Multiseparability and superintegrability for classical and quantum systems

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1 Abstract

We outline the basic ideas relating to the notion of superintegrable potentials and how they are related to separability in multiple coordinate systems. We give examples and indicate how superintegrability can be of use, particularly in relation to bound states. Virtually all of the special functions of mathematical physics (in one and several variables) arise in this study and formulas expanding one type of special function as a series in another type emerge as a byproduct. We describe how one can, in principle, classify all such systems.

2 Issues

- WHAT IS SUPERINTEGRABILITY?
- WHAT ARE SEPARABILITY AND MULTISEPARABILITY?
- HOW ARE THEY RELATED?
- WHY DO WE CARE?
  - Quadratic algebras (algebraic dervation of eigenvalues)
  - Interbasis expansions
- HOW CAN WE CLASSIFY SUPERINTEGRABLE SYSTEMS IN A SYSTEMATIC FASHION?

3 The Hamilton-Jacobi and Schrödinger equations

- n-dimensional Riemannian manifold R_n. Local coordinates $q_1, \cdots, q_n$, metric tensor $\left(g^{jk}(\mathbf{q})\right)$. Potential function $V(\mathbf{q})$. The Hamilton-Jacobi equation is

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = E$$

where

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = \sum_{j,k=1}^{n} g^{jk}(\mathbf{q}) p_j p_k + V(\mathbf{q}) = \sum_{j,k=1}^{n} g^{jk}(\mathbf{q}) \frac{\partial S}{\partial q_j} \frac{\partial S}{\partial q_k} + V(\mathbf{q}),$$

and $S(\mathbf{q})$ is the action function.

- Quantum analog. The Schrödinger equation

$$H\Psi(\mathbf{q}) = E\Psi(\mathbf{q})$$

$$H = \Delta_n + V(\mathbf{q}), \qquad \Delta_n = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial q_j} \cdot (\sqrt{g} g^{jk}) \frac{\partial}{\partial q_k}$$

and $g = \det(g^{jk})^{-1}$.

4 Classical symmetries/Constants of the motion

- Hamiltonian

$$\mathcal{H} = \sum g^{jk} p_j p_k + V(\mathbf{q})$$

The p_j are the momenta conjugate to the coordinates q_j.

- The *Poisson bracket*

$$\{f_1, f_2\}(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^{n} (\frac{\partial f_1}{\partial q_j}\frac{\partial f_2}{\partial p_j} - \frac{\partial f_1}{\partial p_j}\frac{\partial f_2}{\partial q_j}).$$

- A *second-order constant of the motion* is a function

$$\mathcal{L} = \sum a^{jk}(\mathbf{q}) p_j p_k + W(\mathbf{q}), \qquad a^{jk} = a^{kj},$$

such that $\{\mathcal{L}, \mathcal{H}\} = 0$.

- $\mathcal{L}$ is constant along a classical trajectory:

$$\frac{d}{dt}\mathcal{L} = \{\mathcal{L}, \mathcal{H}\} = 0$$

where $\frac{d}{dt}\mathbf{q} = \partial_{\mathbf{p}}\mathcal{H}$, $\frac{d}{dt}\mathbf{p} = -\partial_{\mathbf{q}}\mathcal{H}$.

5 Superintegrability

- The null space of the map

$$T: \quad df(\mathbf{q},\mathbf{p}) \to \{f,\mathcal{H}\}(\mathbf{q},\mathbf{p})$$

  is $2n-1$ dimensional. Thus (locally) there are $2n-1$ functionally independent constants of the motion (but not necessarily second-order).

- Definition (for this talk). The classical system $\mathcal{H}=E$ is *superintegrable* or *maximal* if there are $2n-1$ functionally independent second-order constants of the motion:

$$\mathcal{L}_\ell = \sum a^{jk}(\mathbf{q})p_jp_k + W_\ell(\mathbf{q})$$

$$\mathcal{L}_0 = \mathcal{H}, \qquad \ell = 0,1,\cdots,2n-2$$

$$\{\mathcal{L}_\ell,\mathcal{H}\} = 0.$$

- The quantum system $H\Psi = E\Psi$ is *superintegrable* or *maximal* if there are $2n-1$ linearly independent second-order symmetry operators:

$$L_\ell = \sum \frac{1}{\sqrt{g}}\partial_{q_j}(\sqrt{g}a^{jk}(\mathbf{q}))\partial_{q_k} + W_\ell(\mathbf{q})r$$

$$L_0 = H, \qquad \ell = 0,1,\cdots,2n-2$$

$$[L_\ell,H] \equiv L_\ell H - HL_\ell = 0.$$

6 Separability

- A *complete integral* $S(\mathbf{q}, \lambda_1, \cdots, \lambda_n)$ of the Hamilton-Jacobi equation solves the associated classical mechanical system. (A complete integral is a solution such that locally

$$\det\left(\frac{\partial^2 S}{\partial q_j \partial \lambda_k}\right) \neq 0.$$

- Any solution of the Hamilton-Jacobi equation via (additive) separation of variables

$$S(\mathbf{q}, \lambda_1, \cdots, \lambda_n) = \sum_{j=1}^{n} S^{(j)}(q_j, \lambda)$$

where $\lambda_1 = E, \lambda_2, \cdots, \lambda_n$ are the separation constants, yields a complete integral.

- Quantum case: If the Schrödinger equation $H\Psi = E\Psi$ (multiplicatively) separates in the coordinates $\mathbf{q}$ then

$$\Psi(\mathbf{q}) = \Pi_{j=1}^{n} \Psi^{(j)}(q_j, \lambda)$$

and the equation decomposes into n ordinary differential equations, one for each of the factors $\Psi^{(j)}$.

- Many of the special functions of mathematical physics occur as solutions of these ordinary differential equations.

- For orthogonal coordinates $\mathbf{q}$ on an n-dimensional constant curvature space the Hamilton-Jacobi equation is additively separable if and only if the Schrödinger equation is multiplicatively separable.

7 Stäckel construction

- All orthogonal separable coordinate systems are obtained from the Stäckel construction. Each such system is characterized by a set of n quadratic functions $\mathcal{L}_1 = \mathcal{H}, \mathcal{L}_2, \cdots, \mathcal{L}_n$. The separable solutions satisfy equations $\mathcal{L}_j = \lambda_j$, where the λ_j, $j = 1, \cdots, n$ are the *separation constants.*

- $$\{\mathcal{L}_\ell, \mathcal{L}_j\} = 0, \qquad \ell \neq j$$

  Thus the $\mathcal{L}_\ell, \quad 2 \leq \ell \leq n$, are *constants of the motion* for the *Hamiltonian* $\mathcal{L}_1$.

- An analogous Stäckel construction, replacing the separation equations by n second-order linear ODE's for factors $\Psi^{(i)}(q_i)$ leads to second order linear partial differential operators $L_1 = H, L_2, \cdots, L_n$ such that

  $$H\Psi = E\Psi, \qquad L_\ell\Psi = \lambda_\ell\Psi, \qquad \ell = 2, \cdots, n$$

  and

  $$\Psi(\mathbf{q}) = \Pi_{k=1}^{n}\Psi^{(i)}(q_i).$$

- Then

  $$[L_k, H] = 0, \qquad [L_k, L_j] = 0.$$

8 Orthogonal separable coordinate systems

- How to find all orthogonal separable coordinate systems $\mathbf{q}$ for a given space R_n for zero potential, $V \equiv 0$?
  A difficult problem in differential geometry.

- Answer known for some constant curvature spaces. Kalnins and Miller have solution for n-dimensional real Euclidean space, spheres and hyperboloids of two sheets, for all n.

- EXAMPLE: COMPLEX EUCLIDEAN 2-SPACE.

  For complex Euclidean 2-space, including real Euclidean space and real Minkowski space, there are six separable systems: Cartesian, polar, parabolic, elliptic, hyperbolic and semi-hyperbolic.

- Coordinates that differ by an Euclidean motion are identified.

- We describe these coordinate systems and their corresponding free particle constants of the motion L.

- Adopt basis $p_x, p_y, M = xp_y - yp_x$ for Lie algebra $e(2, C)$ and $p_\pm = p_x \pm ip_y$. Here,

$$H = p_x^2 + p_y^2.$$

- There is one orbit of constants of the motion, with representative Mp_+, that is not associated with variable separation.

- The separable systems are:

Cartesian coordinates

$$x, y, \qquad L = p_x^2$$

Polar Coordinates

$$x = r\cos\theta \ , \qquad y = r\sin\theta \ , \qquad L = M^2$$

Parabolic Coordinates.

$$x_P = \frac{1}{2}(\xi^2 - \eta^2) \ , \qquad y_P = \xi\eta \ , \qquad L = Mp_y$$

Elliptic Coordinates (in algebraic form)

$$x_E^2 = c^2(u-1)(v-1), \quad y_E^2 = -c^2uv, \quad L = M^2 + c^2p_x^2$$

Hyperbolic Coordinates

$$x_H = \frac{c(r^2 + r^2s^2 + s^2)}{2rs}, \quad y_H = i\frac{c(r^2 - r^2s^2 + s^2)}{2rs}, \quad L = M^2 + c^2p_+^2$$

Semi-Hyperbolic Coordinates

$$x_{SH} = -\frac{c}{4}(w-u)^2 + \frac{c}{2}(w+u), \quad iy_{SH} = -\frac{c}{4}(w-u)^2 - \frac{c}{2}(w+u),$$

$$L = 2Mp_+ + cp_-^2$$

9 Intrinsic characterization of variable separation for the Hamilton-Jacobi equation

Theorem 1 *Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system* $\{x^i\}$ *for the Hamilton-Jacobi equation* $\mathcal{H}^1 = E$ *on an* N*-dimensional pseudo-Riemannian manifold are that there exist* N *quadratic forms* $\mathcal{H}^k = \sum_{i,j=1}^{N} H_{ij}^{(k)} p_i p_j$ *on the manifold such that:*

- $\{\mathcal{H}^k, \mathcal{H}^\ell\} = 0, \quad 1 \leq k, i \leq N,$
- *The set* $\{\mathcal{H}^k\}$ *is linearly independent (as* N *quadratic forms).*
- *There is a basis* $\{\omega_{(j)} : 1 \leq j \leq N\}$ *of simultaneous eigenforms for the* $\{\mathcal{H}^k\}$*. If conditions (1)-(3) are satisfied then there exist functions* $g^i(x)$ *such that:*

$$\omega_{(j)} = g^j dx^j, \quad j = 1, \cdots, N.$$

10 Intrinsic characterization of variable separation for the Schrödinger equation

Theorem 2 *Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system* $\{x^i\}$ *for the Helmholtz equation* $\Delta_N \Psi = E\Psi$ *on an* N*-dimensional pseudo-Riemannian manifold are that there exists a linearly independent set* $\{A_1 = \Delta_N, A_2, \cdots, A_N\}$ *of second-order differential operators on the manifold such that:*

- $[A_k, A_\ell] = 0, \quad 1 \le k, \ell \le N,$
- *Each* A_k *is in self-adjoint form,*
- *There is a basis* $\{\omega_{(j)} : 1 \le j \le N\}$ *of simultaneous eigenforms for the* $\{A_k\}$.

If conditions (1)-(3) are satisfied then there exist functions $g^i(x)$ *such that:*

$$\omega_{(j)} = g^j dx^j, \quad j = 1, \cdots, N.$$

- The main point of the theorems is that, under the required hypotheses the eigenforms ω^ℓ of the quadratic forms a^{ij} are normalizable, i.e., that up to multiplication by a nonzero function, ω^ℓ is the differential of a coordinate. This fact permits us to compute the coordinates directly from a knowledge of the symmetry operators.

11 Superintegrabilty versus separability

Hamiltonian in R_n

$$\mathcal{H} = \sum g^{jk} p_j p_k + V(\mathbf{q})$$

- SUPERINTEGRABLE: There are $2n-1$ functionally independent 2nd order constants of the motion $L_0 = H, L_1, L_2, \cdots, L_{2n-2}$,

$$\{H, L_j\} = 0, \qquad j = 1, \cdots, 2n-2.$$

- SEPARABLE: There are n linearly independent 2nd order constants of the motion $L_0 = H, L_1, L_2, \cdots, L_{n-1}$,

$$\{L_j, L_k\} = 0, 0 \leq j, k \leq n-1.$$

  The symmetries must also satisfy eigenform conditions.

- Analogous statements hold for the Schrödinger equation.

REMARK: One of the most effective methods of finding superintegrable systems is to search for systems that are multiseparable, i.e., that separate in more than one coordinate system.

12 Superintegrable examples in real Euclidean 2-space

EXAMPLE: ($n = 2$, $2n - 1 = 3$, so each separable system yields one new symmetry)

- Schrödinger equation with potential

$$V(x,y) = \frac{1}{2}\left(\omega^2(x^2+y^2) + \frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right),$$

  i.e.,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi - \left(\omega^2(x^2+y^2) + \frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right)\Psi = -2E\Psi.$$

- Equation separates in three coordinate systems: *Cartesian* coordinates (x, y); *polar* coordinates $x = r\cos\theta$, $y = r\sin\theta$, and *elliptical* coordinates

$$x^2 = c^2\frac{(u_1 - e_1)(u_2 - e_1)}{(e_1 - e_2)}, \qquad y^2 = c^2\frac{(u_1 - e_2)(u_2 - e_2)}{(e_2 - e_1)}.$$

- Bound state energies are given by

$$E_n = \omega(2n + 2 + k_1 + k_2)$$

  for integer n.

- The wave functions for each of these coordinate systems are:

  1. *Cartesian* coordinates

$$\Psi_{n_1,n_2}(x,y) = 2\omega^{\frac{1}{2}(k_1+k_2+2)}\sqrt{\frac{n_1!n_2!}{\Gamma(n_1+k_1+1)\Gamma(n_2+k_2+1)}}\, x^{(k_1+\frac{1}{2})} y^{(k_2+\frac{1}{2})}$$

$$e^{-\frac{\omega}{2}(x^2+y^2)} L_{n_1}^{k_1}(\omega x^2) L_{n_2}^{k_2}(\omega y^2)$$

  where $n = n_1 + n_2$, and the $L_n^k(x)$ are Laguerre polynomials.

  2. *polar* coordinates

$$\Psi(r,\theta) = \Phi_q^{(k_1,k_2)}(\theta)\omega^{\frac{1}{2}(2q+k_1+k_2+1)}\sqrt{\frac{2m!}{\Gamma(m+2q+k_1+k_2+1)}}$$

$$e^{(-\omega r^2/2)} r^{(2q+k_1+k_2+1)} L_m^{2q+k_1+k_2+1}(\omega r^2)$$

  where $n = m + q$,

$$\Phi_q^{(k_1,k_2)}(\theta) = \sqrt{2(2q+k_1+k_2+1)\frac{q!\Gamma(k_1+k_2+q+1)}{\Gamma(k_2+q+1)\Gamma(k_1+q+1)}}$$

$$\times(\cos\theta)^{k_1+(1/2)}(\sin\theta)^{k_2+(1/2)} P_q^{(k_1,k_2)}(\cos 2\theta),$$

  and the $P_q^{(k_1,k_2)}(\cos 2\theta)$ are Jacobi polynomials, .

  3. *elliptical* coordinates

$$\Psi = e^{-\omega(x^2+y^2)} x^{k_1+\frac{1}{2}} y^{k_2+\frac{1}{2}} \prod_{m=1}^{n}\left(\frac{x^2}{\theta_m - e_1} + \frac{y^2}{\theta_m - e_2} - c^2\right)$$

  where

$$\frac{x^2}{\theta - e_1} + \frac{y^2}{\theta - e_2} - c^2 = -c^2\frac{(u_1-\theta)(u_2-\theta)}{(\theta-e_1)(\theta-e_2)}.$$

13 Quadratic algebras of symmetries

Associated with the separability of the Schrödinger equation in these coordinate systems there are second order symmetry operators. A basis for such operators is

$$L_1 = \partial_x^2 + \frac{(\frac{1}{4} - k_1^2)}{x^2} - \omega^2 x^2, \qquad L_2 = \partial_y^2 + \frac{(\frac{1}{4} - k_2^2)}{y^2} - \omega^2 y^2$$

$$M^2 = (x\partial_y - y\partial_x)^2 + (\frac{1}{4} - k_1^2)\frac{y^2}{x^2} + (\frac{1}{4} - k_2^2)\frac{x^2}{y^2} - \frac{1}{2}.$$

(Note that $H = L_1 + L_2$.)

- Separable solutions are eigenfunctions of the symmetry operators L_1, M^2 and $M^2 + e_2 L_1 + e_1 L_2$ with eigenvalues

$$\lambda_c = -\omega(2n_1 + k_1 + 1), \qquad \lambda_p = (2q + k_1 + k_2 + 1)^2 + (1 + k_1^2 + k_2^2),$$

$$\lambda_e = 2(1 - k_1)(1 - k_2) - 2e_2\omega(k_1 + 1) - 2e_1\omega(k_2 + 1) - \omega^2 e_1 e_2 - 4\sum_{m=1}^{q}[e_2\frac{k_1 + 1}{\theta_m - e_1} + e_1\frac{k_2 + 1}{\theta_m - e_2}].$$

- The algebra constructed by repeated commutators is (R is defined by the first relation)

$$[L_1, M^2] = [M^2, L_2] = R, \qquad [L_i, R] = -4\{L_i, L_j\} + 16\omega^2 M^2, \qquad i \neq j,$$

$$[M^2, R] = 4\{L_1, M^2\} - 4\{L_2, M^2\} + 8(1 - k_2^2)L_1 - 8(1 - k_1^2)L_2,$$

$$R^2 = \frac{8}{3}\{M^2, L_1, L_2\} + \frac{64}{3}\{L_1, L_2\} + 16\omega^2 M^4 - 16(1 - k_2^2)L_1^2 - 16(1 - k_1^2)L_2^2 - \frac{128}{3}\omega^2 M^2 - 64\omega^2(1 - k_1^2)(1 - k_2^2).$$

- These relations are quadratic. Here $\{A, B\} = AB + BA$ is a symmetrizer.

- The important fact to observe about the algebra generated by L_1, L_2, M^2, R is that it is closed under commutation.

- In real Euclidean two-space there are precisely four potentials that have the multiseparation property. The second potential is

$$V(x,y) = \omega^2(4x^2 + y^2) - \frac{(\frac{1}{4} - k_2^2)}{y^2}.$$

The corresponding Schrödinger equation is separable in two coordinate systems: *Cartesian* coordinates and *parabolic* coordinates

$$x = \frac{1}{2}(\xi^2 - \eta^2), y = \xi\eta.$$

- The third potential is

$$V(x,y) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{B_1}{4}\frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}} + \frac{B_2}{4}\frac{\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}}.$$

Separation occurs here in *parabolic* and *parabolic coordinates of the second type*

$$x = \mu\nu, \;\; y = \frac{1}{2}(\mu^2 - \nu^2).$$

- The fourth potential is

$$V(x,y) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{1}{4\sqrt{x^2+y^2}}\left(\frac{(k_1^2 - \frac{1}{4})}{\sqrt{x^2+y^2+x}} + \frac{(k_2^2 - \frac{1}{4})}{\sqrt{x^2+y^2-x}}\right).$$

The corresponding Schrödinger equation is separable in three coordinate systems: *polar*, *parabolic* and *modified elliptic* coordinates.

14 What good is a quadratic algebra?

EXAMPLE: The third potential.

- A basis for the quadratic algebra is L_1, L_2 and H with defining relations

$$[R, L_1] = -4L_2H + B_1B_2, \qquad [R, L_2] = 4L_1H + \frac{1}{2}(B_1^2 - B_2^2)$$

$$R^2 = 4L_1^2H + 4L_2^2H - 16\alpha^2H + (B_2^2 - B_1^2)L_1 - 2B_1B_2L_2 - 2\alpha^2(B_1^2 + B_2^2)$$

with $R = [L_1, L_2]$.

- If we look for eigenfunctions of the operators L_1, L_2 respectively, we have

$$L_1\varphi_m = \lambda_m\varphi_m, \qquad L_2\psi_n = \rho_n\psi_n.$$

- If we write

$$L_1\psi_n = \sum_\tau C_{n\tau}\psi_\tau$$

then the quadratic algebra relations imply

$$[(\rho_n - \rho_\tau)^2 + 8E]C_{n\tau} = -[\frac{1}{2}(B_1^2 - B_2^2) - 16\alpha E]\delta_{n\tau}$$

$$\sum_\tau C_{n\tau}C_{\tau\sigma}(2\rho_\tau - \rho_n - \rho_\sigma) = (8E\rho_n + B_1B_2 + 16\alpha E)\delta_{n\sigma}.$$

- These relations in turn imply that

$$C_{nn} = -\frac{\frac{1}{2}(B_1^2 - B_2^2) + 16\alpha E}{8E}$$

and $C_{nn+1} = C^*_{n+1n}$ are the only nonzero coefficents. They can essentially be determined by the relation

$$4\sqrt{-2E}\,(|C_{n,n+1}|^2 - |C_{n-1,n}|^2) = 8E\rho_n + B_1 B_2 + 16\alpha E$$

where the eigenvalues λ_m and ρ_n are

$$\lambda_m = 2\alpha - \frac{B_1^2}{8E} - (2m+1)\sqrt{-2E}, \quad \rho_n = 2\alpha - \frac{(B_1 + B_2)^2}{16E} - (2n+1)\sqrt{-2E}$$

and the quantisation condition for E is

$$4\alpha - \frac{B_1^2 + B_2^2}{8E} = -(q+2)\sqrt{-2E}$$

for integer q.

15 Basic features of superintegrability (n=2)

- The potential V permits separability of the Hamilton-Jacobi equation $\mathcal{H} = E$ and the Schrödinger equation $H\Psi = E\Psi$ in at least two coordinate systems, characterized by symmetry conditions

$$\mathcal{L}_1 = \lambda_1, \mathcal{L}_2 = \lambda_2$$

  in the first case and

$$L_1\Psi = \lambda_1\Psi, L_2\Psi = \lambda_2\Psi$$

  in the second.

- One can obtain alternate spectral resolutions $\{\Psi_j^{(1)}\}, \{\Psi_k^{(2)}\}$ for the multiply-degenerate eigenspaces of H,

$$L_1\Psi_j^{(1)} = \lambda_1^{(1)}\Psi_j^{(1)}, \qquad L_2\Psi_k^{(2)} = \lambda_2^{(2)}\Psi_k^{(2)}.$$

  These alternate resolutions resolve the degeneracy problem.

- The interbasis expansions

$$\Psi_k^{(2)} = \sum_j a_{jk}\Psi_j^{(1)}$$

  yield important special function identities. In many cases, these become expansions of one set of multivariable orthogonal polynomials in terms of another set.

- The operators H, L_1, L_2 generate a quadratic algebra. With $R = [L_1, L_2]$ we have that R^2 is a polynomial of order 3 in H, L_1, L_2, whereas $[L_1, R]$ and $[L_2, R]$ are polynomials of order 2 in H, L_1, L_2.
- Corresponding statement is true for algebra generated by the symmetries $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ under the Poisson bracket.
- **This is a remarkable property**, and is false for general symmetries.
- The quadratic algebra structure can be used to compute the interbase expansion coefficients.

16 A rational approach to superintegrability: Complex Euclidean 2-space in detail

We consider the case

$$\mathcal{H} = p_1^2 + p_2^2 + V(x, y)$$

where $p_1 = p_x, p_2 = p_y$ and all variables are complex.
Here $2n - 1 = 3$, so for superintegrability, in addition to the

classical Hamiltonian we have two quadratic constants of the motion

$$\mathcal{L}_h = \sum_{k,j=1}^{2} a_{(h)}^{kj}(x, y) p_k p_j + W_{(h)}(x, y) \equiv \ell_h + W_{(h)}, \quad h = 1, 2,$$

$$\{\mathcal{H}, \mathcal{L}_h\} = 0.$$

We require that the set $\{d\mathcal{H}, d\mathcal{L}_1, d\mathcal{L}_2\}$ is linearly independent, so that $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ is a maximal set of functionally independent constants of the motion.

16.1 Conditions for L to be a constant of the motion

- $$\mathcal{L} = \sum_{j,k=1}^{2} a^{jk}(x,y) p_k p_j + W(x,y), \quad a^{jk} = a^{kj},$$

  The requirement is $\{\mathcal{H}, \mathcal{L}\} = 0$ where

  $$\mathcal{H} = p_1^2 + p_2^2 + V(x,y).$$

- Thus

  $$\frac{\partial a^{11}}{\partial x} = 0 \qquad 2\frac{\partial a^{12}}{\partial x} + \frac{\partial a^{11}}{\partial y} = 0$$
  $$\frac{\partial a^{22}}{\partial y} = 0 \qquad \frac{\partial a^{22}}{\partial x} + 2\frac{\partial a^{12}}{\partial y} = 0,$$

  and

  $$\frac{\partial W}{\partial x} - a^{11}\frac{\partial V}{\partial x} - a^{12}\frac{\partial V}{\partial y} = 0, \quad \frac{\partial W}{\partial y} - a^{12}\frac{\partial V}{\partial x} - a^{22}\frac{\partial V}{\partial y} = 0.$$

- The solution for the terms quadratic in the p_j is

  $$a^{11} = \alpha_1 y^2 + \alpha_2 y + \alpha_3'$$
  $$a^{12} = -\alpha_1 xy - \frac{1}{2}\alpha_2 x - \frac{1}{2}\alpha_4 y + \frac{1}{2}\alpha_5$$
  $$a^{22} = \alpha_1 x^2 + \alpha_4 x + \alpha_3'',$$

  where the α_k are constants.

16.2 Integrability condition

- The requirement that $\partial_x W_y = \partial_y W_x$ leads to the second order PDE for the potential

$$\frac{1}{2}(2\alpha_1 xy + \alpha_2 x + \alpha_4 y - \alpha_5)(V_{xx} - V_{yy}) + (\alpha_1[y^2 - x^2] + \alpha_2 y - \alpha_4 x + \alpha_3)V_{xy}$$

$$= (-3\alpha_1 y - \frac{3}{2}\alpha_2)V_x + (3\alpha_1 x + \frac{3}{2}\alpha_4)V_y,$$

where $\alpha_3 = \alpha_3' - \alpha_3''$.

- We denote the solution space of this equation by

$$(*) \qquad [\alpha_1, \cdots, \alpha_5].$$

This corresponds to the 2nd order symmetry

$$\mathcal{L} = \alpha_1 M^2 - \alpha_2 p_x M + \alpha_3 p_x^2 + \alpha_4 p_y M + \alpha_5 p_x p_y + W.$$

- The Hamilton-Jacobi equation admits two constants of the motion:

$$\mathcal{L}_h = \sum_{j,k=1}^{2} a_{(h)}^{jk} p_k p_j + W_{(h)}, \qquad h = 1, 2.$$

These two operators together with $\mathcal{H}$ are functionally independent. The constant of the motion $\mathcal{L}_1$ leads to the condition (*) on the potential V; whereas $\mathcal{L}_2$ leads to the second condition

$$[\beta_1, \cdots, \beta_5].$$

- The potential must lie in the intersection of the solution spaces for these two conditions. Thus equations

$$V_{xx} - V_{yy} = AV_x + BV_y, \qquad V_{xy} = CV_x + DV_y$$

must hold, where

$$A\mathcal{E} = \frac{3}{2}H_{12}(x^2+y^2) - 3H_{14}xy + 3H_{13}y - \frac{3}{2}H_{24}x + \frac{3}{2}H_{23}$$
$$B\mathcal{E} = \frac{3}{2}H_{14}(x^2+y^2) - 3H_{12}xy - 3H_{13}x + \frac{3}{2}H_{24}y + \frac{3}{2}H_{34}$$
$$2C\mathcal{E} = -3H_{14}y^2 + (-\frac{3}{2}H_{24} + 3H_{15})y + \frac{3}{2}H_{25}$$
$$2D\mathcal{E} = 3H_{12}x^2 + (-\frac{3}{2}H_{24} - 3H_{15})x - \frac{3}{2}H_{45}$$
$$2\mathcal{E} = -H_{12}xy^2 + H_{14}x^2y - H_{12}x^3 + H_{14}y^3 - 2H_{13}xy + H_{24}(x^2+y^2)$$
$$+H_{15}(x^2-y^2) + (H_{34} - H_{25})y + (H_{45} - H_{23})x - H_{35},$$

and $H_{k\ell} = -H_{\ell k} = \alpha_k\beta_\ell - \alpha_\ell\beta_k$.

- From the fundamental equations we can compute all of the third and higer order partial derivatives of V.

- V can depend on at most 3 parameters, in addition to a trivial additive constant. We can choose these parameters to be $V_x(x_0,y_0), V_y(x_0,y_0), V_{yy}(x_0,y_0)$ for any fixed regular point (x_0,y_0). Then $V_{xx}(x_0,y_0)$ and all higher derivatives can be computed by successive differentiation of relations . We require that our potential be *nondegenerate*, i.e., that it depend on 3 arbitrary parameters.

- Conditions $\partial_x V_{xxy} = \partial_y V_{xxx}, \partial_y V_{xxy} = \partial_x V_{xyy}, \partial_y V_{xyy} = \partial_x V_{yyy}$ for the fourth partial derivatives lead to the *integrability conditions*

$$\partial_x(2C - B) = \partial_y(2D + A) \quad (satisfied\ identically)$$

$$C_{xx} - C_{yy} - A_{xy} = 2CC_y - DA_y - 2CD_x + AA_y - AC_x$$
$$+CB_y + BC_y$$
$$D_{xx} - D_{yy} - B_{xy} = -2DD_x - CB_x + 2DC_y - BB_x$$
$$-BD_y + DA_x + AD_x.$$

- Use these conditions to classify the possible potentials V and the corresponding constants of the motion $\mathcal{L}_1, \mathcal{L}_2$. It is only the three-dimensional subspace spanned by $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ that matters; we can choose any basis for this subspace. Hence we can replace the conditions by linear combinations of themselves without changing the potential. Also we can always subject the coordinates (x, y), and $\mathcal{L}_1, \mathcal{L}_2$ to a simultaneous Euclidean motion, i.e., we regard all translated and rotated potentials as members of the same equivalence class.

- We exploit these conditions, and Euclidean motions to classify the possibilities for the $\mathcal{L}_j$. The full conditions take several pages to list and are complicated to solve directly. By dividing the problem up into special cases and using Euclidean motions, we can simplify the conditions and obtain a full solution.

THE LIST IS AS FOLLOWS:

1.

$$L_1 = 4M^2 + W^{(1)}, \qquad L_2 = -2Mp_y + W^{(2)}$$

$$V(x) = \frac{\alpha}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{x^2+y^2}}\left[\frac{\beta}{\sqrt{x^2+y^2+x}} + \frac{\gamma}{\sqrt{x^2+y^2-x}}\right].$$

This potential allows separation in parabolic or polar coordinates.

2.

$$L_1 = M^2 + c^2p_x^2 + W^{(1)}, \qquad L_2 = Mp_+ + \frac{ic}{2}p_+^2 + W^{(2)}$$

$$V(x) = \frac{\alpha z}{(c^2-z^2)^{\frac{1}{2}}} + \frac{\beta}{\sqrt{(c-z)(c+\bar{z})}} + \frac{\gamma}{\sqrt{(c+z)(c+\bar{z})}}.$$

The corresponding Hamilton-Jacobi and Schrödinger equations for this system separates in elliptical coordinates, as well as shifted elliptical coordinates.

3.

$$L_1 = M^2 + p_+^2 + W^{(1)}, \qquad L_2 = (M + 2ip_+)^2 + p_+^2 + W^{(2)}$$

$$V(x) = \frac{\alpha}{z^2} + \frac{\beta}{\sqrt{z^3(\bar{z}+2)}} + \frac{\gamma}{\sqrt{z(\bar{z}+2)}}.$$

This system separates in terms of hyperbolic coordinates and displaced hyperbolic coordinates.

4.

$$L_1 = M^2 + p_+^2 + W^{(1)}, \qquad L_2 = Mp_+ + W^{(2)}$$

$$V(x) = \frac{\alpha}{\sqrt{\bar{z}(z+2)}} + \frac{\beta}{\sqrt{\bar{z}(z-2)}} + \frac{\gamma z}{\sqrt{z^2-4)}}.$$

This system separates in terms of hyperbolic coordinates and displaced elliptic coordinates.

5.

$$L_1 = M^2 + W^{(1)}, \qquad L_2 = p_x^2 + W^{(2)}$$
$$V(x) = \alpha(x^2+y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}.$$

This potential permits separation in polar, elliptic and cartesian coordinates.

6.

$$L_1 = M^2 + W^{(1)}, \qquad L_2 = p_+^2 + W^{(2)}$$
$$V(x) = \alpha\frac{x^2+y^2}{(x+iy)^4} + \frac{\beta}{(x+iy)^2} + \gamma(x^2+y^2).$$

(There is a similar solution where the term p_+^2 in L_2 is replaced by p_-^2.) The potential permits separation in hyperbolic and polar coordinates.

7.

$$L_1 = M^2 + c^2p_x^2 + W^{(1)}, \qquad L_2 = p_+^2 + W^{(2)}$$
$$V(x) = \frac{\alpha z}{\sqrt{z^2-c^2}} + \frac{\beta\bar{z}}{\sqrt{z^2-c^2}(z+\sqrt{z^2-c^2})^2} + \gamma z\bar{z}$$

The potential permits separation in hyperbolic and elliptic coordinates.

8.

$$L_1 = -2Mp_x + W^{(1)}, \qquad L_2 = -2Mp_y + W^{(2)}$$
$$V(x) = \frac{\alpha}{\sqrt{x^2+y^2}} + \beta\frac{(\sqrt{x^2+y^2}+x)^{\frac{1}{2}}}{\sqrt{x^2+y^2}} + \gamma\frac{(\sqrt{x^2+y^2}-x)^{\frac{1}{2}}}{\sqrt{x^2+y^2}}.$$

Separation of variables is possible in two types of parabolic coordinates, the usual parabolic coordinates and the interchanged parabolic coordinates $x = \mu\nu, y = \frac{1}{2}(\mu^2-\nu^2)$.

9.

$$L_1 = 4iMp_- + p_+^2 + W^{(1)}, \quad L_2 = p_-^2 + W^{(2)}$$

$$V(x) = \alpha(x - iy) + \beta(x + iy - \frac{3}{2}(x - iy)^2) + \gamma(x^2 + y^2 - \frac{1}{2}(x - iy)^3).$$

The possible separable coordinates are semihyperbolic coordinates corresponding to operator $Mp_- + p_+^2$ and shifted semihyperbolic coordinates with operator $Mp_- + \delta p_-^2 + p_+^2$. This corresponds to the standard coordinates shifted via the transformation $x \to x + \delta, y \to y + i\delta$.

10.

$$L_1 = Mp_- + W^{(1)}, \quad L_2 = p_+^2 + W^{(2)}$$

$$V(x) = \alpha \bar{z} + \frac{\beta \bar{z}}{\sqrt{z}} + \frac{\gamma}{\sqrt{z}}.$$

The possible separable coordinates are a one-parameter family of semihyperbolic coordinates corresponding to operators $Mp_- + \delta p_+^2$ for $\delta \neq 0$.

11.

$$L_1 = p_x^2 + W^{(1)}, \qquad L_2 = -2Mp_y + W^{(2)}$$

$$V(x) = \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y^2}.$$

The possible separable coordinates are cartesian and parabolic.

12.

$$L_1 = 2p_y p_+ + W^{(1)}, \qquad L_2 = Mp_y + W^{(2)}$$

$$V(x) = \frac{\alpha}{\sqrt{x + iy}} + \beta x + \gamma \frac{2x + iy}{\sqrt{x + iy}}.$$

There is the possibility of separability in parabolic coordinates $\{Mp_y\}$ or displaced parabolic coordinates $\{(M + \delta(p_x + ip_y))p_y\}$ for suitable δ.

- This is the complete list of superintegrable systems in complex Euclidean 2-space. It includes real Euclidean space and Minkowski space as special cases.

- We can demonstrate expicitly that there is a quadratic algebra associated with each nondegenerate potential. It is sufficient to give the relation $R^2 = F(L_0, L_1, L_2)$ for each case. An example is (for system 1)

$$\mathcal{R}^2 = 16\mathcal{L}_1^2\mathcal{H} - 16\mathcal{L}_2^2\mathcal{L}_1 - 32(\beta+\gamma)\mathcal{L}_2^2 + 64\alpha(\beta-\gamma)\mathcal{L}_2 + 16\alpha^2\mathcal{L}_1 - 256\beta\gamma\mathcal{H} - 32\alpha^2(\beta+\gamma).$$

16.3 Quantum quadratic algebras

- Analogous quantum algebras for superintegrable systems arising from the potentials we have already computed. The only difference is that the Poisson bracket is now replaced by the commutator bracket $[A, B] = AB - BA$ and the operators H, L_1 and L_2 are the obvious (formally self-adjoint) symmetry partial differential operators.

$$H = \partial_x^2 + \partial_y^2 + V(x,y), \quad L_h = \sum_{k,j=1}^{2} \partial_k (a^{kj}_{(h)}) \partial_j + W_{(h)}(x,y), \quad h = 1, 2.$$

17 Summing up

We have used the concept of a "nondegenerate potential" to add structure to the study of superintegrable classical and quantum mechanical systems in $E_{2,C}$. (We have obtained similar results for systems on the complex 2-sphere, and separation for maximally symmetric 2-dimensional spaces that are not of constant curvature.) We have shown how to classify all such systems in a straightforward manner. Furthermore:

- Each system is associated with a pair of constants of the motion in the classical case, and a pair of symmetry operators in the quantum case, that generate a quadratic algebra.
- There is a one-to-one correspondence between superintegrable systems and free-field symmetry operators that generate quadratic algebras.
- Superintegrability implies multiseparability. NOTE: This is not the

  case for degenerate potentials. For example:

  $$H = p_x^2 + p_y^2 + \alpha(x - iy)^2$$

  is superintegrable. It admits symmetries $(px - ip_y)^2$ and $M(p_x - ip_y) + \frac{i}{3}\alpha(x - iy)^3$, but separates in a single (non-orthogonal) coordinate system. However, this potential is degenerate.

The next major challenge is to extend this analysis to higher dimensional systems.