# Superintegrability and Special Functions in Mathematical Physics

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**Abstract**

In this talk we outline the basic ideas relating to the notion of superintegrability. The fundamental notion we introduce is that of simultaneous separability of the Schrödinger equation in more than one coordinate system. The energy observable is degenerate for potentials of this type and the corresponding observables that arise from the simultaneous separability close quadratically under repeated commutation. We give examples of these systems and indicate how superintegrability can be of use, particularly in relation to bound states.

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# 1 Introduction

It has long been known that there are potentials for which a given Hamiltonian system in classical mechanics admits a solution via separation of variables in more than one coordinate system [1,2]. The methodical search for such potentials was initiated by Smorodinsky, Winternitz et.al. [3,4] in two and three dimensions. A subset of such systems is called *maximal* in dimension $N$ if there exist 2N-1 functionally independent integrals of motion. In this talk we show how to find the finite solutions (bound state eigenfunctions) for those quantum mechanical systems in two dimensions which admit separation of variables in at least two coordinate systems. We do this for the corresponding systems defined in two dimensional Euclidean space and indicate how these ideas work for quantum mechanical systems defined on the two dimensional sphere [5].

# 2 The Euclidean plane and the two dimensional sphere

**2.1** To illustrate the basic idea we can consider the example of the Schrödinger equation with potential

$$V(x,y) = \frac{1}{2}\left(\omega^2(x^2+y^2) + \frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right),$$

i.e.,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi - \left(\omega^2(x^2+y^2) + \frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right)\Psi = -2E\Psi.$$

This equation can be solved by the separation of variables ansatz $\Psi = \psi_1(u)\psi_2(v)$ for a suitable choice of coordinates $x = x(u,v), y = y(u,v)$. In fact this equation separates in three coordinate systems: *Cartesian* coordinates $(x,y)$; *Polar* coordinates $x = r\cos\theta,\ y = r\sin\theta$, and *Elliptical* coordinates

$$x^2 = c^2\frac{(u_1-e_1)(u_2-e_1)}{(e_1-e_2)}, \qquad y^2 = c^2\frac{(u_1-e_2)(u_2-e_2)}{(e_2-e_1)}.$$

The bound state energies are given by

$$E_n = \omega(2n+2+k_1+k_2)$$

for integer $n$. The wave functions for each of these coordinate systems are:

1. *Cartesian* coordinates

$$\Psi_{n_1,n_2}(x,y) = 2\omega^{\frac{1}{2}(k_1+k_2+2)}\sqrt{\frac{n_1!n_2!}{\Gamma(n_1+k_1+1)\Gamma(n_2+k_2+1)}}\,x^{(k_1+\frac{1}{2})}y^{(k_2+\frac{1}{2})}$$
$$e^{-\frac{\omega}{2}(x^2+y^2)}L_{n_1}^{k_1}(\omega x^2)L_{n_2}^{k_2}(\omega y^2)$$

where $n = n_1 + n_2$,
2. *polar* coordinates

$$\Psi(r,\theta) = \Phi_q^{(k_1,k_2)}(\theta)\omega^{\frac{1}{2}(2q+k_1+k_2+1)}\sqrt{\frac{2m!}{\Gamma(m+2q+k_1+k_2+1)}}$$
$$e^{(-\omega r^2/2)}r^{(2q+k_1+k_2+1)}L_m^{2q+k_1+k_2+1}(\omega r^2)$$

where $n = m + q$ and

$$\Phi_q^{(k_1,k_2)}(\theta) = \sqrt{2(2q+k_1+k_2+1)\frac{q!\Gamma(k_1+k_2+q+1)}{\Gamma(k_2+q+1)\Gamma(k_1+q+1)}}$$
$$\times(\cos\theta)^{k_1+(1/2)}(\sin\theta)^{k_2+(1/2)}P_q^{(k_1,k_2)}(\cos 2\theta),$$

3. *elliptical* coordinates

$$\Psi = e^{-\omega(x^2+y^2)}x^{k_1+\frac{1}{2}}y^{k_2+\frac{1}{2}}\prod_{m=1}^{n}\left(\frac{x^2}{\theta_m - e_1} + \frac{y^2}{\theta_m - e_2} - c^2\right)$$

where use has been made of the identity

$$\frac{x^2}{\theta - e_1} + \frac{y^2}{\theta - e_2} - c^2 = -c^2\frac{(u_1-\theta)(u_2-\theta)}{(\theta-e_1)(\theta-e_2)}.$$

The zeros $\theta_j$ satisfy the relations

$$\frac{k_1+1}{\theta_m - e_1} + \frac{k_2+1}{\theta_m - e_2} + \sum_{j\neq m}\frac{2}{(\theta_m - \theta_j)} - \omega = 0.$$

Because of the separability properties of the Schrödinger equation in these coordinate systems there are second order symmetries. These are second order partial differential operators $L$ which are such that if $\Psi$ is a solution then so is $L\Psi$. A basis for such operators is

$$L_1 = \partial_x^2 + \frac{(\frac{1}{4} - k_1^2)}{x^2} - \omega^2 x^2, \qquad L_2 = \partial_y^2 + \frac{(\frac{1}{4} - k_2^2)}{y^2} - \omega^2 y^2$$

$$M = (x\partial_y - y\partial_x)^2 + (\frac{1}{4} - k_1^2)\frac{y^2}{x^2} + (\frac{1}{4} - k_2^2)\frac{x^2}{y^2} - \frac{1}{2}.$$

The separable eigenfunctions already given are eigenfunctions of the symmetry operators $L_1, M$ and $M + e_2L_1 + e_1L_2$ with eigenvalues

$$\lambda_c = -\omega(2n_1 + k_1 + 1), \qquad \lambda_p = (2q + k_1 + k_2 + 1)^2 + (1 + k_1^2 + k_2^2),$$

$$\lambda_e = 2(1 - k_1)(1 - k_2) - 2e_2\omega(k_1 + 1) - 2e_1\omega(k_2 + 1) - \omega^2 e_1 e_2 -$$
$$4\sum_{m=1}^{q}[e_2\frac{k_1 + 1}{\theta_m - e_1} + e_1\frac{k_2 + 1}{\theta_m - e_2}].$$

The algebra constructed by repeated commutators is

$$[L_1, M] = [M, L_2] = R, \qquad [L_i, R] = -4\{L_i, L_j\} + 16\omega^2 M, \qquad i \neq j,$$

$$[M, R] = 4\{L_1, M\} - 4\{L_2, M\} + 8(1 - k_2^2)L_1 - 8(1 - k_1^2)L_2,$$

$$R^2 = \frac{8}{3}\{M, L_1, L_2\} + \frac{64}{3}\{L_1, L_2\} + 16\omega^2 M^2 - 16(1 - k_2^2)L_1^2$$
$$-16(1 - k_1^2)L_2^2 - \frac{128}{3}\omega^2 M - 64\omega^2(1 - k_1^2)(1 - k_2^2).$$

These relations are quadratic.

**2.2** In two real dimensions there are four potentials that have the multiseparation property. The second is

$$V(x, y) = \omega^2(4x^2 + y^2) - \frac{(\frac{1}{4} - k_2^2)}{y^2}.$$

The corresponding Schrödinger equation is separable in two coordinate systems: *Cartesian* coordinates and *parabolic* coordinates

$$x = \frac{1}{2}(\xi^2 - \eta^2), y = \xi\eta.$$

**2.3** The third potential is

$$V(x, y) = -\frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{1}{4\sqrt{x^2 + y^2}}\left(\frac{(k_1^2 - \frac{1}{4})}{\sqrt{x^2 + y^2} + x} + \frac{(k_2^2 - \frac{1}{4})}{\sqrt{x^2 + y^2} - x}\right).$$

The corresponding Schrödinger equation is separable in two coordinate systems: *polar*, *parabolic* and *modified elliptic* coordinates, where

$$\xi^2 = c^2\frac{(u_1 - e_1)(u_2 - e_1)}{(e_1 - e_2)}, \qquad \eta^2 = c^2\frac{(u_1 - e_2)(u_2 - e_2)}{(e_2 - e_1)}.$$

This last coordinate system can be written as

$$x = \sqrt{\frac{(U_1 - E_1)(U_2 - E_1)}{4(E_1 - E_2)}}, \qquad y = \sqrt{\frac{(U_1 - E_2)(U_2 - E_2)}{4(E_2 - E_1)}} - 2\sqrt{E_1 - E_2},$$

where $E_1 = -e_1 e_2$, $E_2 = -\frac{1}{4}(e_1 + e_2)^2$ and $U_j = u_j^2 - u_j(e_1 + e_2)$.

**2.4** The fourth potential is

$$V(x, y) = -\frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{B_1}{4}\frac{\sqrt{\sqrt{x^2 + y^2} + x}}{\sqrt{x^2 + y^2}} + \frac{B_2}{4}\frac{\sqrt{\sqrt{x^2 + y^2} - x}}{\sqrt{x^2 + y^2}}.$$

Separation occurs here in *parabolic* and *parabolic coordinates of the second type*

$$x = \mu\nu, \ \ y = \frac{1}{2}(\mu^2 - \nu^2).$$

As an illustration of the utility of the notion of a quadratic algebra consider the last potential given. A basis for the quadratic algebra consists of $L_1, L_2$ and $H$ with defining relations

$$[R, L_1] = -4L_2 H + B_1 B_2, \qquad [R, L_2] = 4L_1 H + \frac{1}{2}(B_1^2 - B_2^2)$$

$$R^2 = 4L_1^2 H + 4L_2^2 H - 16\alpha^2 H + (B_2^2 - B_1^2)L_1 - 2B_1 B_2 L_2 - 2\alpha^2(B_1^2 + B_2^2)$$

with $R = [L_1, L_2]$. If we look for eigenfunctions of the operators $L_1, L_2$ respectively, we have

$$L_1\varphi_m = \lambda_m\varphi_m, \qquad L_2\psi_n = \rho_n\psi_n.$$

If we write

$$L_1\psi_n = \sum_\tau C_{n\tau}\psi_\tau$$

then the quadratic algebra relations imply

$$[(\rho_n - \rho_\tau)^2 + 8E]C_{n\tau} = -[\frac{1}{2}(B_1^2 - B_2^2) - 16\alpha E]\delta_{n\tau}$$

$$\sum_\tau C_{n\tau}C_{\tau\sigma}(2\rho_\tau - \rho_n - \rho_\sigma) = (8E\rho_n + B_1 B_2 + 16\alpha E)\delta_{n\sigma}.$$

These relations in turn imply that

$$C_{nn} = -\frac{\frac{1}{2}(B_1^2 - B_2^2) + 16\alpha E}{8E}$$

and $C_{nn+1} = C^*_{n+1n}$ are the only nonzero coefficents. Indeed they can essentially be determined by the relation

$$4\sqrt{-2E}\,(|C_{n,n+1}|^2 - |C_{n-1,n}|^2) = 8E\rho_n + B_1B_2 + 16\alpha E$$

where the eigenvalues $\lambda_m$ and $\rho_n$ are given by

$$\lambda_m = 2\alpha - \frac{B_1^2}{8E} - (2m+1)\sqrt{-2E}, \quad \rho_n = 2\alpha - \frac{(B_1+B_2)^2}{16E} - (2n+1)\sqrt{-2E}$$

and the quantisation condition for $E$ is

$$4\alpha - \frac{B_1^2 + B_2^2}{8E} = -(q+2)\sqrt{-2E}$$

for integer $q$.

These ideas work also for separable coordinates on the two dimensional sphere. Two important examples are

**2.5** The potential

$$V = \frac{1}{2}\left[\frac{k_1^2 - \frac{1}{4}}{s_1^2} + \frac{k_2^2 - \frac{1}{4}}{s_2^2} + \frac{k_3^2 - \frac{1}{4}}{s_3^2}\right]$$

where as usual $s_1^2 + s_2^2 + s_3^2 = 1$. The corresponding Schrödinger equation has the form

$$\left[(s_1\frac{\partial}{\partial s_2} - s_2\frac{\partial}{\partial s_1})^2 + (s_1\frac{\partial}{\partial s_3} - s_3\frac{\partial}{\partial s_1})^2 + (s_3\frac{\partial}{\partial s_2} - s_2\frac{\partial}{\partial s_3})^2\right]\Psi +$$
$$\left[\frac{(k_1^2 - \frac{1}{4})}{s_1^2} + \frac{(k_2^2 - \frac{1}{4})}{s_2^2} + \frac{(k_3^2 - \frac{1}{4})}{s_3^2}\right]\Psi = -2E\Psi.$$

This equation admits solution via separation of variables in two coordinate systems: *spherical* coordinates

$$s_1 = \sin\theta\cos\varphi, \quad s_2 = \sin\theta\sin\varphi, \quad s_3 = \cos\theta$$

and *elliptical* coordinates

$$s_i^2 = \frac{(u_1 - e_i)(u_2 - e_i)}{(e_1 - e_j)(e_i - e_j)}, \qquad i,j,k = 1,2,3 \qquad i \neq j \neq k.$$

Indeed a basis for the symmetries of Schrödinger's equation with this potential is

$$L_{ij} = (s_i\partial_{s_j} - s_j\partial_{s_i})^2 + (\frac{1}{4} - k_i^2)\frac{s_j^2}{s_i^2} + (\frac{1}{4} - k_j^2)\frac{s_i^2}{s_j^2} - \frac{1}{2}, \quad i \neq j.$$

These symmetries satisfy the quadratic algebra relations

$$[L_{ij},R]=4\{L_{ij},L_{jk}\}-4\{L_{ij},L_{ik}\}+8(1-k_i^2)L_{jk}-8(1-k_j)^2L_{ik},$$

$$R^2=-\frac{4}{3}\{L_{ij},L_{ik},L_{jk}\}+\frac{64}{3}\{L_{ij},L_{ik}\}+\frac{64}{3}\{L_{ij},L_{jk}\}+\frac{64}{3}\{L_{ik},L_{jk}\}-$$
$$16(1-k_k^2)L_{ij}^2-16(1-k_j^2)L_{ik}^2-16(1-k_i^2)L_{jk}^2+$$
$$\frac{128}{3}(1-k_i^2)L_{jk}+\frac{128}{3}(1-k_j^2)L_{ik}+\frac{128}{3}(1-k_k^2)L_{ij},\quad i\neq j\neq k.$$

The eigenfunctions with bound state energy eigenvalues

$$E_p=\frac{1}{2}(2p+2+k_1+k_2+k_3)^2-\frac{1}{8}$$

in these coordinate systems are: *polar*

$$\Psi=(\sin\theta)^{-1}\Phi_n^{(k_2,k_1)}(\varphi)\Phi_m^{(2n+k_1+k_2,k_3)}(\theta)$$

where $p=m+n$, and *elliptical*

$$\Psi=(\prod_{\ell=1}^{3}s_\ell^{(k_\ell+\frac{1}{2})})\prod_{j=1}^{q}\left(\frac{s_1^2}{\theta_j-e_1}+\frac{s_2^2}{\theta_j-e_2}+\frac{s_3^2}{\theta_j-e_3}\right)$$

where

$$\frac{k_1+1}{\theta_m-e_1}+\frac{k_2+1}{\theta_m-e_2}+\frac{k_3+1}{\theta_m-e_3}+\sum_{j\neq m}\frac{2}{(\theta_m-\theta_j)}=0$$

and $q=p$. Here we have made use of the identity

$$\frac{s_1^2}{\theta_j-e_1}+\frac{s_2^2}{\theta_j-e_2}+\frac{s_3^2}{\theta_j-e_3}=\frac{\Pi_{\ell=1}^2(u_\ell-\theta_j)}{\Pi_{m=1}^3(\theta_j-e_m)}.$$

The separable eigenfunctions already given are eigenfunctions of the symmetry operators $L_{12}$ and $e_3L_{12}+e_2L_{13}+e_1L_{23}$:

$$\lambda_S=(2n+k_1+k_2+1)^2,$$

$$\lambda_E=-2[k_1(e_2+e_3)+k_2(e_1+e_3)+k_3(e_2+e_1)+e_3k_1k_2+e_1k_2k_3+e_2k_1k_3]-$$
$$\frac{3}{2}(e_1+e_2+e_3)-4e_2e_3(k_1+1)\sum_{m=1}^{q}\frac{1}{\theta_m-e_1}-$$

$$4e_2e_1(k_3+1)\sum_{m=1}^{q}\frac{1}{\theta_m-e_3}-4e_1e_3(k_2+1)\sum_{m=1}^{q}\frac{1}{\theta_m-e_2}.$$

**2.6** A second multiseparable potential on the sphere is

$$V=-\frac{\alpha s_3}{\sqrt{s_1^2+s_2^2}}+\frac{1}{4\sqrt{s_1^2+s_2^2}}\left[\frac{(k_1^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{(k_2^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}-s_1}\right].$$

Schrödinger's equation for this potential has the form

$$\left[(s_1\frac{\partial}{\partial s_2}-s_2\frac{\partial}{\partial s_1})^2+(s_1\frac{\partial}{\partial s_3}-s_3\frac{\partial}{\partial s_1})^2+(s_3\frac{\partial}{\partial s_2}-s_2\frac{\partial}{\partial s_3})^2\right]\Psi+$$

$$\left(-\frac{\alpha s_3}{\sqrt{s_1^2+s_2^2}}+\frac{1}{4\sqrt{s_1^2+s_2^2}}\left[\frac{(k_1^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{(k_2^2-\frac{1}{4})}{\sqrt{s_1^2+s_2^2}-s_1}\right]\right)\Psi=-2E\Psi.$$

This equation admits solution via separation of variables in two coordinates systems: *spherical* and *elliptical coordinates of modified type*

$$s_1'=\cos f\ s_1+\sin f\ s_3,\qquad s_2'=s_2,\qquad s_3'=-\sin f\ s_1+\cos f\ s_3$$

where

$$s_i^2=\frac{(y_1-e_i)(y_2-e_i)}{(e_1-e_j)(e_i-e_j)},\qquad i,j,k=1,2,3\quad i\neq j\neq k,$$

and

$$\sin f=\sqrt{\frac{(e_2-e_1)}{(e_3-e_1)}},\qquad e_1=e_2+\frac{1}{4}(E_++E_-)^2,\qquad e_3=e_2+\frac{1}{4}(E_+-E_-)^2,$$

$$y_j=e_2+\frac{1}{4}(E_+^2+E_-^2)+\frac{1}{4}E_+E_-(Z_j+\frac{1}{Z_j}),\quad j=1,2.$$

Indeed if we use the variables

$$U_3^2=Z_1Z_2,\quad U_1^2=\frac{(Z_1+\Omega_-)(Z_2+\Omega_-)}{(\Omega_-^2-1)},\quad U_2^2=\frac{(Z_1+\Omega_+)(Z_2+\Omega_+)}{(\Omega_+^2-1)}$$

where

$$\Omega_+=\frac{E_+}{E_-},\Omega_-=\frac{E_-}{E_+},$$

then, putting $k_3 = \sqrt{2(E - i\alpha) + 1/4}$ and $\hat{E} = i\alpha + E$ and multiplying the Schrödinger equation by $(Z_1 Z_2)^{-1} - 1$, we see that the resulting equation has the form

$$H\Psi = (U_1 \frac{\partial}{\partial U_2} - U_2 \frac{\partial}{\partial U_1})^2 + (U_1 \frac{\partial}{\partial U_3} - U_3 \frac{\partial}{\partial U_1})^2 + (U_3 \frac{\partial}{\partial U_2} - U_2 \frac{\partial}{\partial U_3})^2)\Psi +$$
$$\left[\frac{(\frac{1}{4} - k_1^2)}{U_1^2} + \frac{(\frac{1}{4} - k_2^2)}{U_2^2} + \frac{(\frac{1}{4} - k_3^2)}{U_3^2}\right]\Psi + 2\hat{E}\Psi = 0$$

which is essentially the same form as for the first potential. The bound state quantisation condition has the form

$$E = \frac{1}{2}\left[2q + 2 + k_1 + k_2 + \sqrt{\frac{1}{4} + 2(E - i\alpha)}\right]^2 - \frac{1}{8}.$$

**Outlook for this subject**

Some natural questions arise from the outline of superintegrability given above.

[1]. What in general do the quadratic algebra relations imply?

[2]. How does one characterize the quadratic algebras?

[3] Can we make systemmatic statements about superintegrability of Schrödinger's equation with a given potential in $n$ dimensions?

[4]. What are the complete sets of superintegrable potentials in complex spaces?

These are all questions for which a solution seems possible. In addition to these considerations are the relations between the special functions that appear in the solutions of these problems. This is a topic that is currently also being investigated.

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