Tensor products of q-superalgebras and q-series identities. I

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Abstract

We introduce and work out examples of tensor products for distinct q-generalizations of $s\ell(2)$ type superalgebras, usually with both factors from the positive discrete series of representations. We show that various q-analogs of the exponential function can be used to mimic the exponential mapping from a Lie algebra to its Lie group and we compute the corresponding matrix elements of the "group operators" on these representation spaces. We show that the matrix elements themselves transform irreducibly under the action of the quantum superalgebra. The most important q-series identities derived here are interpreted as the expansion of the matrix elements of a "group operator" (via the exponential mapping) in a tensor product basis in terms

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of the matrix elements in a reduced basis. They involve q-hypergeometric series with base $-q,\ 0<q<1$.

1 Introduction

Zhedanov and others have introduced a product of generalized $s\ell_q(2)$ algebras that allows one to take tensor products of representations corresponding to two distinct algebras, [13, 5, 4]. Their generalization is an algebra (v,u) with generators $H,\ E_+,\ E_-$ which obey the commutation relations

$$[H,E_+]=E_+,\quad [H,E_-]=-E_-,\quad [E_+,E_-]=-uq^{-H}-vq^H. \tag{1}$$

Here, u and v are real numbers and $0<q<1$. For $uv\neq 0$ this algebra is isomorphic to one of the true $s\ell_q(2)$ type algebras, for $uv=0,\ u^2+v^2>0$ it is isomorphic to a special realization of the q-oscillator algebra, and for $u=v=0$ it is isomorphic to the Euclidean Lie algebra $m(2)$, [6]. This algebra has an invariant element

$$C=E_+E_-+\frac{vq^H-uq^{1-H}}{1-q}. \tag{2}$$

As pointed out by Zhedanov and others [13, 5, 4], the family of algebras admits a multiplication $(v,u)\otimes(-u,t)\cong(v,t)$, defined by

$$F_\pm=\Delta(E_\pm)=E_\pm\otimes q^{\frac{1}{2}H}+q^{-\frac{1}{2}H}\otimes E_\pm,\quad L=\Delta(H)=H\otimes I+I\otimes H. \tag{3}$$

The operators $F_\pm$, L satisfy the commutation relations (1). Using (3) we can easily define the tensor product $\rho\otimes\mu$ of a representation ρ of (v,u) and the representation μ of $(-u,t)$, thereby obtaining a representation of (v,t). This construction yields a convenient generalization of the tensor product computations in, for example, [7, 8, 9]. We follow this idea to study generalizations of the $osp_q(1/2)$ algebra, [2, 10, 12].

In this paper the q-superalgebra $[v,u]$ is defined by the generators $H,V_\pm$ and relations

$$[H,V_\pm]=\pm\frac{1}{2}V_\pm,\quad \{V_+,V_-\}=-uq^{-2H}-vq^{2H}, \tag{4}$$

where $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$. Here, u, v are parameters. The "Casimir" operator C for $[v, u]$ is

$$C = V_+ V_- + \frac{v}{1+q} q^{2H} + \frac{u}{1+q^{-1}} q^{-2H} \tag{5}$$

and satisfies the relations

$$[H, C] = 0, \quad \{V_\pm, C\} = 0. \tag{6}$$

Note that C^2 is an invariant operator, i.e., C^2 commutes with all elements of the $[v, u]$ algebra.

We define the coproduct

$$L = \Delta(H) = H \otimes I + I \otimes H, \quad F_\pm = \Delta(V_\pm) = V_\pm \otimes q^H + q^{-H} \otimes V_\pm, \tag{7}$$

where

$$(A \otimes B)(C \otimes D) = (-1)^{p(B)p(C)} AC \otimes BD. \tag{8}$$

Here $p(A)$ is the *parity* of the operator A. In this case the bosonic variables H, q^H have parity 0 and the fermionic variables $V_\pm$ have parity 1, [1, 10]. It follows that this algebra admits a multiplication of the form

$$[v, u] \otimes [-u, t] = [v, t], \tag{9}$$

where the first factor corresponds to the algebra $[v, u]$ and the second factor to algebra $[-u, t]$. Indeed the relations

$$[L, F_\pm] = \pm \frac{1}{2} F_\pm, \quad \{F_+, F_-\} = -tq^{-2L} - vq^{2L}, \tag{10}$$

are satisfied.

In §2 and §3 we study irreducible representations from the positive discrete series of the general algebra $[v, u]$ for $v^2 + u^2 > 0$ and work out the tensor product decomposition for (9). We compute the Clebsch-Gordan coefficients for the expansion and use them to derive q-series identities for the special functions that appear naturally in the theory. They are interpreted here as expansions of the matrix elements of a "group operator" in a tensor product basis in terms of the matrix elements in a reduced basis. The q-series are base $-q$, as follows naturally from the relations (7). In §4 we study finite dimensional irreducible representations and their tensor products.

This paper represents an introduction to the representation theory of q-superalgebras from a q-series point of view. Due to space limitations we have not discussed the representation theory of the Euclidean superalgebra $[0, 0]$ or decompositions of the form (9) where the representation of one factor lies in the positive discrete series and the representation of the other factor lies in the negative discrete or continuous series. In the latter case there is an intimate relationship with the orthogonality relations of a class of Askey-Wilson polynomials, and in both cases the tensor product decomposition may not be unique. We will discuss these examples in a forthcoming paper.

The notation used for q-series and q-integrals in this paper follows that of Gasper and Rahman [3].

2 Representations of the q-supersymmetric algebras $[v, u]$

We introduce a family (λ, v, u) of irreducible representations of the q-superalgebras $[v, u]$, belonging to the positive discrete series. A convenient basis $\{f_n : n = 0, 1, \cdots\}$ for the representation space is determined by the algebra action

$$V_- f_n = F(n) f_{n-1}, \quad V_+ f_n = G(n) f_{n+1}, \quad H f_n = (-\lambda + n/2) f_n \tag{11}$$

where λ is a complex parameter and $f_{-1} \equiv 0$. Now, inserting (11) into (4), we obtain

$$a_{n+1} + a_n = -uq^{2\lambda - n} - vq^{-2\lambda + n}, \quad F(0) = 0 \tag{12}$$

where $a_n = F(n)G(n-1)$. This equation is easily solved, and modulo a change of normalization of the basis, the solution is unique;

$$F(n) = \frac{q^{-n/2} - (-1)^n q^{n/2}}{q^{-1/2} + q^{1/2}}, \quad G(n) = (-1)^{n+1} v q^{n/2 - 2\lambda} - uq^{2\lambda - n/2}. \tag{13}$$

Therefore we get

$$V_- f_n = \frac{q^{-n/2} - (-1)^n q^{n/2}}{q^{-1/2} + q^{1/2}} f_{n-1}, \quad V_+ f_n = \left[(-1)^{n+1} v q^{n/2 - 2\lambda} - uq^{2\lambda - n/2}\right] f_{n+1},$$

$$H f_n = (-\lambda + n/2) f_n. \tag{14}$$

(For irreducibility of these infinite dimensional representations we require that the real constants λ, u, v are such that $-uq^{4\lambda} \neq v(-q)^k$ for any $k = 0, 1, \cdots$.) The "Casimir" operator (5) has the action

$$Cf_n = (-1)^n \mu f_n, \quad \mu = \frac{vq^{2\lambda}}{1+q} + \frac{uq^{-2\lambda}}{1+q^{-1}}. \tag{15}$$

Note that $C^2 = \mu^2 I$ is a multiple of the identity operator.

We investigate under what conditions we can introduce an inner product $< \cdot, \cdot >$ on the Hilbert space spanned by the basis vectors f_n, such that $H^* = H$ and $V_+^* = -V_-$. From relations (14) we see that the first condition implies that the basis is orthogonal and the second that

$$[(-1)^n vq^{n/2-2\lambda} + uq^{2\lambda-n/2}] \cdot ||f_{n+1}||^2 = \frac{q^{-(n+1)/2} - (-1)^{n+1} q^{(n+1)/2}}{q^{-1/2} + q^{1/2}} ||f_n||^2$$

for $n = 0, 1, \cdots$. This is possible if and only if

$$u > 0, \ -vq^{-4\lambda}, \ qvq^{-4\lambda} \tag{16}$$

and, for convenience in the following computations, we restrict ourselves to this case. Moreover, if we normalize the inner product such that $||f_0|| = 1$ then it follows that

$$||f_n||^2 = \left(\frac{q^{-2\lambda}}{u(1+q)}\right)^n \frac{(-q;-q)_n}{(\frac{-v}{u}q^{-4\lambda};-q)_n}, \quad n = 0, 1, \cdots. \tag{17}$$

Thus, an orthonormal basis for the representation space is given by the vectors

$$e_n = ||f_n||^{-1} f_n, \quad n = 0, 1, \cdots, \tag{18}$$

with algebra action

$$V_- e_n = \left[\frac{(q^{-n/2} - (-1)^n q^{n/2})(uq^{2\lambda+(1-n)/2} - v(-1)^n q^{-2\lambda-(1-n)/2})}{q^{-1/2} + q^{1/2}}\right]^{\frac{1}{2}} e_{n-1},$$

$$V_+ e_n = -\left[\frac{(q^{-(n+1)/2} - (-1)^{n+1} q^{(n+1)/2})(uq^{2\lambda-n/2} + v(-1)^n q^{-2\lambda+n/2})}{q^{-1/2} + q^{1/2}}\right]^{\frac{1}{2}} e_{n+1},$$

$$He_n = (-\lambda + n/2) e_n. \tag{19}$$

A functional realization of (λ, v, u) is

$$V_- = z^{-1}\frac{T_z^{-1/2} - R_z T_z^{1/2}}{q^{-1/2}+q^{1/2}}, \qquad V_+ = z(-uq^{2\lambda}T_z^{-1/2} - vq^{-2\lambda}R_z T_z^{1/2})$$

$$H = -\lambda + \frac{1}{2}z\frac{d}{dz}, \quad f_n(z) = z^n. \tag{20}$$

We now define an explicit inner product $< \cdot, \cdot >$ on our function space such that

$$(f,g) \ = \frac{1}{2\pi}\int_0^{2\pi}\int_{-k}^{k} f(re^{i\theta})\overline{g(re^{i\theta})}\rho(r^2)\ d_p r^2\ d\theta$$

where $z = re^{i\theta}$ and

$$\int_{-k}^{k} h(r^2)\ d_p r^2 = k(1-p)\sum_{n=0}^{\infty} h(kp^n)p^n.$$

Requiring that $(V_+f,g) = -(f,V_-g)$, $(Hf,g) = (f,Hg)$ for all polynomials f,g we find that the (essentially) unique solution is

$$k^{-1} = (1+q)uq^{2\lambda}, \quad \rho(r^2) = \frac{(uq^{2\lambda} - vq^{-2\lambda-1})(-q(1+q)uq^{2\lambda}r^2;p)_\infty}{((1+q)vq^{-2\lambda-1}r^2;p)_\infty},$$

normalized so that $||1||^2 = <1,1> = 1$. We see from (17) that the Hilbert space spanned by the basis vectors consists of all functions $f(z) = \sum_{n=0}^{\infty} c_n z^n$ such that

$$\sum_{n=0}^{\infty} |c_n|^2 \left(\frac{q^{-2\lambda}}{u(1+q)}\right)^n \frac{(-q;-q)_n}{(\frac{-v}{u}q^{-4\lambda};-q)_n} < \infty.$$

These are just the functions $f(z)$ analytic in the disk $|z| < \sqrt{u(1+q)}q^\lambda$. The Hilbert space has corresponding kernel function

$$S(\overline{z'},z) = \sum_{n=0}^{\infty} e_n(\overline{z'})e_n(z) = \frac{(-v(1+q)q^{-2\lambda}\overline{z'}z;-q)_\infty}{(u(1+q)q^{2\lambda}\overline{z'}z;-q)_\infty},$$

so that $(g, S(\overline{z'},\cdot)) \ = g(z')$ for $|z'| < \sqrt{u(1+q)}q^\lambda$ and g in the Hilbert space.

A second functional realization of (λ, v, u) is

$$V_- = z^{-1}\frac{T_z^{-1} - R_z}{q^{-1/2}+q^{1/2}}, \qquad V_+ = -zq^{2\lambda+1/2}(u - vq^{-4\lambda}R_z T_z)$$

$$H=-\lambda+\frac{1}{2}z\frac{d}{dz},\quad f_n(z)=q^{n(n+1)/4}z^n. \tag{21}$$

We employ the model (21) to define the following q-analogs of matrix elements of (λ, v, u):

$$a)\ e_{-q}(\beta V_+)E_{-q}(\alpha V_-)f_n=\sum_{n'=0}^{\infty}T^{(e+,E-)}_{n'n}(\alpha,\beta)f_{n'}, \tag{22}$$

$$b)\ E_{-q}(\beta V_+)e_{-q}(\alpha V_-)f_n=\sum_{n'=0}^{\infty}T^{(E+,e-)}_{n'n}(\alpha,\beta)f_{n'}, \tag{23}$$

$$c)\ e_{-q}(\beta V_-)E_{-q}(\alpha V_+)f_n=\sum_{n'=0}^{\infty}T^{(e-,E+)}_{n'n}(\alpha,\beta)f_{n'},\ |q^{2\lambda-n}\alpha\beta u/(1+q)|<1, \tag{24}$$

$$d)\ E_{-q}(\beta V_-)e_{-q}(\alpha V_+)f_n=\sum_{n'=0}^{\infty}T^{(E-,e+)}_{n'n}(\alpha,\beta)f_{n'}, \tag{25}$$

where $0<q<1$ and $\alpha,\beta\in \mathbb{C}$. Here $e_q(x)$, $E_q(x)$ are the standard q-exponential functions, [3]. The motivation for the choice of base $-q$ comes from the relations

$$(V_\pm\otimes q^H)(q^{-H}\otimes V_\pm)=-q^{\pm 1}(q^{-H}\otimes V_\pm)(V_\pm\otimes q^H),$$

relevant for the coproduct. Subject only to the conditions on (24), the left-hand sides of these expressions are analytic functions of z, convergent in a neighborhood of $z=0$. (For model (20) however, all series except (23) diverge.) Note that the matrix elements $T_{n'n}$ of the operator T with respect to the f_n basis and the matrix elements $\mathcal{T}_{n'n}$ with respect to the normalized basis e_n are related by

$$T_{n'n}\frac{||f_{n'}||}{||f_n||}=(||f_n||\cdot||f_{n'}||)^{-1}<Tf_n,f_{n'}>=<Te_n,e_{n'}>=\mathcal{T}_{n'n}. \tag{26}$$

Since $V_+^*=-V_-$ the following relationships hold:

$$T^{(e\pm,E\mp)}_{n'n}(\alpha,\beta)A_{n'n}=\overline{T^{(E\pm,e\mp)}_{nn'}(-\beta,-\alpha)}. \tag{27}$$

Here

$$A_{n'n}=\frac{||f_{n'}||^2}{||f_n||^2}=\left(\frac{q^{-2\lambda}}{u(1+q)}\right)^{n'-n}\frac{(-q;-q)_{n'}(-\frac{v}{u}q^{-4\lambda};-q)_n}{(-q;-q)_n(-\frac{v}{u}q^{-4\lambda};-q)_{n'}}.$$

Since the "group" operators that we are considering are unbounded, it is not clear that relations (26) and (27) make more than formal sense. However, by truncating the operator power series expressions for $e_p(\beta V_+)$, $E_p(\alpha V_-)$, etc. at N terms, we see that (26) and (27) hold rigorously for the truncated series. The final result follows in the limit as $N \to \infty$.

Since $e_p(z)E_p(-z) = 1$, we have the identity

$$\sum_\ell T^{(e+,E-)}_{n'\ell}(\alpha,\beta)T^{(e-,E+)}_{\ell n}(-\beta,-\alpha) = \delta_{n'n}, \quad |q^{2\lambda-n}\alpha\beta u/(1+q)| < 1. \quad (28)$$

Using the model (21) to compute the matrix elements (which are model independent) we obtain the explicit results

$$\begin{aligned} T^{(e+,E-)}_{n'n} &(\alpha,\beta) = \quad (29)\\ &(-u\beta)^{n'-n}q^{(n'-n)(8\lambda-n'-n+1)/4}\frac{(\frac{(-1)^{n+1}v}{u}q^{n-4\lambda};-q)_{n'-n}}{(-q;-q)_{n'-n}}\\ &\times\Big[\frac{(1-i)}{2}{}_2\phi_1\left(\begin{matrix}(-q)^{-n}, & \frac{u}{v}q^{4\lambda+1}(-q)^{-n}\\ (-q)^{n'-n+1} & \end{matrix};-q;\frac{-\alpha\beta iv}{q^{2\lambda-n}(1+q)}\right)\\ &+\frac{(1+i)}{2}{}_2\phi_1\left(\begin{matrix}(-q)^{-n}, & \frac{u}{v}q^{4\lambda+1}(-q)^{-n}\\ (-q)^{n'-n+1} & \end{matrix};-q;\frac{\alpha\beta iv}{q^{2\lambda-n}(1+q)}\right)\Big] \end{aligned}$$

$$\begin{aligned} T^{(e-,E+)}_{n'n} &(\alpha,\beta) = (-1)^{(n'-n)(n'-n+1)/2} \quad (30)\\ &\times(u\alpha)^{n'-n}q^{(n'-n)(8\lambda+n'-3n-1)/4}\frac{(\frac{(-1)^{n+1}v}{u}q^{n-4\lambda};-q)_{n'-n}}{(-q;-q)_{n'-n}}\\ &{}_2\phi_1\left(\begin{matrix}(-q)^{n'+1}, & \frac{u}{v}(-1)^{n+1}q^{n'-4\lambda}\\ (-q)^{n'-n+1} & \end{matrix};-q;\frac{(-1)^{n-n'+1}\alpha\beta u}{q^{n-2\lambda}(1+q)}\right), \end{aligned}$$

where $|q^{2\lambda-n}\alpha\beta u/(1+q)| < 1$. These results make sense for $n > n'$ as well as $n' \geq n$. Indeed, for $k > 0$ we have

$$\lim_{m\to -k}\frac{1}{(p;p)_m}{}_2\phi_1\left(\begin{matrix}A & B\\ p^{m+1} & \end{matrix};p,C\right) = \frac{(A,B;p)_k C^k}{(p;p)_k}{}_2\phi_1\left(\begin{matrix}Ap^k & Bp^k\\ p^{k+1} & \end{matrix};p,C\right). \quad (31)$$

Each of these families of matrix elements determines models of the irreducible representations (λ, v, u). To see this we make use of relations (4) and the following formal power series results for linear operators X, Y:

Lemma 1

$$E_p(\alpha X) Y e_p(\alpha X) = \sum_{n=0}^{\infty} \frac{\alpha^n}{(p;p)_n} \{X, Y\}_n$$

where

$$\{X, Y\}_0 = Y, \quad \{X, Y\}_{n+1} = X\{X, Y\}_n p^n + \{X, Y\}_n X, \ n = 0, 1, \cdots.$$

Lemma 2

$$e_p(\beta X) Y E_p(\beta X) = \sum_{n=0}^{\infty} \frac{\beta^n}{(p;p)_n} \{X, Y\}'_n$$

where

$$\{X, Y\}'_0 = Y, \quad \{X, Y\}'_{n+1} = X\{X, Y\}'_n + p^n \{X, Y\}'_n X, \ n = 0, 1, \cdots.$$

As a consequence of these lemmas we have, (for $p = -q$),

$$E_p(\alpha V_-)(V_+ + \frac{\alpha v}{1+q} q^{2H}) \ = \ (V_+ - \frac{\alpha u}{1+q} q^{-2H}) E_p(-\alpha V_-) \tag{32}$$

$$e_p(\beta V_+)(V_- + \frac{\beta v}{1+q} q^{2H}) \ = \ (V_- - \frac{\beta u}{1+q} q^{-2H}) e_p(-\beta V_+). \tag{33}$$

Note also the easily verified power series identities

$$f(\beta V_+) q^{-2H} = q^{-2H} f(\beta q V_+), \quad f(\beta q V_-) q^{-2H} = q^{-2H} f(\beta V_-). \tag{34}$$

For the matrix elements $(e-, E+)$ the operator identities

$$e_p(\beta V_-) E_p(\alpha V_+) V_+ = \frac{p}{\alpha}(T_\alpha^{-1} - I) e_p(\beta V_-) E_p(\alpha V_+), \tag{35}$$

$$e_p(\beta V_-) E_p(\alpha V_+) V_- = \frac{1}{\beta}(I - T_\beta) e_p(\beta V_-) E_p(\alpha V_+)$$

$$- \left(\frac{u \alpha q^{-2H}}{1+q} T_\beta^{-1} R_\beta T_\alpha R_\alpha + \frac{v \alpha q^{2H}}{1+q} T_\beta R_\beta R_\alpha \right) e_p(\beta V_-) E_p(\alpha V_+), \tag{36}$$

$$[H, e_p(\beta V_-) E_p(\alpha V_+)] = \frac{1}{2}(\alpha \partial_\alpha - \beta \partial_\beta) e_p(\beta V_-) E_p(\alpha V_+), \tag{37}$$

imply

$$\begin{aligned}
&\left[(-1)^{n+1}vq^{n/2-2\lambda}-uq^{2\lambda-n/2}\right]T^{(e-,E+)}_{n',n+1}(\alpha,\beta)=\frac{p}{\alpha}(T_\alpha^{-1}-I)T^{(e-,E+)}_{n'n}(\alpha,\beta)\\
&\frac{q^{-n/2}-(-1)^nq^{n/2}}{q^{-1/2}+q^{1/2}}T^{(e-,E+)}_{n',n-1}(\alpha,\beta)=\Big(\frac{1}{\beta}(I-T_\beta)R_\alpha- \qquad (38)\\
&(\frac{u\alpha q^{-(n'+2\lambda)}}{1+q}T_\beta^{-1}R_\beta T_\alpha R_\alpha+\frac{v\alpha q^{n'-2\lambda}}{1+q}T_\beta R_\beta R_\alpha)\Big)T^{(e-,E+)}_{n'n}(\alpha,\beta)\\
&(-\lambda+\frac{n}{2})T^{(e-,E+)}_{n'n}(\alpha,\beta)=\left(-\lambda+n'+\frac{1}{2}(\alpha\partial_\alpha-\beta\partial_\beta)\right)T^{(e-,E+)}_{n'n}(\alpha,\beta)
\end{aligned}$$

where $T_\alpha f(\alpha,\beta)=f(p\alpha,\beta)$,$R_\alpha f(\alpha,\beta)=f(-\alpha,\beta)$. Similar relations hold for the other matrix elements.

These relations are equivalent to q-difference relations satisfied by various q-hypergeometric series. Furthermore, it is easy to verify from the series that the relations hold also for λ and n' complex. Thus we have a wide variety of two variable models of algebraically irreducible representations of $[v,u]$. We note that this approach is closely related to the factorization method of quantum mechanics [11].

The following are simple consequences of operator identities (32) and (33):

$$\begin{aligned}
E_p(\beta V_-)e_p(\alpha R_\beta V_+) \quad & e_p\left(\frac{\alpha\beta v}{1+q}q^{2H}R_\beta\right) \qquad (39)\\
&= e_p\left(-\frac{\alpha\beta u}{1+q}q^{-2H}R_\beta\right)e_p(\alpha R_\beta V_+)E_p(\beta V_-),\\
e_p(\beta V_+)E_p(\alpha R_\beta V_-) \quad & E_p\left(\frac{\alpha\beta v}{1+q}q^{2H}R_\beta\right) \qquad (40)\\
&= E_p\left(-\frac{\alpha\beta u}{1+q}q^{-2H}R_\beta\right)E_p(\alpha R_\beta V_-)e_p(\beta V_+).
\end{aligned}$$

Using the formulas

$$f(XR_\beta)=\frac{1}{2}\left(f(X)+f(-X)\right)+\frac{1}{2}\left(f(X)-f(-X)\right)R_\beta,$$

$$f(\beta XR_\beta)=\frac{1}{2}\left(f(\beta iX)+f(-\beta iX)\right)+\frac{i}{2}\left(f(-\beta iX)-f(\beta iX)\right)R_\beta,$$

valid for formal power series f in the operator X, such that $[X, R_\beta] = 0$ and $R_\beta^2 = I$, we can derive identies relating the matrix elements. These results and (27) allow all 4 of our matrix elements to be computed in terms of any one of them.

3 Tensor products of $[v, u]$ q-superalgebra representations

Given functional realizations of the factors $[v, u]$ and $[-u, t]$ in terms of variables z_1, z_2, respectively, the functional realization of the representation

$$(\lambda_1, v, u) \otimes (\lambda_2, -u, t) \tag{41}$$

of $[v, t]$ is given by the coproduct

$$\begin{aligned} F_+ = & - q^{-\lambda_2} z_1 (u q^{2\lambda_1} T_1^{-\frac{1}{2}} + v q^{-2\lambda_1} R_1 T_1^{\frac{1}{2}}) T_2^{\frac{1}{2}} \\ & - q^{\lambda_1} z_2 T_1^{-\frac{1}{2}} (t q^{2\lambda_2} T_2^{-\frac{1}{2}} - u q^{-2\lambda_2} R_2 T_2^{\frac{1}{2}}), \end{aligned} \tag{42}$$

$$F_- = \frac{q^{-\lambda_2}}{z_1(q^{-\frac{1}{2}} + q^{\frac{1}{2}})} (T_1^{-\frac{1}{2}} - R_1 T_1^{\frac{1}{2}}) T_2^{\frac{1}{2}} + \frac{q^{\lambda_1} T_1^{-\frac{1}{2}}}{z_2(q^{\frac{1}{2}} + q^{-\frac{1}{2}})} (T_2^{-\frac{1}{2}} - R_2 T_2^{\frac{1}{2}}), \tag{43}$$

$$L = -\lambda_1 - \lambda_2 + \frac{1}{2}\frac{d}{dz_1} + \frac{1}{2}\frac{d}{dz_2}. \tag{44}$$

Here, $T_1 f(z_1, z_2) = f(z_1 q, z_2)$, $R_1 f(z_1, z_2) = f(-z_1, z_2)$, with similar definitions for the operators T_2, R_2. The z_i anticommute: $z_1 z_2 = -z_2 z_1$. With these assumptions the usual relations (10) hold.

To decompose (41) into a direct sum of irreducible representations we follow the procedure of [7] and use our functional tensor product model to compute the lowest weight vectors. The standard tensor product basis for our space is $F_{n_1 n_2}(z_1, z_2) = f_{n_1} \otimes f_{n_2} = z_1^{n_1} z_2^{n_2}$. Note that $z_1^{n_1} z_2^{n_2} = (-1)^{n_1 n_2} z_2^{n_2} z_1^{n_1}$. In order to make use of the inner product structure on this space, we assume that

$$u > 0,\ -vq^{-4\lambda_1},\ qvq^{-4\lambda_1} \qquad t > 0,\ uq^{-4\lambda_2},\ -quq^{-4\lambda_2}. \tag{45}$$

Thus,

$$
\begin{aligned}
< F_{n_1 n_2}, F_{m_1 m_2} > \quad &= \delta_{n_1 m_1}\delta_{n_2 m_2}\left(\frac{q^{-2\lambda_1}}{u(1+q)}\right)^{n_1} \frac{(-q;-q)_{n_1}}{(\frac{-v}{u}q^{-4\lambda_1};-q)_{n_1}} \\
&\times \left(\frac{q^{-2\lambda_2}}{t(1+q)}\right)^{n_2} \frac{(-q;-q)_{n_2}}{(\frac{u}{t}q^{-4\lambda_2};-q)_{n_2}}.
\end{aligned} \tag{46}
$$

It is easy to verify that $< F_+ f, g >= - < f, F_- g >$, $< Lf, g >= < f, Lg >$ for all polynomials f, g. The *lowest weight vectors* are the eigenvectors f_0 of L such that $F_- f_0 = 0$. The terms for the expansion of f_0 in the basis $z_1^{n_1} z_2^{n_2}$ can easily be computed but, due to occurence of certain powers of -1 they are not recognizable as terms in the expansion of a q-hypergeometric function. To correct for this one needs to take the basis in the alternate form $z_2^{n_2} z_1^{n_1} = (-1)^{n_1 n_2} z_1^{n_1} z_2^{n_2}$. Once this choice is made, one can compute as if the variables commute. It is only in the final stages of computing the Clebsch-Gordan coefficients from the expansion that the order is important.) With this choice of phase the series can be summed to get

$$
f_{s,0} = z_1^s \frac{(\frac{z_2}{z_1}q^{-\lambda_1-\lambda_2};-q)_\infty}{(\frac{z_2}{z_1}(-1)^s q^{-\lambda_1-\lambda_2+s};-q)_\infty} = z_1^s(\frac{z_2}{z_1}q^{-\lambda_1-\lambda_2};-q)_s \tag{47}
$$

$$
Lf_{s,0} = (\frac{s}{2} - \lambda_1 - \lambda_2) f_{s,0}, \quad s = 0, 1, \cdots. \tag{48}
$$

It follows that

$$
< f_{s,0}, f_{s,0} >= \left(\frac{q^{-2\lambda_1}}{u(1+q)}\right)^s \frac{(-q, \frac{v}{t}q^{-4\lambda_1-4\lambda_2-1}(-q)^s;-q)_s}{(-\frac{v}{u}q^{-4\lambda_1}, \frac{u}{t}q^{-4\lambda_2};-q)_s}.
$$

Vectors $f_{s,k}$, can now be defined recursively, by

$$
\begin{aligned}
f_{s,k+1} &= \left[(-1)^{k+1} v q^{\frac{k}{2}-2\lambda_1-2\lambda_2+s} - t q^{2\lambda_1+2\lambda_2-s-\frac{k}{2}}\right]^{-1} F_+ f_{s,k}. \\
s &\ , k = 0, 1, \cdots.
\end{aligned} \tag{49}
$$

Lemma 3

- $F_+ f_{s,k} = \left[(-1)^{k+1} v q^{\frac{k}{2}-2\lambda_1-2\lambda_2+s} - t q^{2\lambda_1+2\lambda_2-s-\frac{k}{2}}\right] f_{s,k+1}$,
- $F_- f_{s,k} = \frac{q^{-\frac{k}{2}} - (-1)^k q^{\frac{k}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} f_{s,k-1}, \quad L f_{s,k} = (-\lambda_1 - \lambda_2 + \frac{s}{2} + \frac{k}{2}) f_{s,k}$.

For fixed s the $\{f_{s,k}\}$ form an orthogonal basis for a subspace transforming according to the irreducible representation $(\lambda_1+\lambda_2-\frac{s}{2},v,t)$.

Lemma 4

$$(\lambda_1,v,u)\otimes(\lambda_2,-u,t)\cong\sum_{s=0}^{\infty}\oplus(\lambda_1+\lambda_2-\frac{s}{2},v,t).$$

Lemma 5

$$\begin{aligned}< f_{s,k},f_{s',k'} > \quad &= \quad \delta_{ss'}\delta_{kk'}\left(\frac{q^{-2\lambda_1}}{u(1+q)}\right)^s\frac{(-q,\frac{v}{t}q^{-4\lambda_1-4\lambda_2-1}(-q)^s;-q)_s}{(-\frac{v}{u}q^{-4\lambda_1},\frac{u}{t}q^{-4\lambda_2};-q)_s}\\ &\times\left(\frac{q^{-2\lambda_1-2\lambda_2+s}}{u(1+q)}\right)^k\frac{(-q;-q)_k}{(-\frac{v}{t}q^{-4\lambda_1-4\lambda_2+2s};-q)_k}. \end{aligned} \qquad (50)$$

Passing to the orthonormal basis $\{e_k^\mu\}$ where

$$e_k^\mu=||f_{s,k}||^{-1}f_{s,k},\quad \mu=\lambda_1+\lambda_2-s/2, \qquad (51)$$

we see that these basis vectors satisfy relations (19).

To derive a generating function for the Clebsch-Gordan coefficients we apply a q-analog of the exponential of F_+ to $f_{s,0}$ and expand the resulting expression in terms of the monomial $f_{n_1}\otimes f_{n_2}$ basis, $(p=-q)$:

$$(\exp_p\tau F_+)f_{s,0}=\sum_{k=0}^{\infty}\frac{q^{k(k+1)/4}}{(p;p)_k}\tau^kF_+^kf_{s,0}. \qquad (52)$$

To compute the righthand side of (52) we need to evaluate the terms $F_+^kf_{n_1}\otimes f_{n_2}$. We find

$$(\exp_p\tau F_+)f_{s,0}=\sum_{k=0}^{\infty}\frac{(-\frac{v}{t}q^{-4\lambda_1-4\lambda_2+2s};p)_k}{(p;p)_k}\left(-t\tau q^{2\lambda_1+2\lambda_2+1/2-s}\right)^kf_{s,k}. \qquad (53)$$

Thus,

$$\begin{aligned} f_{s,k}\qquad (z_1,z_2)&=\sum_{r=0}^{s}\frac{(q^{-3\lambda_2+s/2}\frac{u}{t})^k(-\frac{v}{u}q^{-4\lambda_1+s-r}(-1)^{s+r};p)_k(p^{-s};p)_r}{(p;p)_r(-\frac{v}{t}q^{-4\lambda_1-4\lambda_2+2s};p)_k}\\ &\times((-1)^{k+s}q^{-\lambda_1-\lambda_2+s+k})^rz_2^rz_1^{s+k-r}\\ &\times{}_3\phi_2\left(\begin{matrix}p^{-k}, & \frac{t}{u}(-1)^{r+1}q^{4\lambda_2-r+1}, & p^{-r}\\ p^{s-r+1}, & -\frac{v}{u}(-1)^{s+r}q^{-4\lambda_1+s-r} & \end{matrix};p;(-1)^{s+1}q^{2+s/2}\right).\end{aligned} \qquad (54)$$

From this result we can easily expand the orthonormal basis $\{e^\mu_m\}$ for the Hilbert space (51), in terms of the orthonormal basis

$$e^{\lambda_1}_{n_1} \otimes e^{\lambda_2}_{n_2} = \left[\frac{(\frac{-v}{u}q^{-4\lambda_1};p)_{n_1}(\frac{u}{t}q^{-4\lambda_2};p)_{n_2}}{(p;p)_{n_1}(p;p)_{n_2}}\right]^{\frac{1}{2}} \left(\frac{u(1+q)}{q^{-2\lambda_1}}\right)^{\frac{n_1}{2}} \left(\frac{t(1+q)}{q^{-2\lambda_2}}\right)^{\frac{n_2}{2}} \times f_{n_1n_2} = ||f_{n_1n_2}||^{-1} \cdot f_{n_1n_2}. \tag{55}$$

Thus,

$$e^\mu_m = \sum_{n_1,n_2} \begin{bmatrix} \lambda_1 & \lambda_2 & \mu \\ n_1 & n_2 & m \end{bmatrix}_p e^{\lambda_1}_{n_1} \otimes e^{\lambda_2}_{n_2}. \tag{56}$$

This defines the Clebsch-Gordan coefficients for the tensor product $(\lambda_1, v, u) \otimes (\lambda_2, -u, t)$. It follows from (54) that these coefficients vanish unless $2(\lambda_1 + \lambda_2 - \mu) + m = n_1 + n_2$. Furthermore, the orthogonality of the two bases implies the identities

$$\sum_{n_1,n_2} \begin{bmatrix} \lambda_1 & \lambda_2 & \mu \\ n_1 & n_2 & m \end{bmatrix}_p \begin{bmatrix} \lambda_1 & \lambda_2 & \mu' \\ n_1 & n_2 & m \end{bmatrix}_p^* = \delta_{\mu\mu'}, \tag{57}$$

$$\sum_{\mu} \begin{bmatrix} \lambda_1 & \lambda_2 & \mu \\ n_1 & n_2 & m \end{bmatrix}_p \begin{bmatrix} \lambda_1 & \lambda_2 & \mu \\ n'_1 & n'_2 & m \end{bmatrix}_p^* = \delta_{n_1n'_1}. \tag{58}$$

(In the second sum we require $n_1 + n_2 = n'_1 + n'_2 = 2(\lambda_1 + \lambda_2 - \mu) + m$, and $*$ is complex conjugation.) From (51), (54), (56) we find

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \mu \\ n_1 & n_2 & m \end{bmatrix}_p = \tag{59}$$

$$\left[\frac{(-\frac{v}{u}q^{-4\lambda_1}, \frac{u}{t}q^{-4\lambda_2};p)_{2(\lambda_1+\lambda_2-\mu)}(-\frac{v}{t}q^{-4\mu};p)_m(p;p)_{n_1}(p;p)_{n_2}}{(p, \frac{v}{t}q^{-4\lambda_1-4\lambda_2-1}p^{2(\lambda_1+\lambda_2-\mu)};p)_{2(\lambda_1+\lambda_2-\mu)}(p;p)_m(-\frac{v}{u}q^{-4\lambda_1};p)_{n_1}(\frac{u}{t}q^{-4\lambda_2};p)_{n_2}}\right]^{\frac{1}{2}}$$
$$\times(1+q)^{\lambda_1+\lambda_2-\mu+\frac{m}{2}-\frac{n_1}{2}-\frac{n_2}{2}}u^{\lambda_1+\lambda_2-\mu+\frac{m}{2}-\frac{n_1}{2}}t^{-\frac{n_2}{2}}q^{2\lambda_1(\lambda_1+\lambda_2-\mu)+\mu m-n_1\lambda_1-n_2\lambda_2}$$
$$\times\frac{(q^{\lambda_1-2\lambda_2-\mu}\frac{u}{t})^m(-\frac{v}{u}q^{-2\lambda_1+2\lambda_2-2\mu-n_2}(-1)^{2\lambda_1+2\lambda_2-2\mu+n_2};p)_m}{(p;p)_{n_2}(-\frac{v}{t}q^{-4\mu};p)_m}$$
$$\times(p^{-2\lambda_1-2\lambda_2+2\mu};p)_{n_2}((-1)^{2\lambda_1+2\lambda_2-2\mu+n_2}q^{\lambda_1+\lambda_2-2\mu+n_2})^{n_2}(-1)^{n_1n_2}$$
$$\times{}_3\phi_2\left(\begin{matrix} p^{-m}, & \frac{t}{u}(-1)^{n_2+1}q^{4\lambda_2-n_2+1}, & p^{-n_2} \\ p^{2\lambda_1+2\lambda_2-2\mu-n_2+1}, & -\frac{v}{u}(-1)^{2\lambda_1+2\lambda_2-2\mu+n_2}q^{-2\lambda_1+2\lambda_2-2\mu-n_2} & \end{matrix};\right.$$
$$\left. p;(-1)^{2\lambda_1+2\lambda_2-2\mu+1}q^{2+\lambda_1+\lambda_2-\mu}\right).$$

On the other hand, we have the explicit result for the generating function

$$(\exp_p \tau F_+) f_{s,0} = \tag{60}$$

$$z_1^s \frac{(\tau v q^{-2\lambda_1-\lambda_2+s/2+1/2}(-1)^s z_1, -\tau u q^{\lambda_1-2\lambda_2+s/2+1/2}(-1)^s z_2, q^{-\lambda_1-\lambda_2} z_2/z_1; p)_\infty}{(-\tau t q^{\lambda_1+2\lambda_2-s/2+1/2} z_2, -\tau u q^{2\lambda_1-\lambda_2-s/2+1/2} z_1, (-1)^s q^{-\lambda_1-\lambda_2+s} z_2/z_1; p)_\infty}.$$

Here we assume that the variables z_1 and z_2 commute for purposes of computing the coefficient of $z_2^n z_1^m$. Then we use the property $z_2^n z_1^m = (-1)^{mn} z_1^m z_2^n$ to read off the Clebsch Gordan coefficients from the expansion.

Furthermore, we note that the operator $S_+ = F_+(u, 0, 0)$ in our tensor product model, where $F_+(u, v, t)$ is the general operator, has the property that it maps between lowest weight vectors $f_0^{(s)}(z_1, z_2)$. Indeed,

$$S_+ f_{s,0} = u q^{-s/2} f_{s+1,0}. \tag{61}$$

From relations (7) and and the well known property that if X and Y are linear operators such that $YX = pXY$ then, [3], page 28,

$$e_p(X+Y) = e_p(X) e_p(Y), \quad E_p(X+Y) = E_p(Y) E_p(X). \tag{62}$$

we have the identity

$$T^{(e-,E+)}_{n'm',nm}(\alpha, \beta) = \tag{63}$$

$$= \frac{1}{2}[T^{(e-,E+),\lambda_1}_{n'n}(\alpha q^{-\lambda_2+\frac{m'}{2}}, \beta q^{-\lambda_2+\frac{m'}{2}}) T^{(e-,E+),\lambda_2}_{m'm}(\alpha(-1)^n q^{\lambda_1-\frac{n}{2}}, \beta(-1)^n q^{\lambda_1-\frac{n}{2}})$$

$$+T^{(e-,E+),\lambda_1}_{n'n}(\alpha q^{-\lambda_2+\frac{m'}{2}}, \beta q^{-\lambda_2+\frac{m'}{2}}) T^{(e-,E+),\lambda_2}_{m'm}(\alpha(-1)^n q^{\lambda_1-\frac{n}{2}}, -\beta(-1)^n q^{\lambda_1-\frac{n}{2}})$$

$$+T^{(e-,E+),\lambda_1}_{n'n}(-\alpha q^{-\lambda_2+\frac{m'}{2}}, \beta q^{-\lambda_2+\frac{m'}{2}}) T^{(e-,E+),\lambda_2}_{m'm}(\alpha(-1)^n q^{\lambda_1-\frac{n}{2}}, \beta(-1)^n q^{\lambda_1-\frac{n}{2}})$$

$$-T^{(e-,E+),\lambda_1}_{n'n}(-\alpha q^{-\lambda_2+\frac{m'}{2}}, \beta q^{-\lambda_2+\frac{m'}{2}}) T^{(e-,E+),\lambda_2}_{m'm}(\alpha(-1)^n q^{\lambda_1-\frac{n}{2}}, -\beta(-1)^n q^{\lambda_1-\frac{n}{2}})]$$

where the functions $T^{(e-,E+)}_{n'm',nm}(\alpha, \beta)$ are the matrix elements of the operator $e_p(\beta F_-) E_p(\alpha F_+)$ in the tensor product basis $\{f_n^{\lambda_1} \otimes f_m^{\lambda_2}\}$. Thus, for this operator the matrix elements in a tensor product basis take a simple form.

Defining normalized matrix elements for the representation (λ, v, u) by (26), or

$$\mathcal{T}^{(e-,E+),\lambda}_{n'n}(\alpha, \beta) = < e_p(\beta V_-) E_p(\alpha V_+) e_n, e_{n'} > = q^{\lambda(n-n')} \tag{64}$$

$$\times \left[\frac{(1+q)^{n-n'}}{u^{n'-n}} \frac{(p;p)_{n'}}{(p;p)_n} \frac{(-\frac{v}{u} q^{-4\lambda}; p)_n}{(-\frac{v}{u} q^{-4\lambda}; p)_{n'}} \right]^{\frac{1}{2}} T^{(e-,E+),\lambda}_{n'n}(\alpha, \beta),$$

we note from (56) that the following identity, relating the matrix elements of the operator $e_p(\beta F_-)E_p(\alpha F_+)$ in two distinct orthonormal bases, must hold:

$$\begin{aligned} \mathcal{T}^{(e-,E+)}_{n'm',nm}(\alpha,\beta) &= \sum_{\mu} \left[\begin{array}{ccc} \lambda_1 & \lambda_2 & \mu \\ n & m & M \end{array} \right]_p \mathcal{T}^{(e-,E+),\mu}_{M',M}(\alpha,\beta) \\ &\quad \times \left[\begin{array}{ccc} \lambda_1 & \lambda_2 & \mu \\ n' & m' & M' \end{array} \right]_p , \end{aligned} \tag{65}$$

where $M-n-m = M'-n'-m' = 2(\mu-\lambda_1-\lambda_2)$.

4 Finite dimensional $[v,u]$ q-superalgebra representations

For $uv \neq 0$ relations (14) define finite dimensional irreducible representations of $[v,u]$ if and only if $q^{4\lambda} = -\frac{v}{u}p^{N_0}, p=-q$, for integer $N_0 \geq 0$. As usual, λ, v, u are real constants, so if $vu > 0$ we must have N_0 an odd integer, while if $vu < 0$ then N_0 must be an even integer. We denote these representations by $(N_0, v, u)'$; their dimension is N_0+1. The algebra action is

$$V_- f_n = \frac{q^{-n/2}(1-p^n)}{q^{-1/2}+q^{1/2}} f_{n-1}, \quad V_+ f_n = uq^{2\lambda-n/2}\left[p^{n-N_0}-1\right] f_{n+1},$$
$$H f_n = (-\lambda+n/2) f_n. \tag{66}$$

The basis space is spanned by the vectors $\{f_n : n=0,1,\cdots N_0\}$, and we have $V_+ f_{N_0} = 0$. Using the computations of §2 we can define a nondegenerate real bilinear form $<\cdot,\cdot>'$ on this space such that

$$< f_n, f_m >' = \delta_{nm}\ \xi(n), \quad \xi(n) = \left(\frac{q^{-2\lambda}}{u(1+q)}\right)^n \frac{(p;p)_n}{(p^{-N_0};p)_n} \quad 0 \leq m,n \leq N_0. \tag{67}$$

The form satisfies the conditions

$$< Hf, g >' = < f, Hg >', \qquad < V_+ f, g >' = - < f, V_- g >',$$

but it does not define an inner product since $\xi(n)$ oscillates in sign. With this bilinear form we can mimic the computations in §2. The matrix elements

$T_{n'n}$ are exactly the same as those given in §2, with the proviso that $0 \leq n', n \leq N_0$. Moreover, $< Tf_n, f_{n'} >' = T_{n'n}\ \xi(n')$.

Now we consider the tensor product of the finite dimensional representation $(N_1, v, u)'$ of $[v, u]$ and the representation $(N_2, -u, t)'$ of $[-u, t]$. (Here, $q^{4\lambda_1} = -\frac{v}{u}p^{n_1}$, $q^{4\lambda_2} = \frac{u}{t}p^{n_2}$.) This is a $(N_1+1)(N_2+1)$ dimensional representation of $[v, t]$.

Lemma 6

$$(N_1, v, u)' \otimes (N_2, -u, t)' \cong \sum_{s=0}^{\min(N_1,N_2)} \oplus (N_1 + N_2 - 2s, v, t)'.$$

Indeed lemma 3 and its proof remain valid for the construction of the vectors $f_{s,k}$, $s = 0, 1, \cdots, \min(N_1, N_2)$ and $k = 0, 1, \cdots, N_1 + N_2 - 2s$. However, $F_+ f_{s,N_1+N_2-2s} = 0$. For fixed s the $\{f_{s,k}\}$ form an orthogonal basis (with respect to the bilinear form) for a subspace transforming according to the finite dimensional irreducible representation $(N_1 + N_2 - 2s, v, t)'$.

$$\begin{aligned} < f_{s,k}, f_{s',k'} >' &= \delta_{ss'}\delta_{kk'} \left(\frac{q^{-2\lambda_1}}{u(1+q)}\right)^s \frac{(p, p^{-N_1-N_2+s-1}; p)_s}{(p^{-N_1}, p^{-N_2}; p)_s} \\ &\times \left(\frac{q^{-2\lambda_1-2\lambda_2+s}}{u(1+q)}\right)^k \frac{(p;p)_k}{(p^{-N_!-N_2+2s}; p)_k} = \delta_{ss'}\delta_{kk'}\ \xi(s,k). \end{aligned} \tag{68}$$

The alternate tensor product basis is also orthogonal:

$$\begin{aligned} < F_{n_1n_2}, F_{m_1m_2} > &= \delta_{n_1m_1}\delta_{n_2m_2} \left(\frac{q^{-2\lambda_1}}{u(1+q)}\right)^{n_1} \frac{(p;p)_{n_1}}{(p^{-n_1};p)_{n_1}} \\ &\times \left(\frac{q^{-2\lambda_2}}{t(1+q)}\right)^{n_2} \frac{(p;p)_{n_2}}{(p^{-n_2};p)_{n_2}} = \delta_{n_1m_1}\delta_{n_2m_2}\nu(n_1,n_2). \end{aligned} \tag{69}$$

The Clebsch-Gordan coefficients can be computed from the generating function (52) in strict analogy with the procedure of §3. In particular, we find

$$\begin{aligned} f_{s,k}(z_1, z_2) &= \sum_{r=0}^{s} \frac{(q^{-3\lambda_2+s/2}\frac{u}{t})^k (p^{-N_1+s-r}; p)_k (p^{-s}; p)_r}{(p;p)_r (p^{-N_1-N_2+2s}; p)_k} \\ &\times (q^{-\lambda_1-\lambda_2} p^{s+k})^r z_2^r z_1^{s+k-r} \\ &\times {}_3\phi_2\left(\begin{matrix} p^{-k}, & p^{N_2-r+1}, & p^{-r} \\ p^{s-r+1}, & p^{-N_1+s-r} & \end{matrix}; p; (-1)^{s+1} q^{2+s/2}\right). \end{aligned} \tag{70}$$

From this result we can easily expand the orthogonal basis $\{f_{s,k}\}$ in terms of the orthogonal basis $\{F_{n_1n_2}\}$:

$$f_{s,k} = \sum_{n_1,n_2} \begin{bmatrix} N_1 & N_2 & s \\ n_1 & n_2 & k \end{bmatrix}'_p F_{n_1n_2}. \tag{71}$$

This defines the Clebsch-Gordan coefficients for the tensor product $(N_1, v, u)' \otimes (N_2, -u, t)'$. It follows from (70) that these coefficients vanish unless $s + k = n_1 + n_2$. Furthermore, the orthogonality of the two bases implies the identities

$$\sum_{n_1,n_2} \nu(n_1, n_2) \begin{bmatrix} N_1 & N_2 & s \\ n_1 & n_2 & k \end{bmatrix}'_p \begin{bmatrix} N_1 & N_2 & s' \\ n_1 & n_2 & k \end{bmatrix}'_p = \delta_{ss'}\ \xi(s, k), \tag{72}$$

$$\sum_{s} \xi(s, k) \begin{bmatrix} N_1 & N_2 & s \\ n_1 & n_2 & k \end{bmatrix}'_p \begin{bmatrix} N_1 & N_2 & s \\ n'_1 & n'_2 & k \end{bmatrix}'_p = \delta_{n_1n'_1}\nu(n_1, n_2). \tag{73}$$

(In the second sum we require $n_1 + n_2 = n'_1 + n'_2 = s + k$.)

Note the explicit result for the generating function

$$(\exp_p \tau F_+)f_{s,0} = \sum_{k=0}^{\infty} \frac{(p^{-N_1-N_2+2s}; p)_k}{(p; p)_k} \left(-t\tau q^{2\lambda_1+2\lambda_2+1/2-s}\right)^k f_{s,k} =$$
$$z_1^s \frac{(q^{-\lambda_1-\lambda_2} z_2/z_1; p)_s}{(-\tau t q^{\lambda_1+2\lambda_2-s/2+1/2} z_2; p)_{N_2+s}(-\tau u q^{2\lambda_1-\lambda_2-s/2+1/2} z_1; p)_{N_1+s}}. \tag{74}$$

Here again we assume that the variables z_1 and z_2 commute for purposes of computing the coefficient of $z_2^n z_1^m$. Then we use the property $z_2^n z_1^m = (-1)^{mn} z_1^m z_2^n$ to read off the Clebsch-Gordan coefficients from the expansion.

Furthermore, we note that the operator $S_+ = F_+(u, 0, 0)$ in our tensor product model, where $F_+(u, v, t)$ is the general operator, has the property that it maps between lowest weight vectors $f_0^{(s)}(z_1, z_2)$. Indeed, $S_+ f_{s,0} = uq^{-s/2} f_{s+1,0}$.

Finally, computing the expression $< e_p(\beta F_-)E_p(\alpha F_+)F_{nm}, F_{n'm'} >'$ in terms of our two distinct orthogonal bases, we obtain the identity

$$\frac{1}{2}[T^{(e-,E+),\lambda_1}_{n'n}(\alpha q^{-\lambda_2+\frac{m'}{2}}, \beta q^{-\lambda_2+\frac{m'}{2}}) T^{(e-,E+),\lambda_2}_{m'm}(\alpha(-1)^n q^{\lambda_1-\frac{n}{2}}, \beta(-1)^n q^{\lambda_1-\frac{n}{2}})$$

$$+T^{(e-,E+),\lambda_1}_{n'n}(\alpha q^{-\lambda_2+\frac{m'}{2}}, \beta q^{-\lambda_2+\frac{m'}{2}}) T^{(e-,E+),\lambda_2}_{m'm}(\alpha(-1)^n q^{\lambda_1-\frac{n}{2}}, -\beta(-1)^n q^{\lambda_1-\frac{n}{2}})$$

$$+T^{(e-,E+),\lambda_1}_{n'n}(-\alpha q^{-\lambda_2+\frac{m'}{2}},\beta q^{-\lambda_2+\frac{m'}{2}})T^{(e-,E+),\lambda_2}_{m'm}(\alpha(-1)^n q^{\lambda_1-\frac{n}{2}},\beta(-1)^n q^{\lambda_1-\frac{n}{2}})$$

$$-T^{(e-,E+),\lambda_1}_{n'n}(-\alpha q^{-\lambda_2+\frac{m'}{2}},\beta q^{-\lambda_2+\frac{m'}{2}})T^{(e-,E+),\lambda_2}_{m'm}(\alpha(-1)^n q^{\lambda_1-\frac{n}{2}},-\beta(-1)^n q^{\lambda_1-\frac{n}{2}})]$$

$$=\sum_s \left[\begin{array}{ccc} N_1 & N_2 & s \\ n & m & M \end{array}\right]'_p T^{(e-,E+),S}_{M',M}(\alpha,\beta)\left[\begin{array}{ccc} N_1 & N_2 & s \\ n' & m' & M' \end{array}\right]'_p \frac{\nu(n,m)}{\xi(s,M)}, \qquad (75)$$

where $n+m-M=n'+m'-M'=s$ and $S=N_1+N_2-2s$.

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