Complete sets of invariants for classical systems

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Abstract: We consider the general problem of determining exactly when a classical Hamiltonian H in n dimensions admits a constant of the motion that is polynomial in the momenta. If the associated Hamilton-Jacobi equation admits an orthogonal separation of variables, then it is possible to generate algorithmically a canonical basis Q, P where $P_1 = H, P_2, \cdots, P_n$ are the other 2nd-order constants of the motion associated with the separable coordinates, and $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$, $\{Q_i, P_j\} = \delta_{ij}$. The $2n-1$ functions $Q_2, \cdots Q_n, P_1, \cdots, P_n$ form a basis for the invariants. We show how to determine for exactly which spaces and potentials the invariant Q_j is a polynomial in the original momenta.

1 Complete sets of invariants

BASIC QUESTION: Given a classical Hamiltonian
$H = H(x, p)$ where $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{p} = (p_1, ..., p_n)$ how can one find *all* the solutions to the Poisson bracket condition

$$\{H, L\} = \sum_{i=1}^{n} (\frac{\partial H}{\partial p_i} \frac{\partial L}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial L}{\partial p_i}) = 0 \qquad (1)$$

where $L = L(\mathbf{x}, \mathbf{p})$?

- No known general solution.
- If the Hamilton-Jacobi equation $H(\mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}) = E$ is additively separable in the orthogonal variables $\mathbf{x}$ then a complete integral of the equation can be constructed by quadratures and one can find a basis of $2n - 1$ functionally independent solutions to equation (1). Here n of the equations are 2nd-order in the momenta.

- Indeed there is an explicit canonical change of coordinates from the variables $\mathbf{x}, \mathbf{p}$ with $\{x_i, p_j\} = \delta_{ij}$ to variables $\mathbf{Q}, \mathbf{P}$ where $P_1 = H$, $P_2, \cdots, P_n$ are the other 2nd-order constants of the motion associated with the orthogonal separable x-coordinates, and $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$, $\{Q_i, P_j\} = \delta_{ij}$. Thus the $2n - 1$ functions $Q_2, \cdots, Q_n, P_1, \cdots, P_n$ form a basis for the invariants. Each invariant Q_j can be expressed as a sum of the form

$$Q_j = \sum_{k=1}^{n} M_k(x_k, \mathbf{P}). \tag{2}$$

- IMPORTANT PROBLEMS REMAIN: When are the constants of the motion L *polynomial* in the canonical momenta $p_i, i = 1, \cdots, n$?

- NO KNOWN ALGORITHMIC WAY OF GENERATING ALL POLYNOMIAL SOLUTIONS.

- From the x-based integrals in (2) it is difficult to tell if Q_j is a polynomial in the momenta p_i.

- We adopt a p-based approach to the calculation of the invariants Q_j in which the term M_k take the form $M_k = M(p_k, \mathbf{P})$, and we can say in advance for exactly which separable metrics and potentials Q_j is a polynomial in the momenta. We give, in principle, a complete solution to this problem. Moreover, we show how to characterize each term M_k in (2) by the Poisson brackets $\{M_k, P_j\}$.

- The system could admit a polynomial invariant

$$L = R(\mathbf{P}, Q_2, \cdots, Q_n)$$

  such that $L, \mathbf{P}$ is functionally independent, even if $Q_2, \cdots, Q_n$ are not polynomials. It is a much more difficult problem to classify all such possibilities for polynomial L as functions of possibly nonpolynomial Q_j.

- OUR APPROACH: Assume orthogonal separation in some coordinate system. This gives n 2nd-order commuting invariants. We can compute a remaining $n-1$ by quadratures. How to determine if these are polynomials in the momenta?

2 Cartesian systems in two dimensions

- In Cartesian coordinates the Hamiltonian has the form

$$H = p_x^2 + p_y^2 + V(x, y).$$

If we have separation of variables in Cartesian coordinates the potential must take the form

$$V(x, y) = X(x) + Y(y). \tag{3}$$

- We immediately observe that there are already two invariants arising from the separation, namely $L_1 = p_x^2 + X(x)$ and $L_2 = p_y^2 + Y(y)$.

- Problem: Calculate a third invariant and determine when it can be chosen to be a polynomial in the canonical momenta.

- To do this we compute two functions $M(x, p_x)$ and $N(y, p_y)$ that satisfy the conditions

$$\{H, M\} = 1, \qquad \{H, N\} = 1. \tag{4}$$

These equations can be solved in principle if we know the original functions X and Y.

- Indeed if we write out the first of these conditions, $\{H, M\} = 1$, we obtain

$$2p_x \frac{\partial M}{\partial x} - X' \frac{\partial M}{\partial p_x} = 1.$$

Since we are interested in the dependence of the invariants on the momenta, we take p_x to be the independent variable and consider x to be a function of p_x. Here, L_1, L_2 are treated as constants. Thus the equation takes the form

$$\frac{dM}{d\ p_x} = -\frac{1}{X'}.$$

This equation can be solved to give

$$M = -\int X'^{-1} dp_x$$

where $L_1 = p_x^2 + X$. (We consider $X'^{-1} = \frac{dx}{dX}$ to be a function of $X = L_1 - p_x^2$ to compute the integral.

- Once M and N have been determined, we see that $L_3 = N - M$ must be an invariant. It is immediately clear that if $X = x^{\frac{1}{p}}$ where p is an integer then M is a polynomial in p_x. In general, we want dx/dX to be a polynomial in X.

EXAMPLES:

1. $p = 3$.

$$M = -3x^{\frac{2}{3}}p_x - 4x^{\frac{1}{3}}p_x^3 - \frac{8}{5}p_x^5.$$

2. $p = 4$.

$$M = -4x^{\frac{3}{4}}p_x - 8x^{\frac{1}{2}}p_x^3 - \frac{32}{5}x^{\frac{1}{4}}p_x^5 - \frac{64}{35}p_x^7.$$

- It follows from these two examples that the Hamiltonian

$$H = p_x^2 + p_y^2 + x^{\frac{1}{3}} + y^{\frac{1}{4}}$$

has in addition to the obvious invariants

$$L_1 = p_x^2 + x^{\frac{1}{3}}, \qquad L_2 = p_y^2 + y^{\frac{1}{4}},$$

the additional invariant

$$L_3 = 3x^{\frac{2}{3}}p_x + 4x^{\frac{1}{3}}p_x^3 + \frac{8}{5}p_x^5 - 4y^{\frac{3}{4}}p_y - 8y^{\frac{1}{2}}p_y^3 - \frac{32}{5}y^{\frac{1}{4}}p_y^5 - \frac{64}{35}p_y^7. \quad (5)$$

- All potentials of the form

$$V = \alpha x^{\frac{1}{p}} + \beta y^{\frac{1}{q}} \quad (6)$$

have the superintegrability property with three functionally independent invariants which are polynomial in p_x and p_y. This includes the known examples corresponding to $p = 1, 2$.

- If $X(x)$ is determined by a polynomial relation of the form

$$\sum_{j=1}^{n} a_j X^j = x$$

we can go even further. Then the function M is always a polynomial in the canonical momentum p_x.

- EXAMPLE:

$$X(x) = 2^{-1/3}[\{x + \sqrt{x^2+1}\}^{1/3} - \{x + \sqrt{x^2+1}\}^{-1/3}]. \tag{7}$$

The inverse function is

$$x = X^3 + \frac{3}{2^{2/3}} X$$

and the corresponding function $M(x, p_x)$ is given by

$$-M(x, p_x) = \frac{8}{5} p_x^5 + 4X p_x^3 + 3X^2 p_x + \frac{3}{2^{2/3}} p_x.$$

- One further Cartesian case for which polynomial invariants can be generated. Consider case when $X(x) = \omega_1^2 x^2$. The corresponding function $M(x, p_x)$ is given by

$$M(x, p_x) = \frac{1}{4\omega_1} \arcsin(\frac{\omega_1^2 x^2 - p_x^2}{\omega_1^2 x^2 + p_x^2}).$$

If $Y(y) = \omega_2^2 y^2$ this establishes that the Hamiltonian

$$H = p_x^2 + p_y^2 + \omega_1^2 x^2 + \omega_2^2 y^2 \tag{8}$$

has the constant of motion

$$L_3 = \frac{1}{4\omega_1} \arcsin(\frac{\omega_1^2 x^2 - p_x^2}{\omega_1^2 x^2 + p_x^2}) - \frac{1}{4\omega_2} \arcsin(\frac{\omega_2^2 y^2 - p_y^2}{\omega_2^2 y^2 + p_y^2}), \tag{9}$$

in addition to the constants $L_1 = p_x^2 + \omega_1^2 x^2$ and $L_2 = p_y^2 + \omega_2^2 y^2$. In general this invariant is not polynomial in the canonical momenta.

However, if ω_1/ω_2 is a fraction p/q for integers p, q then $\omega_1 = ps, \omega_2 = qs$ and $L_3' = \sin(4spqL_3)$ will be a rational invariant whose common denominator is a product of powers of L_1 and L_2. The numerator is then an additional polynomial invariant, e.g., consider $\omega_1 = 1, \omega_2 = 2$. Then

$$L_3' = \sin(8L_3) = \frac{L_1L_2^2 - 2(xp_y^2 - 4yp_xp_y - 4xy^2)^2}{L_1L_2^2},$$

which indicates that $L_3'' = xp_y^2 - 4yp_xp_y - 4xy^2$ is an additional invariant. In general, $L_1^pL_2^q\sin(4spqL_3)$ will be a polynomial invariant, functionally independent of L_1 and L_2.

3 General two-dimensional orthogonal separable systems

- General Riemannian space.

- Separable potential has the form

$$H = L_1 = \frac{p_x^2 + p_y^2 + v_1(x) + v_2(y)}{f_1(x) + f_2(y)}. \qquad (10)$$

and, due to the separability, there is the invariant

$$L_2 = \frac{f_2(y)(p_x^2 + v_1(x)) - f_1(x)(p_y^2 + v_2(y))}{f_1(x) + f_2(y)}.$$

NOTE: v_1, v_2 are part of the potential, f_1, f_2 are part of the metric.

- Look for a function $M(H, x, p_x)$ which satisfies

$$\{H, M\} = \frac{1}{f_1(x) + f_2(y)}. \qquad (11)$$

The condition has the form

$$(-v_1'(x) + f_1'(x)H)\frac{\partial M}{\partial p_x} + 2p_x\frac{\partial M}{\partial x} = 1. \qquad (12)$$

Note that

$$L_2 = v_1(x) - f_1(x)H + p_x^2.$$

- We consider x to be a function of the independent variable $U = p_x^2$ and write (12) in the form

$$\frac{dM}{dp_x} = \frac{1}{-v_1'(x) + f_1'(x)H} = \frac{dx}{dU}.$$

- Solution

$$M(H, L_2, p_x) = \int U'^{-1} dp_x$$

where

$$U(x) = p_x^2 = -v_1(x) + f_1(x)H + L_2.$$

(We consider $U'^{-1} = \frac{dx}{dU}$ to be a function of $U = p_x^2$.)

Similarly, we compute a function $N(H, y, p_y)$ which satisfies

$$\{H, N\} = \frac{1}{f_1(x) + f_2(y)}. \qquad (13)$$

- The new invariant is $L_3 = N - M$, and also

$$\{L_2, L_3\} = 1. \qquad (14)$$

Indeed, $\{L_2, M\} = f_2/(f_1 + f_2)$, $\{L_2, N\} = -f_1/(f_1 + f_2)$. This implies that the set L_1, L_2, L_3 is functionally independent.

- Similarly we can construct functions $M(H,x,p_x), N(H,y,p_y)$ that satisfy

$$\{H,M\} = \frac{f_1(x)}{f_1(x)+f_2(y)}, \qquad \{H,N\} = \frac{-f_2(y)}{f_1(x)+f_2(y)}, \tag{15}$$

Assuming that $|v_i'| + |f_i'| > 0$ for $i = 1,2$, we see that these equations have the solutions

$$M(H,L_2,p_x) = \int f_1(x) U_1'^{-1} dp_x, \qquad N(H,L_2,p_y) = -\int f_2(y) U_2'^{-1} dp_y$$

where

$$U_i = -v_i + f_i H + L_2.$$

Setting $L_4 = N - M$, we see that L_4, not an invariant, satisfies

$$\{H,L_4\} = 1, \qquad \{L_2,L_4\} = 0. \tag{16}$$

- Thus

$$\{H,L_2\} = \{H,L_3\} = 0, \qquad \{L_2,L_3\} = 1, \qquad \{H,L_4\} = 1, \qquad \{L_2,L_4\} = 0.$$

EXAMPLE:

- We choose parabolic coordinates in Euclidean space $x' = \frac{1}{2}(\xi^2 - \eta^2)$, $y' = \xi\eta$. First consider the parabolic-separable Hamiltonian

$$H = L_1 = \frac{p_\xi^2 + p_\eta^2 + \xi}{\xi^2 + \eta^2}. \tag{17}$$

We can immediately associate with this the extra invariant

$$L_2 = \frac{\eta^2 p_\xi^2 - \xi^2 p_\eta^2 + \eta^2 \xi}{\xi^2 + \eta^2}.$$

- If we look for our functions $M(\xi, p_\xi)$ and $N(\eta, p_\eta)$, we obtain

$$M(\xi, p_\xi) = \frac{1}{4\sqrt{H}} \ln \left(\frac{\sqrt{H} p_\xi + \frac{1}{2} - \xi H}{-\sqrt{H} p_\xi + \frac{1}{2} - \xi H} \right),$$

$$N(\eta, p_\eta) = \frac{1}{4\sqrt{H}} \ln \left(\frac{\sqrt{H}\eta + p_\eta}{\sqrt{H}\eta - p_\eta} \right).$$

If we now consider the constant $\cosh(4(M - N)\sqrt{H})$, we find that it can be written in the form

$$4\cosh(4(M - N)\sqrt{H}) = \frac{L_3^2 H}{(1 - 4HL_2)L_2},$$

where

$$L_3 = \frac{2\xi\eta}{\xi^2 + \eta^2}(p_\xi^2 + p_\eta^2) - 2p_\xi p_\eta + \frac{\eta(\xi^2 - \eta^2)}{\xi^2 + \eta^2} \tag{18}$$

is an additional invariant quadratic in the canonical momenta.

- WHAT IS THE SCOPE OF THIS CONSTRUCTION SO THAT POLYNOMIAL INVARIANTS ARE OBTAINED?

- The solution of the equation (11) can be understood in a more general context. We have the dual relations

$$x = F(U - L_2, H), \qquad U(x, H) = -v_1(x) + f_1(x)H + L_2, \quad U_x \neq 0. \tag{19}$$

  Here, $H = L_1, L_2$ are parameters. Thus we have

$$1 = F_U U_x, \qquad F_U U_H + F_H = 0.$$

- The condition that $U(x, H)$ is linear in H, i.e., $U_{HH} = 0$, leads to the following necessary and sufficient conditions that the function $x = F(U, H)$ correspond to an invariant M on a Riemannian manifold with potential:

$$F_{HH} F_U^2 - 2F_{UH} F_U F_H + F_{UU} F_H^2 = 0, \quad F_U \neq 0. \tag{20}$$

- This equation admits an infinite dimensional conformal symmetry group. Indeed if $V = F(U,H)$ is a solution then $G(V)$ is also a solution, for *any* nonconstant function G. Also, this group contains the subgroup of inhomogeneous affine symmetries: if $F(U,H)$ is a solution then so is $F([a_{11}U + a_{12}H + a_{13}]/A, [a_{21}U + a_{22}H + a_{23} + a_{23}]/A)$, where a_{ij} are constants, $\det(a_{ij}) \neq 0$ and

$$A = a_{31}U + a_{32}H + a_{33}.$$

- In order to have M as a polynomial in the momentum variable p_x we must have $dx/dU = \partial F/\partial U$ a polynomial in U

- What is the most general function F that leads to an invariant that is polynomial in p_x?

Theorem 1 *The function $F(U,H)$ with $F_U \neq 0$ is a solution of equation (20) with polynomial dependence on U if and only if it is of the form*

$$F(U,H) = P\left(\frac{U + \alpha H + \beta}{\gamma H + \delta}\right)$$

where P is a (nonconstant) polynomial and $\alpha, \beta, \gamma, \delta$ are constants with $|\gamma|^2 + |\delta|^2 > 0$.

NOTE: This theorem generalizes to higher dimensions.

- Standard Hamilton-Jacobi theory gives essentially these same constants of the motion, but from a different viewpoint. Our expression for L_3, for example, is

$$L_3 = \int U_x'^{-1} dp_x - \int U_y'^{-1} dp_y = M - N,$$

where $U_x = -v_1(x) + f_1(x)H + L_2$, etc. Standard Hamilton-Jacobi theory gives

$$L_3 = \frac{1}{2}\int \frac{dx}{\sqrt{-v_1 + f_1 H + L_2}} - \frac{1}{2}\int \frac{dy}{\sqrt{-v_2 + f_2 H - L_2}} = \tilde{M} - \tilde{N}.$$

In the standard theory $\tilde{M} = \tilde{M}(H, L_2, x)$, etc., whereas in our approach $M = M(H, L_2, p_x)$, etc. In both cases the condition (12) is satisfied. Our approach makes it easier in some cases to determine if polynomial invariants exist. It also points out the bracket relations between M, N and the operators L_j defining the separation.

- Examples abound of spaces for which these constructions apply. Consider a family of surfaces in Minkowski space: $ds^2 = dz^2 - dy^2 - dx^2$:

$$\mathbf{X}(t,\xi) = (x,y,z) = \Big(2t\xi, g(t) + (\xi^2 - 1)t, g(t) + (\xi^2 + 1)t\Big) . \qquad (21)$$

The metric on the surface is

$$ds^2 = 4[tg'(t)\ dt^2 - t^2\ d\xi^2] = 4t^2[d\rho^2 - d\xi^2] = (f(\rho) + 1)[d\rho^2 - d\xi^2],$$

where $(\frac{d\rho}{dt})^2 = \frac{g'(t)}{t^2}$, and we can construct a polynomial invariant for the surface (and for an appropriate added potential) provided that the function $t^2 = F(\rho)$ has a polynomial inverse function, i.e., $\rho = G(t^2)$ where G is a polynomial. Clearly $g'(t) = 4t^4 G'(t^2)^2$ and any polynomial G will determine a surface with a polynomial invariant. For example, choose $G(t^2) = \frac{1}{2}t^4 + t^2$. Then we can take $g(t) = \frac{4}{9}t^9 + \frac{8}{7}t^7 + \frac{4}{5}t^5$ and $\rho(t) = \frac{1}{2}t^4 + t^2$. The resulting M will be third-order polynomial in p_ξ and p_ρ. Similarly, we can determine a potential term $v(\rho)$ with $v' \neq 0$ such that N is a polynomial in p_ξ and p_ρ.

4 Lie form in two dimensions

- We know that if a Hamiltonian

$$H = \sum_{i,j=1}^{2} g^{ij} p_i p_j$$

admits a constant of the motion L that is quadratic in the momenta

$$L = \sum_{i,j=1}^{2} a^{ij} p_i p_j, \qquad \{H, L\} = 0 \tag{22}$$

and if the roots of the determinant $|a^{ij} - \lambda g^{ij}|$ are distinct, then the eigenforms define new (separable) variables ρ, μ and the Hamiltonian can be written in Liouville form

$$H = \frac{p_\rho^2 + p_\mu^2}{f(\rho) + g(\mu)}.$$

- However, it may be that the roots of this determinant are equal. In this case H cannot be put into Liouville form, but rather Lie form, which for a suitable choice of variables (non-separable) is

$$H = \frac{p_x p_y}{x + B(y)}. \tag{23}$$

The associated quadratic constant of the motion is

$$L = p_x^2 - 2yH. \tag{24}$$

- QUESTION: When the roots of L are equal, how can we calculate the third invariant? We are interested in the the same question when a potential is added to the Hamiltonian.

- ANSWER: If we look for a function $N(H, L, y, p_y)$ that is in involution with H, we obtain the equation

$$(x + B(y))N_y + p_y B'(y)N_{p_y} = 0. \tag{25}$$

Solving (23), (24) for x and p_x in terms of the variables H, L, y and p_y, we obtain

$$p_x = \sqrt{L + 2yH}, \qquad x = \frac{p_y}{H}\sqrt{L + 2yH} - B(y).$$

The equation (25) for N then has the form

$$\frac{\sqrt{L + 2yH}}{HB'(y)}N_y + N_{p_y} = 0.$$

From this condition get a second invariant

$$L' = H \int \frac{B'(y)}{\sqrt{L + 2yH}}dy - p_y. \tag{26}$$

- We extend these considerations by adding a potential. If we do this and have an extra quadratic constant then H and L have the forms

$$H = \frac{p_x p_y + \frac{1}{2}K(y)}{x + B(y)} + \frac{1}{2}U'(y), \quad L = p_x^2 - 2yH + U(y). \tag{27}$$

- Solving (27) for p_x and x gives

$$p_x = \sqrt{L - U(y) + 2yH}, \qquad x = \frac{p_y\sqrt{L - U(y) + 2yH} + \frac{1}{2}K(y)}{H - \frac{1}{2}U'(y)} - B(y).$$

Then the equation for N has the form

$$2\sqrt{L - U(y) + 2yH}(2H - U'(y))N_y \tag{28}$$

$$+[-2U''(y)\sqrt{L - U(y) + 2yH}p_y + B'(y)U'(y)^2 + 4B'(y)H^2 - U''(y)K(y)$$

$$-4B'(y)U'(y)H + K'(y)U'(y) - 2K'(y)H]N_{p_y} = 0.$$

- This equation can, in principle, be solved directly. For suitable redefinition of the variables $y \to Y$, $p_y \to P_y$ equation (28) can be put in the form

$$N_Y + (P_Y + s(Y))N_{P_Y} = 0 \tag{29}$$

that can be solved by the further transformation

$$P_{Y'} = P_Y + t(Y), \qquad Y' = Y.$$

- Then, provided that

$$t'(Y) - t(Y) + s(Y) = 0,$$

(29) reduces to

$$N_{Y'} + P_{Y'} N_{P_{Y'}} = 0.$$

- From this we immediately deduce an extra constant of the motion of the form

$$L' = e^{Y'}/P_{Y'}. \tag{30}$$

The equation for $t(Y)$ has the solution

$$t(Y) = e^Y \int^Y e^{-u} s(u) du.$$

5 Nonorthogonal separation in two dimensions

There is one remaining possibility for a quadratic constant of the motion (22) in two dimensions: the constant may be associated with *nonorthogonal* separation of variables.

In two dimensions there is only one case: separation in light cone (null) coordinates. For this case the Hamiltonian takes the form

$$H = p_z p_{\bar{z}} + f(\bar{z})$$

and there is a Killing vector p_z, so p_z^2 is a second-order constant of the motion. In addition there is a quadratic constant

$$L = M p_z + \frac{i}{2} \int \bar{z} \frac{df}{d\bar{z}} d\bar{z}.$$

Thus we have answered the following questions.

1. If a Hamiltonian with potential admits a quadratic constant of the motion in two dimensions how does one calculate the third constant?

2. A subset of problem 1 is when we require separation only and ask to calculate the third constant.

Let us now look at how the orthogonal separation of variable considerations extend to three dimensions.

- If we have a general separable coordinate system in three dimensions we could take the Hamiltonian to be

$$\begin{aligned} H = L_1 &= \frac{g_2 - g_3}{\Phi}(p_{x_1}^2 + v_1(x_1)) + \frac{g_3 - g_1}{\Phi}(p_{x_2}^2 + v_2(x_2)) \\ &+ \frac{g_1 - g_2}{\Phi}(p_{x_3}^2 + v_3(x_3)) \end{aligned} \tag{31}$$

  where $g_i = g_i(x_i), f_i = f(x_i)$ and Φ is the determinant of the Stäckel matrix

$$\begin{pmatrix} 1 & f_1 & g_1 \\ 1 & f_2 & g_2 \\ 1 & f_3 & g_3 \end{pmatrix} \tag{32}$$

- This automatically gives us two more invariants:

$$\begin{aligned} L_2 &= \frac{f_3 - f_2}{\Phi}(p_{x_1}^2 + v_1(x_1)) + \frac{f_1 - f_3}{\Phi}(p_{x_2}^2 + v_2(x_2)) \\ &+ \frac{f_2 - f_1}{\Phi}(p_{x_3}^2 + v_3(x_3)), \end{aligned} \tag{33}$$

$$\begin{aligned} L_3 &= \frac{f_2 g_3 - f_3 g_2}{\Phi}(p_{x_1}^2 + v_1(x_1)) + \frac{f_3 g_1 - f_1 g_3}{\Phi}(p_{x_2}^2 + v_2(x_2)) \\ &+ \frac{f_1 g_2 - f_2 g_1}{\Phi}(p_{x_3}^2 + v_3(x_3)). \end{aligned} \tag{34}$$

- We need to find an additional two invariants, such that the five form a functionally independent set.

- If we look for a function M_1 such that

$$\{H, M_1\} = \frac{g_2 - g_3}{\Phi}, \tag{35}$$

then this function satisfies the equation

$$2p_{x_1}\partial_{x_1}M_1 + [-v_1'(x_1) + f_1'H + g_1'L_2]\partial_{p_{x_1}}M_1 = 1, \tag{36}$$

which looks like the form we have been using in two dimensions. There are similar equations for the corresponding functions M_i for $i = 2, 3$. For $M_1(H, L_2, L_3, p_{x_1})$ this has the solution

$$M_1 = \int U_1'^{-1} dp_{x_1}$$

where $U_1(x_1) = -v_1(x_1) + f_1H + g_1L_2 + L_3$ and $L_3 = v_1 - f_1H - g_1L_2 + p_{x_1}^2$. (Here, we consider $U_1'^{-1} = \frac{dx_1}{dU_1}$ to be a function of $U_1 = p_{x_1}^2$ to compute the integral. We also assume that $|v_1'| + |f_1'| + |g_1'| > 0$.)

- The corresponding invariant that we can calculate from these three functions is $L_3' = M_1 + M_2 + M_3$. This is based on the obvious identity

$$(g_2 - g_3) + (g_3 - g_1) + (g_1 - g_2) = 0.$$

- The invariant $L_3' = M_1 + M_2 + M_3$ also commutes with L_2. Indeed, from the fact that

$$\partial_{x_1} L_2 = \frac{f_3 - f_2}{\Phi} \left(v'_1 - f_1' H - g_1' L_2\right)$$

we can verify that (36) implies

$$\{L_2, M_1\} = \frac{f_3 - f_2}{\Phi}, . \tag{37}$$

The corresponding conditions are satisfied by M_2 and M_3. Then the fact that $\{L_2, L_3'\} = 0$ is implied by the obvious identity

$$(f_3 - f_2) + (f_1 - f_3) + (f_2 - f_1) = 0.$$

- Finally, from the fact that

$$\partial_{x_1} L_3 = \frac{f_2 g_3 - f_3 g_2}{\Phi} \left(v'_1 - f_1' H - g_1' L_2\right)$$

we can verify that (36) implies

$$\{L_3, M_1\} = \frac{f_2 g_3 - f_3 g_2}{\Phi}, . \tag{38}$$

The corresponding conditions are satisfied by M_2 and M_3. Then the fact that $\{L_3, L_3'\} = 1$ is implied by the identity

$$(f_2 g_3 - f_3 g_2) + (f_3 g_1 - f_1 g_3) + (f_1 g_2 - f_2 g_1) = \Phi. \tag{39}$$

- Similarly, we can define a new invariant L_2' by requiring that a new function M_1 satisfy

$$\{L_1, M_1\} = \frac{g_1 (g_2 - g_3)}{\Phi}, \tag{40}$$

with analogous conditions for M_2 and M_3. For $M_1(H, L_2, L_3, p_{x_1}$ this has the solution

$$M_1 = \int g_1 U_1'^{-1} dp_{x_1}$$

where $U_1(x_1) = -v_1(x_1) + f_1 H + g_1 L_2 + L_3$.

- The corresponding invariant that we can calculate from these three functions is $L_2' = M_1 + M_2 + M_3$. This is based on the obvious identity

$$g_1(g_2 - g_3) + g_2(g_3 - g_1) + g_3(g_1 - g_2) = 0.$$

Then it follows that

$$\{L_2, M_1\} = \frac{g_1(f_3 - f_2)}{\Phi}, \qquad \{L_3, M_1\} = \frac{g_1(f_2 g_3 - f_3 g_2)}{\Phi},$$

with analogous results for M_2, M_3. Thus, from the definition of Φ we see that $\{L_2, L_2'\} = 1$.

- Finally we define a function $L_1' = M_1 + M_2 + M_3$ by requiring

$$\{L_1, M_1\} = \frac{f_1(g_2 - g_3)}{\Phi}, \tag{41}$$

with similar conditions for M_2 and M_3. For $M_1(H, L_2, L_3, p_{x_1})$ this has the solution

$$M_1 = \int f_1 U_1'^{-1} dp_{x_1}.$$

Then it follows that

$$\{L_2, M_1\} = \frac{f_1(f_3 - f_2)}{\Phi}, \qquad \{L_3, M_1\} = \frac{f_1(f_2 g_3 - f_3 g_2)}{\Phi},$$

with analogous relations for M_2 and M_3.

- In summary, all brackets between the six functions L_i, L_i' are zero except that

$$\{L_3, L_3'\} = \{L_2, L_2'\} = \{L_1, L_1'\} = 1. \tag{42}$$

Thus the mapping $(x_1, x_2, x_3, p_{x_1}, p_{x_2}, p_{x_3}) \rightarrow (L_1, L_2, L_3, L_1', L_2', L_3')$ is canonical.

- Our construction generalizes to orthogonal separation in n dimensions
- Many examples and further results are given in the paper "Complete sets of invariants for dynamical systems that admit a separation of variables" by Kalnins, Kress, Miller & Pogosyan, to appear in JMP. See http://www.ima.umn.edu/ miller/bibli.html