

$$V(u_1, ...., u_n) = \frac{\sum_{i=1}^{n} v_i(u_i)}{\Pi_{m\neq i}(u_m - u_i)}. \tag{1.4}$$

In this sense this problem has been solved. However we seek a solution to a more restricted problem here. Indeed we answer the following two questions.

(1) When is the Stäckel multiplier $V(u_1, ..., u_n)$ expressible as a polynomial in the defining coordinates $x_i$?

(2) When is the multiplier in the form of a rational function of the cartesian coordinates $x_i$?

These are two important classes of potentials and a number of physically interesting cases are known. Indeed Eisenhart has shown that in an orthogonal separable system $y^i$ a potential is in the form of a Stäckel multiplier if and only if

$$g^{ii}g^{jj}\frac{\partial^2 V}{\partial u_i \partial u_j} - g^{ii}\frac{\partial g^{jj}}{\partial u_i}\frac{\partial V}{\partial u_j} - g^{jj}\frac{\partial g^{ii}}{\partial u_j}\frac{\partial V}{\partial u_i} = 0, \tag{1.5}$$

for $i \neq j$,[9]. However Benenti [16] has proved that (expressed in cartesian coordinates) these conditions for elliptical coordinates are equivalent to

$$(e_i - e_j)\frac{\partial^2 V}{\partial x_i \partial x_j} + (x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i})(2V + \sum_{k=1}^{n} x_k\frac{\partial V}{\partial x_k}) = 0 \tag{1.6}$$

for $i \neq j$. Thus for elliptical coordinates, the solutions to questions 1) and 2) are just the polynomial or rational solutions to (1.6). We will give a different and more general solution to these questions.

We first look at the problem of polynomial potentials. The crucial observation is the following. Let $V(x_1, ..., x_n)$ be a polynomial solution of (1.6). When expressed as a function of the $u_i$ coordinates via (1.3), $V$ should be symmetric. Therefore when expressed in the form (1.4) each of the functions $v_i(u_i) = v(u_i)$. We can deduce $v(u_i)$ if $V$ is of polynomial form. Indeed if we put $u_i = a_i$ for $i \neq 1$ such that $a_k \neq e_\ell$ for all $k$ and $\ell$ then we see that $v(u)$ should have the form $v(u) = R(u)\sqrt{u - e_1} + S(u)$ where $R$ and $S$ are rational functions of $u$. Doing a similar calculation with $e_1$ replaced by $e_2$ we must then have

$$R(u)\sqrt{u - e_1} + S(u) = R'(u)\sqrt{u - e_2} + S'(u).$$

This can only hold if $R(u) = R'(u) = 0$ and $S(u) = S'(u)$. Before proceeding further we note the following.

**Lemma.** *Let $S_i[u_1, ...., u_n]$ be the symmetric polynomial defined by*

(1.7) $$S_i[u_1, ...., u_n] = \frac{1}{i!} \sum_{j_1,\dots,j_i \neq} u_{j_1} ....u_{j_i}, \quad i = 1, \cdots, n$$

*and $S_0[u_1, ...., u_n] = 0$. Similarly let $E_i[e_1, ..., e_n] = S_i[e_1, ...., e_n]$. Then the symmetric functions $S_i[u_1, ...., u_n]$ are expressible in terms of the cartesian coordinates via*

(1.8)
$$S_\ell[u_1, ...., u_n] = E_\ell[e_1, ..., e_n] + \sum_{k=1}^{n} \frac{x_k^2}{c^2} E_{\ell-1}[e_1, .., e_{k-1}, e_{k+1}, ..., e_n], \quad \ell = 1, ..., n.$$

This follows readily from the identity

(1.9) $$\sum_{k=1}^{n} \frac{x_k^2}{(u - e_k)} - c^2 = -c^2 \frac{\Pi_{k=1}^n (u - u_k)}{\Pi_{j=1}^n (u - e_j)}.$$

Multiplying this expression by $\Pi_{j=1}^n(u-e_j)$ we easily deduce the identity (1.8). The functions $SM_p[u_1,....,u_n] = \sum_{i=1}^n u_i^p/\Pi_{j\neq i}(u_i-u_j)$ for $p$ an integer can be expressed in terms of cartesian coordinates via the recurrence relation

$$SM_q = \sum_{j=1}^n (-1)^{j+1} S_j[u_1,....,u_n] SM_{q-j}, \tag{1.10}$$

for $q \geq n$, where $SM_{n-1} = 1$ and $SM_p = 0$ for $0 \leq p < n-1$.

Consequently if $v(u)$ is a polynomial in $u$ the corresponding potential is a polynomial in the cartesian coordinates. We can conclude that all contributions for which $v(u) = u^p$ give a basis for all Stäckel multipliers which are polynomial in the cartesian coordinates $x_k$. The second type of contribution viz. $1/(u-a)^q$ can be readily expressed in terms of cartesian coordinates. Indeed we need only consider the special case $a = 0$, as the general case can be obtained by translating all the $u_i$ variables. We observe that $SM_{-p}[u_1,...,u_n] = (\Pi_{i=1}^n v_i) SM_{p+n-2}[v_1,...,v_n]$ where $v_i = 1/u_i$. Then using the result $S_q[v_1,...,v_n] = S_{n-q}[u_1,....,u_n]/(\Pi_{i=1}^n u_i)$ we can always evaluate $SM_{-p}[u_1,...,u_n]$. The analysis is now complete. The basic building blocks for potentials which are rational functions of the cartesian coordinates are the Stäckel multipliers $SM_p[u_1,...,u_n]$ where $p = 0, \pm 1, \pm 2, \cdots$.

For real Euclidean spaces there is one other generic coordinate system, paraboloidal coordinates. These coordinates are given by

$$\begin{aligned} x_1 &= (c/2)(-\sum_{j=1}^n u_j + \sum_{k=1}^{n-1} e_k) \\ x_i^2 &= c^2 \frac{\Pi_{j=1}^n (u_j - e_{i-1})}{\Pi_{k\neq i-1}(e_k - e_{i-1})}, \quad i,k = 2,\cdots,n, \quad j = 1,\cdots,n \end{aligned} \tag{1.11}$$

where $e_1 < u_1 < e_2 < u_2 < e_3 < ... < u_{n-1} < e_n < u_n$. The corresponding infinitesmal distance is

$$ds^2 = \frac{c^2}{4}\sum_{k=1}^{n}\frac{\Pi_{m\neq i}(u_m-u_i)}{\Pi_{\ell=1}^{n-1}(u_k-e_\ell)}du_k^2 \tag{1.12}$$

and the symmetrical functions $S_i[u_1,...,u_n]$ are given by the expressions

$$\begin{aligned}
&c^2S_1[u_1,...,u_n] = E_1[e_2,...,e_n] - 2x_1\\
&c^2S_\ell[u_1,...,u_n] = E_\ell[e_2,...,e_n] - 2x_1E_{\ell-1}[e_2,...,e_n] -\\
&\qquad\sum_{k=2}^{n}x_k^2E_{\ell-2}[e_2,.e_{k-1},e_{k+1}..,e_n],\quad \ell=2,...,n-1\\
&c^2S_n[u_1,...,u_n] = -2x_1E_{n-1}[e_2,...,e_n] - \sum_{k=2}^{n}x_k^2E_{n-2}[e_2,.e_{k-1},e_{k+1}..,e_n].
\end{aligned}\tag{1.13}$$

This follows from the identity

$$-2cx_1 - c^2u + \sum_{k=2}^{n}\frac{x_k^2}{(u-e_k)} = c^2\frac{\Pi_{k=1}^{n}(u-u_k)}{\Pi_{j=2}^{n}(u-e_j)}. \tag{1.14}$$

To establish which potentials are Stäckel multipliers for paraboloidal coordinates and are rational functions of the cartesian coordinates we must consider two cases.

(1) $n>2$. In this case we can use essentially the previous argument for elliptical coordinates in $E_n$. Then, as expected, linear combinations of $v_i(u)=v(u)=(u-a)^p$, $p=0,\pm1,\pm2,...$ form a basis for all Stäckel multipliers that are rational functions of cartesian coordinates.

(2) $n=2$. In this case we cannot use the previous argument, as there is only one $e_k$ ocurring, viz $e_2$, which we can take to be 0. We can only deduce that $v(u)=R(u)u^{1/2}+S(u)$ where $R$ and $S$ are rational functions. If $R(u)=0$ then our result follows as for $n>2$. If $R(u)\neq 0$ then we change to the new variable $w=u^{1/2}$ and discard the function $S(u)$. The corresponding

Stäckel multiplier has the form

$$v(w_1, w_2) = \frac{T(w_1) - T(w_2)}{w_1^2 - w_2^2} \tag{1.15}$$

where $T(w)$ is a rational function of $w$ of the form $R(w^2)w$. If we perform a partial fraction decomposition of $R(u)$ with respect to $u$ then the function $T(w)$ typically has terms of the *type*/1 which are monomials $w^p$ and terms of *type*/2 which are of the form $w/(w^2 - a^2)^p$. In the case of monomials, if $p$ is even then the corresponding contribution to the Stäckel multiplier is a polynomial in the cartesian coordinates $x_k$. If $p$ is odd then the corresponding contribution can be written as an algebraic function of the symmetric functions $S_i[w_1, w_2]$. Furthermore these symmetric functions cannot be expressed as rational functions of $x_k$. Indeed they are determined by algebraic relations. For example, in the case of

$$S_{1/2}[u_1, u_2] = S_1[w_1, w_2] = w_1 + w_2 = u_1^{1/2} + u_2^{1/2}$$

the relation between $S_{1/2}$ and the symmetric polynomials $S_i = S_i[u_1, u_2]$ is given by

$$[S_{1/2}^2 - S_1]^2 = 4S_2^2.$$

In a similar way we could argue that contributing terms to $v(w_1, w_2)$ of the form $1/(w - a)^p$ cannot contribute rational functions of $x_k$ to the expression for a separable potential. We are thus left with the possibility that $v(u)$ must be a rational function of $u$. Consequently $v(u)$ admits a partial fraction decomposition with typical terms of the form $u^p$ or $1/(u-a)^q$ for $p$ and $q$ positive integers.

In the following sections we will answer questions 1) and 2) for a variety of classical mechanical systems embeddable into flat space with cartesian coordinates $\{x_i\}$ and for which the Hamilton-Jacobi equation can be solved via separation of variables in a particular curvalinear system $\{u_j\}$. Many examples are given in the appendix.

In summary, the conclusions of section I are as follows. It is known that separable coordinates in $n$ dimensional Euclidean space are constructed from generic separable coordinate systems viz. elliptical coordinates and paraboloidal coordinates,[6]. For these coordinate systems we have asked what potentials rational in the cartesian coordinates can be found such that solution via the separation of variables ansatz is still valid. We have shown that potentials fulfilling this requirement can be constructed from the symmetric quantities $S_\ell[u_1, ..., u_n]$ and that these quantities can always be expressed as a rational function of the cartesian coordinates $x_k$ via (1.8). These ideas are extended in subsequent sections to the case of the sphere and the hyperboloid in $n$ dimensions. The result of these investigations gives all potentials which are rational in terms of cartesian coordinates and have the separability property. In addition we develop these ideas even further to include spaces in which the rational Stäckel form of elliptic coordinates is generalised and show that similar results hold in this case also. This we do via what amounts to flat space embeddings.

**II. Elliptic coordinates on the $n$ sphere.** The application of these ideas to the case of elliptic coordinates on the $n$-sphere proceeds with some modifications.

Elliptic coordinates on $S_n$ are given by

$$x_i^2 = \frac{\Pi_{j=1}^n (u_j - e_i)}{\Pi_{k\neq i}(e_k - e_i)}, \quad i, k = 1, \cdots, n+1, \quad j = 1, \cdots, n, \tag{2.1}$$

subject to $e_1 < u_1 < e_2 < u_2 < .... < e_n < u_n < e_{n+1}$ and $\sum_{i=1}^{n+1} x_i^2 = 1$. A typical form of Stäckel multiplier is

$$V(u_1, ...., u_n) = \sum_{i=1}^{n} \frac{v_i(u_i)}{\Pi_{m\neq i}(u_m - u_i)}. \tag{2.2}$$

Again we ask the question which separable potentials can be expressed as rational functions of the projective coordinates $x_k$. The expression (1.9) is now replaced by

$$\sum_{k=1}^{n+1} \frac{x_k^2}{u - e_k} = \frac{\Pi_{k=1}^n (u - u_k)}{\Pi_{j=1}^{n+1}(u - e_j)}, \tag{2.3}$$

from which we deduce that

$$S_\ell[u_1, ..., u_n] = \sum_{k=1}^{n+1} x_k^2 E_\ell[e_1, ..., e_{k-1}, e_{k+1}, ..., e_n]. \tag{2.4}$$

The argument used for the case of Euclidean elliptic coordinates goes through much as before. Indeed the Stäckel multipliers $SM_p[u_1, ..., u_n]$ with $p = 0, \pm 1, \pm 2, .....$ form a basis for all such potentials. Among these potentials are several well known examples of separable coordinate systems on the $n$ sphere. Indeed $V = S_1[u_1, ...., u_n] = \sum_{i=1}^n u_i$ corresponds to the Neumann potential $\sum_{i=1}^{n+1} e_i(1 - x_i^2)$, [1,3,7]. The Stäckel multiplier $1/x_k^2$ corresponds to the choice of function $v(u) = 1 - \sum_{k=1}^{n+1} \Pi_{\ell\neq k}(u - e_\ell)]/(u - e_k)$. The often quoted Rosochatius potential $V = \sum_{k=1}^{n+1} c_k/x_k^2$, [17] can then be constructed from the corresponding Stäckel multipliers. For the Garnier system with potential

$$V(x_1, ..., x_n) = (\sum_{k=1}^{n} x_k^2)^2 - (\sum_{k=1}^{n} E_1[e_1, .., e_{k-1}, e_{k+1}, ..., e_n] x_k^2)$$

the corresponding function is

$$v_k(u) = u^{n+1} - 2E_1[e_1, ...e_n]u^n + (E_2[e_1, ...e_n] - E_1[e_1, ...e_n]^2)u^{n-1} + 2E_1[e_1, ...e_n]^2.$$

**III. Complex $S_n$, complex $E_n$ and limiting real cases.** Having solved the problem of rational potentials for generic coordinates on the $n$ sphere and in Euclidean $n$ space we can now answer the same question for degenerate versions of these coordinates. We make use of well established limiting procedures, [7,18]. In particular we consider those cases relevent to the real manifolds 1) Euclidean $n$ space $E_n$, 2)the $n$ sphere $S_n$, 3) the $n$ dimensional hyperboloid $H_n$. This can be best done by first considering complex Euclidean $n$ space and the complex $n$ sphere, and then passing to the real cases. For the complex $n$ sphere let us consider what modifications are necessary for the solution of our problems. The process of taking two roots $e_i$ equal best illustrates the general procedure. Indeed if we put

$$x_2 \to \sqrt{a_2}(x'_1 + \epsilon x'_2), \quad x_1 \to \sqrt{a_1}x'_1,$$
$$x_j \to x'_j, \quad j = 3, ..., n \tag{3.1}$$

where $a_1 = -a_2 = 1/\epsilon$ and $e_2 = e_1 + \epsilon$, then the generic elliptic coordinates on the $n$ sphere become

$$\begin{aligned}
{x'_1}^2 &= \Pi_{i=1}^n (u_i - e_1)/\Pi_{k=3}^{n+1}(e_k - e_1) \\
2x'_1x'_2 &= (\partial/\partial e_1)[\Pi_{i=1}^n (u_i - e_1)/\Pi_{k=3}^{n+1}(e_k - e_1)] \\
{x'_j}^2 &= \Pi_{i=1}^n (u_i - e_j)/\Pi_{k\neq j}^{n+1}(e_k - e_j), j = 3, ..., n+1 \\
2x'_1x'_2 &+ \sum_{i=3}^{n+1} {x'_i}^2 = 1.
\end{aligned} \tag{3.2}$$

For this example, the argument used previously in the generic case is readily adapted. Indeed if we take a Stäckel multiplier in the form (1.4) the function $v(u)$ must be a rational function of $u$. Accordingly the basis for the rational Stäckel multipliers in this coordinate system is exactly the same when written in terms of the coordinates $u_i$. To express these Stäckel mulipliers in terms of the coordinates $x_i'$ we need only take the appropriate limits in the expressions (1.9). The functions $S_i[u_1, ...., u_n]$ can be obtained from the new form of (1.9), viz.

$$\frac{2x_1x_2}{u-e_1} + \frac{x_1^2}{(u-e_1)^2} + \sum_{k=3}^{n+1} \frac{x_k^2}{u-e_k} = \frac{\Pi_{k=1}^n (u-u_k)}{(u-e_1)^2 \Pi_{j=3}^{n+1}(u-e_j)}. \tag{3.3}$$

For the explicit form of the functions $S_\ell[u_1, ..., u_n]$ we obtain

$$\begin{aligned} S_\ell[u_1, ..., u_n] &= 2x_1x_2 E_\ell[e_1, e_3, ...., e_{n+1}] + x_1^2 E_{\ell-1}[e_3, ..., e_{n+1}] + \\ &\sum_{k=3}^{n+1} x_k^2 E_\ell[e_1, e_1, ..., e_{k-1}, e_{k+1}, ..., e_{n+1}]. \end{aligned} \tag{3.4}$$

Here we have dropped the primes in the $x_k$. We note in particular from (3.3) that $2x_1x_2 + \sum_{k=3}^{n+1} x_k^2 = 1$. For this particular case the Neumann potential becomes

$$V = 2e_1 + e_3 + ... + e_{n+1} - (x_1^2 + 2e_1x_1x_2 + \sum_{k=3}^{n+1} x_k^2). \tag{3.5}$$

The most general possibility of this kind occurs when the $e_i$'s are equal in groups. Specifically, if we write instead of $x_i$ and $e_k$ the new variables $x_j^J$ and $e_j^J$, where $j = 1, ..., N_J,\ J = 1, ..., P$, respectively, the infinitesmal distance is

$$ds^2 = \sum_{k=1}^n \frac{\Pi_{m\neq i}(u_m - u_i)}{\Pi_{j=1}^{N_J}(u_k - e_j^J)} du_k^2. \tag{3.6}$$

These coordinates can be obtained from the most general elliptic ones by well defined limiting processes, [13]. Under the transformation

$$\begin{aligned} &e_j^J \to e_J +^J \epsilon_{j-1}^1, \quad j = 1, ..., N_J, \quad J = 1, ..., P \\ &x_j^J \to \sqrt{a_j^J}(x_1^J + \sum_{i=2}^{N_J} {}^J\epsilon_{j+1-i}^{i-1} x_i^J), \end{aligned} \tag{3.7}$$

where

$${}^J\epsilon_{j+1-i}^{i-1} = \Pi_{\ell=2}^i ({}^J\epsilon_{j-1}^1 -^J \epsilon_\ell^1), \qquad \sum_{J=1}^P N_J = n+1,$$

and

$$a^J_j = 1/\Pi_{k\neq j}({}^J\epsilon^1_{j-1} - {}^J\epsilon^1_{k-1}), \quad k = 1, ..., N_J,$$

the fundamental formula (1.9) assumes the form

$$\sum_{K=1}^{P}\sum_{\ell=2}^{N_K+1}\frac{S^K_\ell}{(u-e_K)^{N_K+2-\ell}} = \frac{\Pi^n_{i=1}(u-u_i)}{\Pi^P_{K=1}(u-e_K)^{N_K}} \tag{3.8}$$

where $S^J_{j+1} = \sum^j_{i=1} x^J_i x^J_{j+1-i}$. The expressions for $S_i[u_1, ..., u_n]$ can be determined from this result. Indeed (3.8) can be rewritten

$$\Pi^n_{i=1}(u-u_i) = \sum_{K=1}^{P}\sum_{\ell=2}^{N_K+1} S^K_\ell (u-e_k)^{\ell-2}\Pi_{L\neq K}(u-e_L)^{N_L}. \tag{3.9}$$

We then deduce that

$$\begin{aligned} S_m[u_1, ..., u_n] = &\sum_{K=1}^{P}\sum_{\ell=2}^{N_K+1} S^K_\ell(-1)^{A_K+\ell+2-n} \\ &\times S_{A_K+\ell-2+m-n}[e_1(N_1), ..., e_K(\ell-2), ..., e_P(N_P)] \end{aligned} \tag{3.10}$$

where $A_K = \sum_{L\neq K} N_L$. The coordinates determined in this way are said to be generic coordinates of type $\{N_1, ...., N_P\}$. Here,

$$ds^2 = \sum_{k=1}^{n}\frac{\Pi_{m\neq i}(u_m-u_i)}{\Pi^P_{\ell=1}(u_k-e_\ell)^{N_\ell}}du^2_k. \tag{3.11}$$

By cross multiplying in the expression (3.8) we see immediately that

$$\sum_{J=1}^{P}\sum_{i=1}^{N_J} x^J_i x^J_{N_J+1-i} = 1 \tag{3.12}$$

and the coordinates $x^J_j$ are given by

$$S^J_{j+1} = \frac{1}{(j-1)!}(\frac{\partial}{\partial e_J})^{j-1}\frac{\Pi^n_{i=1}(u_i-e_J)}{\Pi_{L\neq J}(e_L-e_J)}, \tag{3.13}$$

where $S^J_{j+1} = \sum^j_{i=1} x^J_i x^J_{j+1-i}$.

The case of generic coordinates in $E_n$ can be treated similarly. The only restriction now is that $\sum_{J=1}^{p} N_J = n+1$ and

$$\sum_{i=1}^{n} x_i^2 \to \sum_{J=1}^{p}\sum_{i=1}^{N_J} x_i^J x_{N_J+1-i}^J$$

under the transformation given above.

The rules for constructing all other coordinate systems on the complex $n$ sphere that correspond to orthogonal coordinates are just as in [18] to which we refer. In the appendix we give some of the potentials for the simplest cases. For the case of hyperbolic $n$ space there are additional coordinate systems which can be regarded as generic. These correspond to the signatures $\{21...1\}$ and $\{31...1\}$, as well as the generic case $\{1...1\}$ in which $e_1 = e_2^*$. The first few potentials which correspond to these cases are given in the appendix.

For the case of complex Euclidean space similar general coordinate systems correspond to a metric of type (3.6) with $\sum_{j=1}^{P} N_j \leq n$. The various possible forms of this metric are determined by coordinates via limiting processes similar to those for the $n$ sphere. Making the same change of designation as before viz. $x_j^J$ and $e_j^J, \quad j = 1, ..., N_J,\ J = 1, ..., P$ and $\sum_{j=1}^{P} N_j = n$, we can calculate the properties of the various coordinates in $n$ dimensions can be calculated by suitable limiting procedures from the generic case in which all $e_i,\ i = 1, ..., n$ are distinct. The only difference in this case is that the condition $e_j^J \to e_J + {}^J\epsilon_{j-1}^1$ , could include the possibility that, say, $e_1 = \infty$. There are two possible signatures that describe coordinate systems of this type: $[N_1, ..., N_J]$, and $[\,{}^{\infty}_{N_1}, ..., N_J]$. All the calculations involving $[N_1, ..., N_J]$ look exactly as with the corresponding coordinates on the $n$ sphere, i.e., the limiting processes are given by (3.7), the fundamental formula by (3.8) and the symmetric functions by (3.10). The only restriction is that $\sum_{j=1}^{P} N_j = n$. For the second type of generic coordinates, which correspond to $[{}^{\infty}_{N_1}, ..., N_J]$, the coordinates are given by

(3.14)
$$S^1_{j+1} = (1/(j-1)!)(\partial/\partial\epsilon)^{j-1}[(\Pi^n_{\ell=1}(\epsilon u_\ell - 1))/(\epsilon e_J - 1)^{N_J}]|_{\epsilon=0}, \quad j = 1, .., N_1$$
$$S^J_{j+1} = [1/(j-1)!](\partial/\partial e_J)^{(j-1)}[\Pi^n_{i=1}(u_i - e_J)/\Pi_{L\neq J}(e_L - e_J)], \quad j = 1, ..., N_J,$$

where $S^J_{j+1}$ is defined as above with the extra condition that $x^1_1 = 1$. The fundamental form defining these coordinates is

(3.15)
$$(1/(N_1-1)!)(\partial/\partial\epsilon)^{N_1-1}[(\Pi^n_{\ell=1}(\epsilon u_\ell - 1))/((\epsilon u - 1)(\epsilon e_J - 1)^{N_J})]|_{\epsilon=0}$$
$$+ \sum_{J=2}^{P}\sum_{j=1}^{N_J} S^J_{j+1}/(u - e_J)^{N_J+1-j} = \Pi^n_{k=1}(u - u_k)/\Pi^P_{J=2}(u - e_J).$$

The symmetric functions $S_\ell[u_1, ..., u_n]$ can be read off from this formula.

**IV. Flat space embeddings.** It is possible to extend these ideas to rational Stäckel metrics not necessarily corresponding to spaces of constant curvature. As an example consider the Riemannian space whose metric is defined by

$$ds^2 = (u_1 - u_2)\left[\frac{du_1^2}{\Pi^5_{j=1}(u_1 - e_j)} - \frac{du_2^2}{\Pi^5_{j=1}(u_2 - e_j)}\right]. \tag{4.1}$$

This space is such that the corresponding Hamilton-Jacobi separation of variables method of solution works. It is indeed a special form of the Stäckel separable form we have given in the introduction. As an analogue of what we have been doing so far, we obtain these coordinates from an embedding in a higher dimensional flat space. Indeed we can obtain suitable coordinates by considering

$$x_i^2 = \frac{(u_1 - e_i)(u_2 - e_i)}{\Pi_{j\neq i}(e_i - e_j)}, \quad i = 1, .., 5, \tag{4.2}$$

subject to the restrictions

(4.3)
$$\sum_{i-1}^{5} x_i^2 = 0, \qquad \sum_{i-1}^{5} e_i x_i^2 = 0, \qquad \sum_{i-1}^{5} e_i^2 x_i^2 = 1.$$

We now ask the question; what form does a potential have to take in order that it be a Stäckel multiplier which is expressible as a rational function of the $x_i$ variables? If the Stäckel multiplier has the form (1.4) and if it is a polynomial in the $x_i$'s then with $u_2 = e_1$, say, $v(u)$ can be written in terms of the functions

$$R_{2345} = \sqrt{\Pi_{i=2}^{5}(u-e_i)}, \quad R_{ijk} = \sqrt{(u-e_j)(u-e_k)(u-e_i)}, \quad i,j,k \neq,$$
$$R_{jk} = \sqrt{(u-e_j)(u-e_k)}, \quad j,k \neq, \quad R_i = \sqrt{u-e_i}$$

as follows:

(4.4)
$$v(u) = B_{2345}(u)R_{2345} + \sum_{i,j,k\neq} B_{ijk}(u)R_{ijk} + \sum_{i,j\neq} B_{ij}(u)R_{ij} + \sum_{i} B_i(u)R_i + B(u),$$

where $i, j, k$ can have the values 2,3,4,5 and all the $B$ functions are rational functions of $u$. If we do this for all possible choices of $u_2 = e_3, e_4, e_5$ then this can only be consistent if $v(u)$ is a polynomial function of $u$. From the relations

(4.5)
$$u_1 + u_2 = E_1[e_1, e_2, e_3, e_4, e_5](\sum_{i=1}^{5} e_i^2 x_i^2) - (\sum_{i=1}^{5} e_i^3 x_i^2)$$
$$u_1 u_2 = (\sum_{i=1}^{5} e_i^4 x_i^2) + E_1[e_1, e_2, e_3, e_4, e_5](\sum_{i=1}^{5} e_i^3 x_i^2) - E_2[e_1, e_2, e_3, e_4, e_5](\sum_{i=1}^{5} e_i^2 x_i^2)$$

we can deduce the form of the corresponding potentials in terms the coordinates $x_i$. A similar argument can also be used to show that if the potential is a rational function of the $x_i$ then the corresponding function $v(u)$ must also be rational.

These results extend easily to the case of general coordinate systems of this type. Indeed let us consider a Riemannian space with infinitesimal distance

$$ds^2 = \sum_{k=1}^{n}[\Pi_{m\neq i}(u_m - u_i)]/[\Pi_{\ell=1}^{p}(u_k - e_\ell)]du_k^2 \tag{4.6}$$

where $p > n+1$. A suitable choice of cartesian coordinates is

$$x_i^2 = (1/4)\Pi_{j=1}^{n}(u_j - e_i)/\Pi_{k\neq i}(e_k - e_i), \quad i,k = 1,...,p. \tag{4.7}$$

The subspace spanned by these coordinates is given by the simultaneous equations

$$\sum_{i=1}^{p} e_i^q x_i^2 = \delta_{q,p-n-1}, \quad q = 1,...,p-n-1. \tag{4.8}$$

The symmetric functions $S_r[u_1,...,u_n]$ are given by the formulas

$$S_r[u_1,...,u_n] = (-1)^{p-n}\sum_{s=0}^{r}(-1)^s E_{r-s}[e_1,...,e_p]\sum_{k=1}^{p} e_k^{p-n-1+s}x_i^2. \tag{4.9}$$

The surfaces thus defined are the intersection of $p-n-1$ quadrics. The result expounded for the particular example given above holds in general for these spaces: If the function $v(u)$ in the corresponding Stäckel multiplier is a polynomial/rational in $u$, the Stäckel multiplier is polynomial/rational in the coordinates $x_i$.

As a corollary let us consider a Riemannian space with infinitesimal distance

$$ds^2 = \sum_{k=1}^{n}[\Pi_{m\neq k}(u_m - u_k)][\Pi_{j=1}^{q}(u_k - E_j)]/[\Pi_{\ell=1}^{p}(u_k - e_\ell)]du_k^2. \tag{4.10}$$

This system can be obtained from one of the previous type with infinitesimal distance

$$ds^2 = \sum_{k=1}^{n+q}[\Pi_{m\neq k}(u_m - u_k)]/[\Pi_{\ell=1}^{p}(u_k - e_\ell)]du_k^2, \tag{4.11}$$

subject to the restrictions $u_{n+1} = E_1, ..., u_{n+q} = E_q$. These can be interpreted as the extra constraints

$$\sum_{i=1}^{n+p} x_i^2/(E_j - e_i) = 0, \qquad j = 1, ..., q. \tag{4.12}$$

Just as before it is possible to make some of the $e_i$ equal. The expressions for the infinitesimal distance, coordinates and the generating function are essentially the same as given previously for systems of type $[N_1, ..., N_P]$. The constraints now have the form

$$\sum_{J=1}^{P}\sum_{k=1}^{N_J} S_{N_J-1+k}[(\partial/\partial e_J)^{N_J-k} e_J^q] = \delta_{q,p-n-1} \tag{4.13}$$

for $q = 1, ..., p - n - 1$.

The results of this section can be summarised as follows. For Stäckel metrics of type

$$ds^2 = \sum_{k=1}^{n} [\Pi_{m\neq i}(u_m - u_i)]/[\Pi_{\ell=1}^P (u_k - e_\ell)^{N_\ell}] du_k^2, \tag{4.14}$$

an embedding into flat space can be distinguished in two ways.

(1) If $\sum_{\ell=1}^P N_\ell = N \geq n$ then the coordinates are given by

$$S_{j+1}^J = [1/(j-1)!](\partial/\partial e_J)^{(j-1)}[\Pi_{i=1}^n (u_i - e_J)/\Pi_{L\neq J}(e_L - e_J)],$$

where $j = 1, ..., N_J$, $J = 1, .., P$. The fundamental form defining these coordinates is

$$\sum_{J=1}^{P}\sum_{j=1}^{N_J} S_{j+1}^J/(u - e_J)^{N_J+1-j} = \Pi_{k=1}^n (u - u_k)/\Pi_{J=2}^P (u - e_J).$$

The coordinates $x_i, i = 1, ..., N$ are subject to the constraints

$$\sum_{J=1}^{P}\sum_{k=1}^{N_J} S_{N_J-1+k}[(\partial/\partial e_J)^{N_J-k} e_J^q] = \delta_{q,p-n-1}.$$

(2) If $\sum_{\ell=2}^{P} N_\ell = N \geq n$ then the coordinates are given by

$$S^1_{j+1} = (1/(j-1)!)(\partial/\partial\epsilon)^{j-1}[(\Pi^n_{\ell=1}(\epsilon u_\ell - 1))/(\epsilon e_J - 1)^{N_J}]|_{\epsilon=0}, \quad j = 1, .., N_1$$

$$S^J_{j+1} = [1/(j-1)!](\partial/\partial e_J)^{(j-1)}[\Pi^n_{i=1}(u_i - e_J)/\Pi_{L\neq J}(e_L - e_J)], \quad j = 1, ..., N_J,$$

where $N_1 = n - N$ and $x^1_1 = 1$. The fundamental form defining these coordinates is

$$(1/(N_1-1)!)(\partial/\partial\epsilon)^{N_1-1}[(\Pi^n_{\ell=1}(\epsilon u_\ell - 1))/((\epsilon u - 1)(\epsilon e_J - 1)^{N_J})]|_{\epsilon=0}$$
$$+ \sum_{J=2}^{P}\sum_{j=1}^{N_J} S^J_{j+1}/(u-e_J)^{N_J+1-j} = \Pi^n_{k=1}(u-u_k)/\Pi^P_{J=2}(u-e_J).$$

In either case the fundamental formula determines the symmetric functions $S_\ell[u_1, ..., u_n]$. Hence the possible Stäckel multipliers that give rise to rational potentials in terms of the cartesian coordinates can be determined.

Metrics of the form

$$ds^2 = \sum_{k=1}^{m}[\Pi_{m\neq k}(u_m - u_k)][\Pi^q_{j=1}(u_k - E_j)^{M_j}]/[\Pi^p_{\ell=1}(u_k - e_\ell)^{N_\ell}]du_k^2$$

can be obtained from (4.11) via the requirement that $u_1 = E_1, ..., u_{M_1} = E_1, u_{M_1+1} = E_2, ..., u_{M_1+M_2} = E_2, ...., u_{M_1+..+M_q} = E_q$. Making suitable redefinitions of the remaining variables we recover the metric given.

In all three cases we can adapt earlier proofs to establish that separable potentials which are rational in the corresponding cartesian coordinates are generated by the functions $v(u) = (u-a)^p$, $p = 0. \pm 1, \pm 2, \cdots$. What has been achieved here is a generalisation of the results of Wojciechowski, [19,20], who obtained families of separable potentials which corresponded to polynomial potentials on the sphere. Indeed these results provide a comprehensive generalisation since we have necessary and sufficent conditions that separable potentials can be obtained which are rational functions of "cartesian coordinates". These also include the rational family of potentials obtained by Wojciechowski.

**V. Further generalizations.** It is possible to generalise further the results presented thus far. Indeed if we consider Stäckel metrics of the form

$$ds^2 = (u_1 - u_2)\left[\frac{du_1^2}{(u_1-e_1)(u_1-e_2)(u_1-e_3)} - \frac{du_2^2}{(u_2-a_1)(u_2-a_2)(u_2-a_3)}\right], \tag{5.1}$$

the corresponding Hamilton-Jacobi and Schrödinger equations afford a separation of variables and the general form of the potential compatible with separation is

$$V(u_1, u_2) = \frac{v_1(u_1) - v_2(u_2)}{u_1 - u_2}. \tag{5.2}$$

The question we now ask is how much the analogy between what we have already done goes through for metrics like this one? We show that an analogue of the embedding into a suitable flat space can be achieved. For the example given above, a suitable choice of Euclidean coordinates is

$$\begin{aligned} z^2_{1\ell_1,2k_1} &= A_{1\ell_1,2k_1} u^2_{1\ell_1,2k_1} \\ &= A_{1\ell_1 2k_1}(u_1 - e_{\ell_1})(u_2 - a_{k_1}) \\ &= A_{1\ell_1 2k_1} v^2_{1\ell_1} w^2_{2k_1}, \\ z^2_{1\ell_1} &= A_{1\ell_1}(u_1 - e_{\ell_1}) = A_{1\ell_1} v^2_{1\ell_1}, \\ z^2_{2k_1} &= A_{2k_1}(u_2 - a_{k_1}) = A_{2k_1} w^2_{1k_1}, \end{aligned} \tag{5.3}$$

where $\ell_1, k_1 = 1, 2, 3$, and a suitable choice of constants is

$$
\begin{aligned}
A_{1\ell_1 2k_2} &= \frac{4}{3}\left[\frac{e_{\ell_2} - e_{\ell_3}}{(e_{\ell_1} - e_{\ell_2})(e_{\ell_1} - e_{\ell_3})} + \frac{a_{k_2} - a_{k_3}}{(a_{k_1} - a_{k_2})(a_{k_1} - a_{k_3})}\right] \\
A_{1\ell_1} &= \frac{4e_{\ell_1} - \frac{4}{3}(a_1 + a_2 + a_3)}{(e_{\ell_1} - e_{\ell_2})(e_{\ell_1} - e_{\ell_3})} \\
A_{2k_2} &= \frac{4a_{k_1} - \frac{4}{3}(e_1 + e_2 + e_3)}{(a_{k_1} - a_{k_2})(a_{k_1} - a_{k_3})}
\end{aligned}
$$

where $\ell_1, \ell_2, \ell_3$ and $k_1, k_2, k_3$ are even permutations of 1,2,3.

There are relations between the various cartesian coordinates thus produced. Indeed

$$
\begin{aligned}
& v_{1\ell_1}^2 - v_{1\ell_2}^2 = e_{\ell_2} - e_{\ell_1} \\
(5.4) \qquad & w_{1k_1}^2 - w_{1k_2}^2 = a_{k_2} - a_{k_1}, \\
& (a_{k_4} - a_{k_3})(u_{1\ell_1 2k_1}^2 - u_{1\ell_1 2k_2}^2) + (a_{k_1} - a_{k_2})(u_{1\ell_2 2k_3}^2 - u_{1\ell_2 2k_4}^2) \\
& \qquad + (a_{k_1} - a_{k_2})(a_{k_4} - a_{k_3})(e_{\ell_1} - e_{\ell_2}) = 0
\end{aligned}
$$

for $k_1 \neq k_2, k_3 \neq k_4$ and $\ell_1 \neq \ell_2$. Clearly, there are similar relations with the $e$ and $a$ interchanged.

For the example metric we are looking at, there are only six independent conditions of this quadratic type. There is also a degree of ambiguity in the choice of the constants $A_{1\ell_1 2k_1}$, $A_{1\ell_1}$ and $A_{2k_1}$. This is the result of the existence of the null forms

$$
\begin{aligned}
& d\omega_1^2 = du_{11,21}^2 - du_{11,23}^2 - du_{13,21}^2 + du_{13,23}^2 + (a_1 - a_3)(du_{11}^2 - du_{31}^2) + \\
& \quad (e_1 - e_3)(du_{21}^2 - du_{23}^2) = 0, \\
& d\omega_2^2 = du_{11,22}^2 - du_{11,23}^2 - du_{13,22}^2 + du_{13,23}^2 + (a_2 - a_3)(du_{11}^2 - du_{13}^2) + \\
& \quad (e_1 - e_3)(du_{22}^2 - du_{23}^2) = 0, \qquad (5.5) \\
& d\omega_3^2 = du_{12,21}^2 - du_{12,23}^2 - du_{13,21}^2 + du_{13,23}^2 + (a_1 - a_3)(du_{12}^2 - du_{13}^2) + \\
& \quad (e_2 - e_3)(du_{21}^2 - du_{23}^2) = 0, \\
& d\omega_4^2 = du_{12,22}^2 - du_{12,23}^2 - du_{13,22}^2 + du_{13,23}^2 + (a_2 - a_3)(du_{11}^2 - du_{13}^2) + \\
& \quad (e_2 - e_3)(du_{22}^2 - du_{23}^2) = 0.
\end{aligned}
$$

Can this example be generalised? The answer is yes. Consider

$$v_{i\ell}^2 = (u_i - A_{i\ell}), \quad i = 1, ..., N; \quad \ell = 1, ..., m_i, \tag{5.6}$$

where we also take $m_i > N$ for all $i$. We define the symbols

$$U_{i_1\ell_{i_1}, i_2\ell_{i_2}, ..., i_k\ell_{i_k}} = v_{i_1\ell_{i_1}}^2 v_{i_2\ell_{i_2}}^2 \cdots, \qquad v_{i_k\ell_{i_k}}^2 \tag{5.7}$$

where $i_m \neq i_n$ for $m \neq n$ and $k = 1, ..., N$. We wish to find cartesian coordinates such that

$$z_{i_1\ell_{i_1}, i_2\ell_{i_2}, ..., i_k\ell_{i_k}}^2 = A_{i_1\ell_{i_1}, i_2\ell_{i_2}, ..., i_k\ell_{i_k}} U_{i_1\ell_{i_1}, i_2\ell_{i_2}, ..., i_k\ell_{i_k}} \tag{5.8}$$

and

$$ds^2 = \sum_{i_j, \ell_{i_j}} dz_{i_1\ell_{i_1}, i_2\ell_{i_2}, ..., i_k\ell_{i_k}}^2 = \sum_{i=1}^{N} \frac{\Pi_{j\neq i}(u_i - u_j)}{\Pi_{\ell_i=1}^{n_i}(u_i - A_{i\ell_i})} du_i^2. \tag{5.9}$$

This is a metric in Stäckel form which separates both Hamilton-Jacobi and Schrödinger equations. It is convenient to write $S_{j\ell_j} = 1/\Pi_{m_j\neq \ell_j}(A_{j\ell_j} - A_{jm_j})$ and also $A_j = \sum_{\ell_j} A_{j\ell_j}$. For these quantities the following identities hold

$$\sum_{\ell_j} S_{j\ell_j} A^k_{j\ell_j} = 0, \quad 0 \le k \le n_j. \tag{5.10}$$

In order to consider cartesian coordinates of the form (5.8) so as to produce the metric, we need to determine the the constants $A_{i_1\ell_{i_1},i_2\ell_{i_2},....,i_k\ell_{i_k}}$. These will be determined in such a way that for fixed $k$ the corresponding cartesian coordinates $z_{i_1\ell_{i_1},i_2\ell_{i_2},....,i_k\ell_{i_k}}$ are orthogonal. If $k = n$, then orthogonality is assured if we have

$$A_{i_1\ell_{i_1},i_2\ell_{i_2},....,i_N\ell_{i_N}} = \sum_{k=1}^{N} ({}_{N,k}\Lambda) S_{k\ell_k}. \tag{5.11}$$

To see this we consider, for instance, the term in $du_1du_2$. The corresponding contribution is $(1/4)(\sum_{k=1}^N ({}_{N,k}\Lambda)S_{k\ell_k})\Pi_{m=3}^N(u_m - A_{m\ell_m})$. Clearly, from the identities (5.10), this type of term summed on the $\ell_j$ gives zero. In general, the correct formula is

$$A_{i_1\ell_{i_1},i_2\ell_{i_2},....,i_k\ell_{i_k}} = \sum_{s=1}^{k}\sum_{q=0}^{N-k}\sum_{p,s} \Lambda^{s_1,\dots s_q}_{i_{k+1},\dots,i_N} A_{s_1}...A_{s_q}(A_{i_j\ell_{i_j}})^{n-k-q} S_{i_s\ell_{i_s}} \tag{5.12}$$

where ${}_{p,s}\Lambda^{s_1,\dots s_q}_{i_{k+1},\dots,i_N}$ is symmetric in the indices $s_1, ..., s_q$ and $i_{k+1}, ..., i_N$. Summation on the indices $s_1, ..., s_q, s_1 \neq s_2 \neq, , , \neq s_q$ and $\{s_1, ..., s_q\} \subset \{i_{k+1}, ...., i_N\}$ is implied. It remains to find the coefficents ${}_{p,s}\Lambda^{s_1,\dots s_q}_{i_{k+1},\dots,i_N}$. Let us consider the coefficent of $du_1^2$. We wish to construct the coefficent to be $\Pi_{j\neq 1}(u_1 - u_j)/\Pi_{\ell_1=1}^{n_1}(u_1 - A_{1\ell_1})$. In particular we consider the term $(-1)^{N-1}u_2...u_N/\Pi_{\ell_1}(u_1 - A_{1\ell_1})$. Typical contributing factors to the numerator are $(\sum_{k=1}^N {}_{N,k}\Lambda\ S_{k\ell_k})(\Pi_{m=2}^N(u_m - A_{m\ell_m}))$. The coefficent of $(-1)^{N-1}u_2...u_N/[\Pi_{\ell_1}(u_1 - A_{1\ell_1})]$ is $\sum_{\ell_1,\ell_2,..,\ell_N\, N,k} \Lambda\ S_{1\ell_1}/(u_1 - A_{1\ell_1}) =$

$n_2 ... n_N/[\Pi_{\ell_1}(u_1 - A_{1\ell_1})]$ from which it follows that ${}_{N,1}\Lambda = (-1)^{N-1}/(n_2 ... n_N)$. Procceding to the calculation of the coefficent of $u_3 ... u_N/\Pi_{\ell_1}(u_1 - A_{1\ell_1})$ we find the contribution of terms of the form (5.12) for $k = N$ is $-A_2 n_3, .., n_N({}_{N,1}\Lambda)$. The other contributing term comes from

$$A_{1\ell_1 3\ell_3,...,N\ell_N} U_{1\ell_1 3\ell_3,...,N\ell_N} = \sum_{k\neq 2}[({}_{N-1,k}\Lambda)(A_{k\ell k}) + ({}_{N-1,k}\Lambda_2^2)A_2]S_{k\ell_k} \tag{5.13}$$

and is ${}_{N-1,1}\Lambda_2^2$. Therefore we have ${}_{N-1,1}\Lambda_2^2 ={}_{N,1} \Lambda$. Proceeding in this way we can establish that

$$ {}_{p,i_1}\Lambda_{i_{p+1},...,i_N}^{i_{p+1},...,i_N} ={}_{N,i_1} \Lambda.$$

Repeating these arguments for the other coeffîcents we obtain the general formula

$$ {}_{N-p,i_p}\Lambda_{i_{p+1},....,i_N} = (-1)^{N-p}/\Pi_{k=1}^{p-1} n_{i_k} \tag{5.14}$$

$$ {}_{N-p-q,i_p}\Lambda^{r_1,...,r_q}_{i_{p+1},....,i_N,r_1,...,r_q} ={}_{N-p,i_p} \Lambda_{i_{p+1},....,i_N}.$$

This completes the embedding of the coordinates $v_{i\ell_i}$ into a flat space of dimension $\Pi_{k=1}^N(n_k + 1)$. The various relations among the coordinates are consequences of the relations $v_{i\ell_i}^2 - v_{j\ell_j}^2 = A_{j\ell_j} - A_{i\ell_i}$. The choice of coordinates given above is unique modulo quadratic forms which are null and diagonal. The set of all such forms is generated by coefficents given by

$$A_{i_1\ell_{i_1}....i_k\ell_{i_k}} = (-1)^{c_k}, \qquad A_{i_s\ell_{i_s}} = \Pi_{j\neq s}(a_{i_j} - b_{i_j})\epsilon_{\ell_{i_s}}, \tag{5.15}$$

where $\ell_{i_s} = a_{i_s}$ or $b_{i_s}$ and $\epsilon_{i_s} = 1$ if $\ell_{i_s} = a_{i_s}$ and $\epsilon_{i_s} = -1$ if $\ell_{i_s} = b_{i_s}$. Further, $c_k = \sum_{j=1}^k \epsilon_{\ell_{i_j}}$.

Separable potentials that are polynomials are constructed in the same way as previously from the symmetric functions $S_\ell[u_1, ..., u_n]$. In particular if we define

$$T_{i_k} = \sum_{\ell_1,...,\ell_{n_{i_k}}} \epsilon^{\ell_1,....,\ell_{n_{i_k}}} A_{i_k\ell_1}....A_{i_k\ell_{n_{i_k}-1}} v_{i_k\ell_{n_{i_k}}}^2 \tag{5.16}$$

and

$$P_{i_k} = \Pi_{\ell_{i_k} > m_{i_k}} (A_{i_k \ell_{i_k}} - A_{i_k m_{i_k}}),$$

then

$$\Pi_{j=1}^{k} u_{i_j} = \Pi_{j=1}^{k} \frac{T_{i_j}}{P_{i_j}}. \tag{5.17}$$

In the above formula $\epsilon^{\ell_1 \cdots \ell_q}$ is the totally skew symmetric tensor. From these formulas, expressions can be constructed for the symmetric functions in terms of quadratic functions of the $x_i$'s. If we look for potentials which are rational functions of the $x_i$'s then the restriction to symmetric functions is no longer necessary. In fact all that is necessary is that the corresponding Stäckel multplier be a rational function of the variables $u_i$.

It is also possible to consider metrics for which some of the constants $A_{i\ell_i}$ are equal. As an example, consider the coordinates defined by

$$\begin{aligned}
\omega_{11,2\ell_2}^2 &= \frac{(u_1 - e_1)(u_2 - a_{\ell_2})}{(e_1 - e_3)} \\
2\omega_{11,2\ell_2}\omega_{12,2\ell_2} &= \frac{2(u_1 - e_1)(u_2 - a_{\ell_2})}{(a_{\ell_3} - a_{\ell_2})(a_{\ell_3} - a_{\ell_2})} + \\
&\quad \frac{\partial}{\partial e_1} \frac{(u_1 - e_1)(u_2 - a_{\ell_2})}{(e_1 - e_3)}, \\
\omega_{13,2\ell_2}^2 &= [\frac{-1}{(e_3 - e_2)^2} + \frac{-1}{(a_{\ell_3} - a_{\ell_2})(a_{\ell_3} - a_{\ell_2})}](u_1 - e_3)(u_2 - a_{\ell_2}) \\
\omega_{11}^2 &= [4(e_1 - (1/3)(a_1 + a_2 + a_3))(u_1 - e_1)/(e_1 - e_3)] \\
2\omega_{11}\omega_{12} &= \frac{\partial}{\partial e_1} \frac{4(e_1 - \frac{1}{3}(a_1 + a_2 + a_3)(u_1 - e_1)}{(e_1 - e_3)} \\
\omega_{13}^2 &= \frac{4((e_3 - \frac{1}{3}(a_1 + a_2 + a_3))(u_1 - e_3)}{(e_1 - e_3)^2} \\
\omega_{2\ell_2}^2 &= \frac{4(a_{\ell_2} - \frac{1}{3}(2e_1 + e_3))(u_2 - a_{\ell_2})}{(a_{\ell_3} - a_{\ell_2})(a_{\ell_3} - a_{\ell_2})}.
\end{aligned} \tag{5.18}$$

The corresponding infinitesmal distance is

$$
\begin{aligned}
ds^2 &= \sum_{\ell_2=1}^{3}(2d\omega_{11,2\ell_2}d\omega_{12,2\ell_2} + d\omega_{13,2\ell_2}^2 + d\omega_{2\ell_2}^2) + 2d\omega_{11}d\omega_{12} + d\omega_{13}^2 \\
&= (u_1-u_2)[\frac{du_1^2}{(u_1-e_1)^2(u_1-e_3)} - \frac{du_2^2}{(u_2-a_1)(u_2-a_2)(u_2-a_3)}].
\end{aligned}
\tag{5.19}
$$

It is also possible to lift the restriction $m_i > N$. We illustrate this with the following example. If we choose coordinates $\omega_i$ such that

$$
\begin{aligned}
\omega_1^2 &= (u_2-a_3)^2, \qquad \omega_1\omega_2 = 2(u_2-a_3)u_1 \\
\omega_3^2 &= u_2-a_3, \qquad \omega_3\omega_4 = (u_2-a_3)(-1+4(u_2-a_3)) \\
\omega_5^2 &= u_1-e_2, \qquad \omega_5\omega_6 = \frac{(u_1-e_2)(-5u_2+4e_2+a_3)}{e_2-e_3} \\
\omega_7^2 &= u_1-e_3, \qquad \omega_7\omega_8 = \frac{(u_1-e_3)(-5u_2+4e_3+a_3)}{e_3-e_2} \\
\omega_9^2 &= -(u_1-e_2)(u_2-a_3))/(e_3-e_2), \qquad \omega_{10}^2 = -(u_1-e_3)(u_2-a_3))/(e_2-e_3) \\
\omega_{11}^2 &= u_2^2,
\end{aligned}
\tag{5.20}
$$

the corresponding infinitesmal distance is

$$
\begin{aligned}
ds^2 &= d\omega_1 d\omega_2 + d\omega_3 d\omega_4 + d\omega_5 d\omega_6 + d\omega_7 d\omega_8 + d\omega_9^2 + d\omega_{10}^2 + d\omega_{11}^2 \\
&= 2(u_1-u_2)[\frac{du_1^2}{(u_1-e_1)(u_1-e_3)} - \frac{du_2^2}{u_2-a_3}].
\end{aligned}
$$

**Appendix.** Here we adopt the shortened notation $S_i = S_i[u_1, ..., u_n]$ and work out a few specific examples of the properties that have been developed above. For the expressions for the Stäckel multipliers we have

$$
SM_{n-1} = 1, \qquad SM_n = S_1, \qquad SM_{n+1} = S_1^2 - S_2, \tag{A.1}
$$

$$
SM_{n+2} = S_1^3 - 2S_1S_2 + S_3, \qquad SM_{n+3} = S_1^4 - 3S_1^2S_2 + 2S_1S_3 + S_2^2 - S_4.
$$

(1) Elliptical coordinates in Euclidean $n$ space

In this case the functions $S_i$ are given by (1.8). Consequently

$$
\begin{aligned}
SM_n &= E_1[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 \\
SM_{n+1} &= (E_1[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2)^2 - (E_2[e_1, ..., e_n] + \\
&\qquad \sum_{k=1}^{n} x_k^2 E_1[e_1, .., e_{k-1}, e_{k+1}, .., e_n]) \\
SM_{n+2} &= (E_1[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2)^3 - 2(E_1[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2) \\
&\qquad \times (E_2[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 E_1[e_1, .., e_{k-1}, e_{k+1}, .., e_n]) + \\
&\qquad (E_3[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 E_2[e_1, .., e_{k-1}, e_{k+1}, .., e_n]).
\end{aligned} \tag{A.2}
$$

(2) Paraboloidal coordinates in $E_n$

In this case the functions $S_i$ are given by (1.13). Consequently

$$
\begin{aligned}
SM_n &= -2x_1 + E_1[e_2, ..., e_n] \\
SM_{n+1} &= (-2x_1 + E_1[e_2, ..., e_n])^2 - (E_2[e_2, ..., e_n] - \\
&\qquad 2x_1 E_1[e_2, ..., e_n] - \sum_{k=2}^{n} x_k^2) \\
SM_{n+2} &= (-2x_1 + E_1[e_2, ..., e_n])^3 - 2(-2x_1 + E_1[e_2, ..., e_n]) \\
&\qquad \times (E_2[e_2, ..., e_n] - 2x_1 E_1[e_2, ..., e_n] - \sum_{k=2}^{n} x_k^2) + \\
&\qquad (E_3[e_2, ..., e_n] - 2x_1 E_2[e_2, ..., e_n] - \\
&\qquad \sum_{k=2}^{n} E_1[e_2, ..., e_{k-1}, e_{k+1}, ..., e_n] x_k^2).
\end{aligned} \tag{A.3}
$$

For the $n$ sphere the corresponding Stäckel multipliers are given by

$$
\begin{aligned}
SM_n &= \sum_{k=1}^{n+1} E_1[e_1,...,e_{k-1},e_{k+1},...,e_{n+1}]x_k^2 \\
SM_{n+1} &= (\sum_{k=1}^{n+1} E_1[e_1,...,e_{k-1},e_{k+1},...,e_{n+1}]x_k^2)^2 - \\
&\quad \sum_{k=1}^{n+1} E_2[e_1,...,e_{k-1},e_{k+1},...,e_{n+1}]x_k^2 \\
SM_{n+2} &= (\sum_{k=1}^{n+1} E_1[e_1,...,e_{k-1},e_{k+1},...,e_{n+1}]x_k^2)^3 - \\
&\quad 2(\sum_{k=1}^{n+1} E_1[e_1,...,e_{k-1},e_{k+1},...,e_{n+1}]x_k^2) \\
&\quad \times (\sum_{k=1}^{n+1} E_2[e_1,...,e_{k-1},e_{k+1},...,e_{n+1}]x_k^2) + \\
&\quad (\sum_{k=1}^{n+1} E_3[e_1,...,e_{k-1},e_{k+1},...,e_{n+1}]x_k^2).
\end{aligned} \tag{A.4}
$$

The basic building blocks for rational potentials in the generic elliptic coordinate systems are obtained from the Stäckel mulipliers of the form (2.2) with $v(u) = u^{-p}$ where $p$ is a positive integer. We list the first few such potentials. In terms of the functions $S_i$ we have

$$
SM_{-1} = \frac{1}{S_n}, \qquad SM_{-2} = \frac{S_{n-1}}{S_n^2}, \qquad SM_{-3} = \frac{(S_{n-1}^2 - S_n S_{n-2})}{S_n^3},
$$

$$
SM_{-4} = \frac{(S_{n-1}^3 - 2S_n S_{n-1} S_{n-2} + S_{n-3} S_n^2)}{S_n^4}. \tag{A.5}
$$

For generic elliptical coordinates in Euclidean $n$ space these potentials have the form

$$SM_{-1} = \left(E_n[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 E_{n-1}[e_1, .., e_{k-1}, e_{k+1}, .., e_n]\right)^{-1}$$

$$SM_{-2} = \frac{E_{n-1}[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 E_{n-2}[e_1, .., e_{k-1}, e_{k+1}, .., e_n]}{(E_n[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 E_{n-1}[e_1, .., e_{k-1}, e_{k+1}, .., e_n])^2}$$

$$\begin{aligned} SM_{-3} = {} & ((E_{n-1}[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 E_{n-2}[e_1, .., e_{k-1}, e_{k+1}, .., e_n])^2 \\ & - (E_n[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 E_{n-1}[e_1, .., e_{k-1}, e_{k+1}, .., e_n]) \\ & \times (E_{n-2}[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 E_{n-3}[e_1, .., e_{k-1}, e_{k+1}, .., e_n]))/ \\ & (E_n[e_1, ..., e_n] + \sum_{k=1}^{n} x_k^2 E_{n-1}[e_1, .., e_{k-1}, e_{k+1}, .., e_n])^3. \end{aligned} \tag{A.6}$$

For paraboloidal coordinates in Euclidean $n$ space these potentials have the form

$$SM_{-1} = -\left(2x_1 E_{n-1}[e_2, ..., e_n] + \sum_{k=2}^{n} x_k^2 E_{n-2}[e_2, .., e_{k-1}, e_{k+1}, .., e_n]\right)^{-1}$$

$$\begin{aligned} SM_{-2} = {} & (E_{n-1}[e_1, ..., e_n] - 2x_1 E_{n-2}[e_1, ..., e_n] - \\ & \sum_{k=2}^{n} x_k^2 E_{n-3}[e_2, .., e_{k-1}, e_{k+1}, .., e_n])/ \\ & (-2x_1 E_{n-1}[e_2, ..., e_n] - \sum_{k=2}^{n} x_k^2 E_{n-2}[e_2, .., e_{k-1}, e_{k+1}, .., e_n])^2 \end{aligned} \tag{A.7}$$

$$\begin{aligned} SM_{-3} = {} & ((E_{n-1}[e_1, ..., e_n] - 2x_1 E_{n-2}[e_1, ..., e_n] - \\ & \sum_{k=2}^{n} x_k^2 E_{n-3}[e_2, .., e_{k-1}, e_{k+1}, .., e_n])^2 - \\ & (-2x_1 E_{n-1}[e_2, ..., e_n] - \sum_{k=2}^{n} x_k^2 E_{n-2}[e_2, .., e_{k-1}, e_{k+1}, .., e_n]) \\ & \times (E_{n-2}[e_1, ..., e_n] - 2x_1 E_{n-3}[e_1, ..., e_n] - \\ & \sum_{k=2}^{n} x_k^2 E_{n-4}[e_2, .., e_{k-1}, e_{k+1}, .., e_n]))/ \\ & (-2x_1 E_{n-1}[e_2, ..., e_n] - \sum_{k=2}^{n} x_k^2 E_{n-2}[e_2, .., e_{k-1}, e_{k+1}, .., e_n])^3. \end{aligned}$$

For the corresponding coordinates on the $n$ sphere

$$
\begin{aligned}
SM_{-1} &= \left(\sum_{k=1}^{n+1} x_k^2 E_n[e_1,..,e_{k-1},e_{k+1},..,e_n]\right)^{-1} \\
SM_{-2} &= \frac{\sum_{k=1}^{n+1} x_k^2 E_{n-1}[e_1,..,e_{k-1},e_{k+1},..,e_n]}{(\sum_{k=1}^{n+1} x_k^2 E_n[e_1,..,e_{k-1},e_{k+1},..,e_n])^2} \\
SM_{-3} &= ((\sum_{k=1}^{n+1} x_k^2 E_{n-1}[e_1,..,e_{k-1},e_{k+1},..,e_n])^2 - \\
&\quad (\sum_{k=1}^{n+1} x_k^2 E_n[e_1,..,e_{k-1},e_{k+1},..,e_n]) \\
&\quad \times (\sum_{k=1}^{n+1} x_k^2 E_{n-2}[e_1,..,e_{k-1},e_{k+1},..,e_n]))/ \\
&\quad (\sum_{k=1}^{n+1} x_k^2 E_n[e_1,..,e_{k-1},e_{k+1},..,e_n])^3.
\end{aligned}
\tag{A.8}
$$

As noted previously there are additional types of generic coordinate systems associated with the $n$ dimensional hyperboloid. We here list the expressions for the symmetric functions $S_i[e_1, ..., e_n]$ in terms of the corresponding coordinates $x_k$.

(1)

$$
\begin{aligned}
S_i[u_1,...,u_n] &= -E_{i-1}[e_3,...,e_{n+1}]x_1^2 + E_i[e_1,e_3,...,e_{n+1}]2x_1x_2 + \\
&\quad \sum_{k=3}^{n+1} E_i[e_1(2),e_3,....,e_{k-1},e_{k+1},..,e_{n+1}]
\end{aligned}
$$

for coordinates of type $\{21...1\}$ and

(2)

$$
\begin{aligned}
S_i[u_1,...,u_n] &= E_{i-2}[e_4,...,e_{n+1}]x_1^2 - E_{i-1}[e_1,e_3,...,e_{n+1}]2x_1x_2 + \\
&\quad E_i[e_1(2),e_4,...,e_{n+1}](x_2^2 + 2x_1x_3) + \\
&\quad \sum_{k=4}^{n+1} E_i[e_1(3),e_4,....,e_{k-1},e_{k+1},..,e_{n+1}]
\end{aligned}
$$

for coordinates of type $\{31...1\}$. These expressions could of course be substituted into the expressions for the Stäckel multipliers given above. The relation of the coordinates $x_k$ to the standard coordinates on the hyperboloid can be deduced from the expressions $2x_1x_2 + \sum_{k=3}^{n+1} x_k^2 = 1$ for the case $\{21...1\}$ and $x_2^2 + 2x_1x_3 + \sum_{k=4}^{n+1} x_k^2 = 1$ for the case $\{31...1\}$. Indeed, in case 1) choose coordinates such that $x_1 = (y_1 - y_2)/\sqrt{2}, x_2 = (y_1 + y_2)/\sqrt{2}, x_k = iy_k, k = 3, ..., n+1$ and in case 2) choose coordinates such that $x_1 = (y_1 - y_3)/\sqrt{2}, x_2 = iy_2, x_3 = (y_1 + y_3)/\sqrt{2}, x_k = iy_k, k = 3, ..., n+1$. With these coordinates we have $y_1^2 - \sum_{k=2}^{n+1} y_k^2 = 1$. For $n = 3$ we give the expressions for the series of Stäckel multipliers.

(1) coordinates of type $\{111\}$

$$SM_2 = (e_2 + e_3)x_1^2 + (e_2 + e_1)x_2^2 + (e_2 + e_1)x_3^2$$

$$SM_3 = ((e_2 + e_3)x_1^2 + (e_2 + e_1)x_2^2 + (e_2 + e_1)x_3^2)^2 - (e_2e_3x_1^2 + e_2e_1x_2^2 + e_2e_1x_3^2)$$

$$SM_{-1} = \left(e_2e_3x_1^2 + e_2e_1x_2^2 + e_2e_1x_3^2\right)^{-1}$$

$$SM_{-2} = (e_2 + e_3)x_1^2 + (e_2 + e_1)x_2^2 + \frac{(e_2 + e_1)x_3^2}{(e_2e_3x_1^2 + e_2e_1x_2^2 + e_2e_1x_3^2)^2}$$

$$SM_{-3} = ((e_2 + e_3)x_1^2 + (e_2 + e_1)x_2^2 + (e_2 + e_1)x_3^2)^2 - (e_2e_3x_1^2 + e_2e_1x_2^2 + e_2e_1x_3^2))/(e_2e_3x_1^2 + e_2e_1x_2^2 + e_2e_1x_3^2)^3$$

(2) coordinates of type {21}

$$SM_2 = -x_1^2 + 2(e_1+e_3)x_1x_2 + 2e_1x_3^2$$

$$SM_3 = (-x_1^2 + 2(e_1+e_3)x_1x_2 + 2e_1x_3^2)^2 - (2e_1e_3x_1x_2 - e_3x_1^2 + e_1^2x_3^2)$$

$$SM_{-1} = \left(2e_1e_3x_1x_2 - e_3x_1^2 + e_1^2x_3^2\right)^{-1}$$

$$SM_{-2} = \frac{-x_1^2 + 2(e_1+e_3)x_1x_2 + 2e_1x_3^2}{(2e_1e_3x_1x_2 - e_3x_1^2 + e_1^2x_3^2)^2}$$

$$SM_{-3} = \frac{(-x_1^2 + 2(e_1+e_3)x_1x_2 + 2e_1x_3^2)^2 - (2e_1e_3x_1x_2 - e_3x_1^2 + e_1^2x_3^2)}{(2e_1e_3x_1x_2 - e_3x_1^2 + e_1^2x_3^2)^3}$$

(3) coordinates of type {3}

$$SM_2 = -2x_1x_2 + 2e_1(x_2^2 + 2x_1x_3)$$

$$SM_3 = (-2x_1x_2 + 2e_1(x_2^2 + 2x_1x_3))^2 - (e_1^2(x_2^2 + 2x_1x_3) + x_1^2 - 2e_1x_1x_2)$$

$$SM_{-1} = \left(e_1^2(x_2^2 + 2x_1x_3) + x_1^2 - 2e_1x_1x_2\right)^{-1}$$

$$SM_{-2} = \frac{-2x_1x_2 + 2e_1(x_2^2 + 2x_1x_3)}{(e_1^2(x_2^2 + 2x_1x_3) + x_1^2 - 2e_1x_1x_2)^2}$$

$$SM_{-3} = ((-2x_1x_2 + 2e_1(x_2^2 + 2x_1x_3))^2 - \frac{e_1^2(x_2^2 + 2x_1x_3) + x_1^2 - 2e_1x_1x_2)}{(e_1^2(x_2^2 + 2x_1x_3) + x_1^2 - 2e_1x_1x_2)^3}.$$

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