Second order superintegrable systems in conformally flat spaces. IV: The classical 3D Stäckel transform and 3D classification theory.

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Abstract

This paper is one of a series that lays the groundwork for a structure and classification theory of second order superintegrable systems, both classical and quantum, in conformally flat spaces. In the first part of the paper we study the Stäckel transform (or coupling constant metamorphosis) as an invertible mapping between classical superintegrable systems on different 3D spaces. We show first that all superintegrable systems with nondegenerate potentials are multiseparable and then that each such system on any conformally flat space is Stäckel equivalent to a system on a constant curvature space. In the second part of the paper we classify all the superintegrable systems that admit separation in generic coordinates. We find that there are 8 families of these systems.

1 Introduction

This is a continuation of the series [1, 2, 3] whose purpose is to lay the groundwork for a structure and classification theory of second order superintegrable systems, both classical and quantum, in complex conformally flat spaces. Real spaces are considered as restrictions of these to the various real forms. In [1, 3] we have given examples in two and three dimensions, described the background as well as the interest and importance of these systems in mathematical physics and given many relevant to such systems on conformally flat spaces. Observed features of the systems are multiseparability, closure of the quadratic algebra of second order symmetries at order 6, use of representation theory of the quadratic algebra to derive spectral properties of the quantum Schrödinger operator, and a close relationship with exactly solvable and quasi-exactly solvable problems. Our approach is, rather than focus on particular spaces and systems, to use a general theoretical method based on integrability conditions to derive structure common to all systems. In distinction to the two-dimensional case, there are relatively few papers considering superintegrabiltiy on spaces of dimension ≥ 3. A few exceptions are [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Except for our own work, no one appears to have studied the detailed structure and classification theory for these higher dimensional systems.

In the first part of this paper we study the Stäckel transform, or coupling constant metamorphosis, [14, 15], for 3D classical superintegrable systems. Recall that for a classical 3D system on a conformally flat space we can always choose local coordinates x, y, z, not unique, such that the Hamiltonian takes the form $H = (p_1^2 + p_2^2 + p_3^2)/\lambda(x, y, z) + V(x, y, z)$. This system is *second order superintegrable* with *nondegenerate* potential $V = V(x, y, z, \alpha, \beta, \gamma, \delta)$ if it admits 5 functionally independent quadratic constants of the motion (i.e., generalized symmetries) $S_k = \sum_{ij} a^{ij}_{(k)} p_i p_j + W_{(k)}(x, y, \alpha, \beta, \gamma)$. As described in [3], the potential V is *nondegenerate* if it satisfies a system of coupled PDEs of the form

$$V_{22} = V_{11} + A^{22}V_1 + B^{22}V_2 + C^{22}V_3,\ V_{33} = V_{11} + A^{33}V_1 + B^{33}V_2 + C^{33}V_3,$$

$$V_{12} = A^{12}V_1 + B^{12}V_2 + C^{12}V_3,\ V_{13} = A^{13}V_1 + B^{13}V_2 + C^{13}V_3,$$
$$V_{23} = A^{23}V_1 + B^{23}V_2 + C^{23}V_3, \tag{1}$$

whose integrability conditions are satisfied identically. The analytic functions A^{ij}, B^{ij}, C^{ij} are determined uniquely from the Bertrand-Darboux equations for the 5 constants of the motion and are analytic except for a finite number of poles. At any regular point $\mathbf{x}_0 = (x_0, y_0, z_0)$, i.e., a point where theA^{ij}, B^{ij}, C^{ij} are defined and analytic and the constants of the motion are functionally independent, we can prescribe the values of $V(\mathbf{x}_0)$, $V_1(\mathbf{x}_0)$,$V_2(\mathbf{x}_0)$,$V_3(\mathbf{x}_0)$,$V_{11}(\mathbf{x}_0)$ arbitrarily and obtain a unique solution of (1). Here, $V_1 = \partial V/\partial x$, $V_2 = \partial V/\partial y$, etc. The 4 parameters for a nondegenerate potential (in addition to the usual additive constant) are the maximum number of parameters that can appear in a superintegrable system. If the number of parameters is fewer than 4, we say that the superintegrable potential is *degenerate*.

The 3D Stäckel transform is a conformal transformation of a superintegrable system on one conformally flat 3D space to a superintegrable system on another such space. We discuss some of the properties of this transform for a classical system and then prove two fundamental results: 1) We show that every superintegrable system with nondegenerate potential is multiseparable. This result uses the structure theory for such systems that we worked out in [3]. 2) We prove that all nondegenerate 3D superintegrable systems are Stäckel transforms of constant curvature systems. Thus, to obtain all nondegenerate conformally flat superintegrable systems, it is sufficient to classify those of constant curvature. The proofs of these fundamental results rest on results obtained in [3], and the careful reader of this paper will need to keep [3] at hand.

In the second part of the paper we use the results of the first part and our explicit knowledge of all separable coordinate systems on 3D constant curvature spaces to make a major advance in the classification of all separable systems with nondegenerate potential on a conformally flat space. Among the separable systems for 3D complex Euclidean space there are 7 that are "generic". We give a precise definition later, but, essentially this means that the coordinates belong to a multiparameter family. The ultimate generic coordinates are the Jacobi elliptic coordinates from which all others can be obtained by limiting processes [16, 17]. We show that each of the generic separable systems uniquely determines a nondegenerate superintegrable system that contains it. We obtain a similar result for the 5 generic separable systems on the complex 3-sphere. However, 4 of these turn out to be Stäckel transforms of Euclidean generic systems. Thus we find 8 Stäckel inequivalent generic systems on constant curvature spaces and all generic systems on 3D conformally flat spaces must be Stäckel equivalent to one of these. (In addition there are 2 nondegenerate superintegrable systems in Euclidean space that are only weakly functionally independent and these give rise to similar systems on a variety of conformally flat spaces.) Thus we exhibit 10 families of superintegrable systems in conformally flat spaces. This doesn't solve the classification problem completely, but it is a major advance. Any remaining nondegenerate superintegrable systems must be multiseparable but separate *only* in degenerate separable coordinates. This remaining problem is still complicated, but much less so than the original problem. This is a technically detailed proof, but the results are quite explicit and easy to grasp. We derive and give a simple characterization of 8 families of separable systems whose Stäckel transforms yield nondegenerate superintegrable systems on a variety of conformally flat spaces.

The next paper in this series will extend all of our classical 2D and 3D results to the quantum case. This is very easy in the 2D case but requires some machinery in 3D.

1.1 Second order conformal Killing tensors

There is a close relationship between the second-order Killing tensors of a conformally flat space in 3D and the second order conformal Killing tensors of flat space. A second order conformal Killing tensor for a space M_3 with metric $ds^2 = \lambda(x_1, x_2, x_3)(dx_1^2 + dx_2^2 + dx_3^2)$ and free Hamiltonian $\mathcal{H} = (p_1^2 + p_2^2 + p_3^2)/\lambda$ is a quadratic form $\mathcal{S} = \sum a^{ij}(x_1, x_2, x_3)p_i p_j$ such that $\{\mathcal{H}, \mathcal{S}\} = f(x_1, x_2, x_3)\mathcal{H}$, for some function f. Since f is arbitrary, it is easy to see that $\mathcal{S}$ is a conformal Killing tensor for M_3 if and only if it is a conformal Killing tensor for flat space $dx_1^2 + dx_2^2 + dx_3^2$. The conformal Killing tensors for flat space are very well known, e.g., [18]. The space of conformal Killing tensors is infinite dimensional. It is spanned by products of the conformal Killing vectors

$$p_1,\ p_2,\ p_3,\ x_3p_2 - x_2p_3,\ x_1p_3 - x_3p_1,\ x_2p_1 - x_1p_2,\ x_1p_1 + x_2p_2 + x_3p_3,$$

$$(x_1^2 - x_2^2 - x_3^2)p_1 + 2x_1x_3p_3 + 2x_1x_2p_2,\ (x_2^2 - x_1^2 - x_3^2)p_2 + 2x_2x_3p_3 + 2x_2x_1p_1,$$
$$(x_3^2 - x_1^2 - x_2^2)p_3 + 2x_3x_1p_1 + 2x_3x_2p_2,$$

and terms $g(x_1, x_2, x_3)(p_1^2 + p_2^2 + p_3^2)$ where g is an arbitrary function. Since every Killing tensor is also a conformal Killing tensor, we see that every second-order Killing tensor for M_3 can be expressed as a linear combination of these second-order generating elements though, of course, the space of Killing tensors is only finite dimensional. This shows in particular that every a^{ij} and every $a^{ii} - a^{jj}$ with $i \neq j$ is a polynomial of order at most 4 in x_1, x_2, x_3, no matter what is the choice of λ.

A straightforward, though tedious, computation from the above results yields the expressions

$$a_{11}^{12} = -a_{22}^{12} = \alpha_3(x_1^2 - x_2^2) + (\delta_1 + \gamma_2)x_1x_2 - \alpha_1x_1x_3 - \alpha_2x_2x_3 + \phi_3x_1 + \xi_3x_2 + \mu_3x_3 + \nu_3, \tag{2}$$

$$a_{11}^{13} = -a_{33}^{13} = \alpha_2(x_1^2 - x_3^2) - \alpha_1x_1x_2 + \gamma_2x_1x_3 - \alpha_3x_2x_3 + \phi_2x_1 + \xi_2x_2 + \mu_2x_3 + \nu_2,$$

$a^{23}_{33} = -a^{23}_{22} = \alpha_1(x_2^2 - x_3^2) - \alpha_2 x_1 x_2 + \alpha_3 x_1 x_3 + \delta_1 x_2 x_3 + \phi_1 x_1 + \xi_1 x_2 + \mu_1 x_3 + \nu_1$, where $\alpha_j, \delta_j, \gamma_j, \phi_j, \xi_j, \mu_j, \nu_j$ are constants. Furthermore. $(a^{ii} - a^{jj})_i = 2a^{ij}_j$ for $i \neq j$, and $a^{12}_3 + a^{13}_2 + a^{23}_1 = 0$.

It is useful to pass to new variables $a^{11},\ a^{24},\ a^{34},\ a^{12},\ a^{13},\ a^{23}$ for the Killing tensor, where $a^{24} = a^{22} - a^{11}$, $a^{34} = a^{33} - a^{11}$. Then we see that $a^{24},\ a^{34}, a^{12},\ a^{13},\ a^{23}$ are polynomials of order ≤ 4. The remaining conditions can be expressed in the form

$$(a^{11}\lambda)_1 = -\lambda_2 a^{12} - \lambda_3 a^{13},\ (a^{11}\lambda)_2 = -\lambda_1 a^{12} - (a^{24}\lambda)_2 - \lambda_3 a^{23}, \quad (3)$$

$$(a^{11}\lambda)_3 = -\lambda_1 a^{13} - \lambda_2 a^{23} - (a^{34}\lambda)_3.$$

Theorem 1 *Necessary and sufficient conditions that the quadratic form $\mathcal{S} = \sum_{ij} a^{ij} p_i p_j + W$ be a second order constant of the motion for the space with metric $ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2)$ and potential V are*

1. *$\sum_{ij} a^{ij} p_i p_j$ is a conformal Killing tensor on the flat space with metric $dx_1^2 + dx_2^2 + dx_3^2$.*

2. *The integrability conditions for (3) hold:*

$$\Big(\lambda_2 a^{12} + \lambda_3 a^{13}\Big)_2 = \Big(\lambda_1 a^{12} + (a^{24}\lambda)_2 + \lambda_3 a^{23}\Big)_1, \quad (4)$$

$$\Big(\lambda_2 a^{12} + \lambda_3 a^{13}\Big)_3 = \Big(\lambda_1 a^{13} + \lambda_2 a^{23} + (a^{34}\lambda)_3\Big)_1,$$

$$\Big(\lambda_1 a^{12} + (a^{24}\lambda)_2 + \lambda_3 a^{23}\Big)_3 = \Big(\lambda_1 a^{13} + \lambda_2 a^{23} + (a^{34}\lambda)_3\Big)_2.$$

3. *The Bertrand-Darboux conditions for the potential hold:*

$$\sum_{s=1}^{3} \Big[V_{sj}\lambda a^{s\ell} - V_{s\ell}\lambda a^{sj} + V_s\Big((\lambda a^{s\ell})_j - (\lambda a^{sj})_\ell\Big)\Big] = 0. \quad (5)$$

These are just the conditions $\partial_{x_\ell} W_j = \partial_{x_j} W_\ell$ for $j \neq \ell$.

2 The Stäckel transform for 3D systems

The Stäckel transform [14] or coupling constant metamorphosis [15] plays a fundamental role in relating superintegrable systems on different manifolds. Suppose we have a superintegrable system

$$H = \frac{p_1^2 + p_2^2 + p_3^2}{\lambda(x, y, z)} + V(x, y, z) \quad (6)$$

in local orthogonal coordinates, with nondegenerate potential $V(x,y,z)$:

$$V_{33} = V_{11} + A^{33}V_1 + B^{33}V_2 + C^{33}V_3,\ V_{22} = V_{11} + A^{22}V_1 + B^{22}V_2 + C^{22}V_3,$$
$$V_{23} = A^{23}V_1 + B^{23}V_2 + C^{23}V_3,\ V_{13} = A^{13}V_1 + B^{13}V_2 + C^{13}V_3,$$
$$V_{12} = A^{12}V_1 + B^{12}V_2 + C^{12}V_3 \tag{7}$$

and suppose $U(x,y,z)$ is a particular solution of equations (7), nonzero in an open set. Then the transformed system $\tilde{H} = (p_1^2 + p_2^2 + p_3^2)/\tilde{\lambda} + \tilde{V}$ with nondegenerate potential $\tilde{V}(x,y,z)$:

$$\tilde{V}_{33} = \tilde{V}_{11} + \tilde{A}^{33}\tilde{V}_1 + \tilde{B}^{33}\tilde{V}_2 + \tilde{C}^{33}\tilde{V}_3,\ \tilde{V}_{22} = \tilde{V}_{11} + \tilde{A}^{22}\tilde{V}_1 + \tilde{B}^{22}\tilde{V}_2 + \tilde{C}^{22}\tilde{V}_3,$$
$$\tilde{V}_{23} = \tilde{A}^{23}\tilde{V}_1 + \tilde{B}^{23}\tilde{V}_2 + \tilde{C}^{23}\tilde{V}_3,\ \tilde{V}_{13} = \tilde{A}^{13}\tilde{V}_1 + \tilde{B}^{13}\tilde{V}_2 + \tilde{C}^{13}\tilde{V}_3,$$
$$\tilde{V}_{12} = \tilde{A}^{12}\tilde{V}_1 + \tilde{B}^{12}\tilde{V}_2 + \tilde{C}^{12}\tilde{V}_3, \tag{8}$$

is also superintegrable, where

$$\tilde{\lambda} = \lambda U,\ \tilde{V} = \frac{V}{U},\ \tilde{A}^{33} = A^{33} + 2\frac{U_1}{U},\ \tilde{C}^{33} = C^{33} - 2\frac{U_3}{U},$$

$$\tilde{A}^{22} = A^{22} + 2\frac{U_1}{U},\ \tilde{B}^{22} = B^{22} - 2\frac{U_2}{U},\ \tilde{B}^{23} = B^{23} - \frac{U_3}{U},\ \tilde{C}^{23} = C^{23} - \frac{U_2}{U},$$
$$\tilde{A}^{13} = A^{13} - \frac{U_3}{U},\ \tilde{C}^{13} = C^{13} - \frac{U_1}{U},\ \tilde{A}^{12} = A^{12} - \frac{U_2}{U},\ \tilde{B}^{12} = B^{12} - \frac{U_1}{U},$$

and $\tilde{A}^{23} = A^{23}$, $\tilde{B}^{33} = B^{33}$, $\tilde{B}^{13} = B^{13}$, $\tilde{C}^{22} = C^{22}4$, $\tilde{C}^{12} = C^{12}$. Let $S = \sum a^{ij}p_ip_j + W = S_0 + W$ be a second order symmetry of H and $S_U = \sum a^{ij}p_ip_j + W_U = S_0 + W_U$ be the special case that is in involution with $(p_1^2 + p_2^2 + p_3^2)/\lambda + U$. Then $\tilde{S} = S - \frac{W_U}{U}H$ is the corresponding symmetry of $\tilde{H}$. Since one can always add a constant to a nondegenerate potential, it follows that $1/U$ defines an inverse Stäckel transform of $\tilde{H}$ to H. See [14] for many examples of this transform.

3 Multiseparability and Stäckel equivalence

From the general theory of variable separation for Hamilton-Jacobi equations, e.g., [20, 21] we know that second order symmetries L_1, L_2 define a separable system for the equation

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{\lambda(x,y,z)} + V(x,y,z) = E$$

if and only if 1) The symmetries H, L_1, L_2 form a linearly independent set as quadratic forms, 2) $\{L_1, L_2\} = 0$, and 3) The three quadratic forms have a common eigenbasis of differential forms. This last requirement means that, expressed in coordinates x, y, z, at least one of the matrices $\mathcal{A}_{(j)}(\mathbf{x})$ (of the quadratic form associated with L_j) can be diagonalized by conjugacy transforms in a neighborhood of a regular point and that $[\mathcal{A}_{(2)}(\mathbf{x}), \mathcal{A}_{(1)}(\mathbf{x})] = 0$. However, for nondegenerate superintegrable potentials in a conformally flat space we see that $\{L_1, L_2\} = 0 \leftrightarrow [\mathcal{A}_{(2)}(\mathbf{x}_0), \mathcal{A}_{(1)}(\mathbf{x}_0)] = 0$, $\mathcal{F}(\mathbf{x}_0) = 0$ at a single regular point $\mathbf{x}_0$, see [3], §5, so that the intrinsic conditions for the existence of a separable coordinate system are simplified.

Let $\mathcal{A} = \sum_{i\leq j} a^{ij}\mathcal{A}^{ij}$, $\mathcal{B} = \sum_{i\leq j} b^{ij}\mathcal{A}^{ij}$, be the matrices of two symmetries at the point $\mathbf{x}_0$. Here, $\mathcal{A}^{ij} = \frac{1}{2}(\mathcal{E}^{ij} + \mathcal{E}^{ji})$ where $\mathcal{E}^{ij}$ is the 3×3 matrix with matrix element 1 in row i, column j, and 0 everywhere else. From the table in §5 of [3] we see that the corresponding symmetries are in involution if and only if the matrices $\mathcal{A}, \mathcal{B}$ commute and the additional condition

$$
\begin{gathered}
(a^{12}b^{11} - b^{12}a^{11})(C^{33} - B^{23} - A^{13}) + (a^{22}b^{12} - a^{12}b^{22})(C^{33} - 2B^{23}) + \\
(a^{13}b^{11} - a^{11}b^{13})(B^{33} + 2A^{12} - B^{22}) + (a^{33}b^{13} - a^{13}b^{33})(2B^{33} + 2A^{12} - B^{22}) + \\
(a^{23}b^{22} - a^{22}b^{23})(-2B^{12} - A^{33}) + (a^{33}b^{23} - b^{33}a^{23})(-2B^{12} + A^{22} - 2A^{33}) + \\
2(a^{11}b^{22} - a^{22}b^{11} + a^{33}b^{11} - a^{11}b^{33} + a^{22}b^{33} - a^{33}b^{22})A^{23} + \\
(a^{23}b^{11} - a^{11}b^{23})(A^{22} - A^{33}) + (a^{33}b^{12} - a^{12}b^{33})(B^{23} - A^{13}) + \\
(a^{13}b^{22} - a^{22}b^{13})B^{33} = 0
\end{gathered}
\tag{9}
$$

holds. Note that the metric G does not appear in these conditions.

Theorem 2 *Let V be a superintegrable nondegenerate potential in a 3D conformally flat space. Then V defines a multiseparable system.*

Proof: From (9) we see that the second order symmetries with matrices $\mathcal{A}^{(33)}$ and $\alpha\mathcal{A}^{(11)} + \beta\mathcal{A}^{(12)}$ will be in involution if and only if $2\alpha A^{23} + \beta(B^{23} - A^{13}) = 0$ at the regular point $\mathbf{x}_0$. If $A^{23}(\mathbf{x}_0) = 0$ we can set $\alpha = 1$, $\beta = 0$ and the symmetries $\mathcal{A}^{(33)}$, $\mathcal{A}^{11}$ will define a separable system. If $A^{23}(\mathbf{x}_0) \neq 0$ we can set $\alpha = -(B^{23} - A^{13})/2A^{23}$, $\beta = 1$. Then the symmetries with nonzero matrices $\mathcal{A}^{(33)}$ and $\alpha\mathcal{A}^{(11)} + \beta\mathcal{A}^{(12)}$ will be in involution. The second case must occur for some regular point $\mathbf{x}_0$ unless $A^{23}(\mathbf{x}) = 0$ for all $\mathbf{x}$. In this last eventuality we can perform a suitable Euclidean rotation (with arbitrarily small complex rotation angle) so that A^{23} doesn't vanish identically in the rotated coordinate system. It is a straightforward exercise to show that this transformation is not possible if and only if

$$B^{33} = C^{22} = 0, \quad A^{13} = B^{23}, \quad A^{12} = C^{23}, \quad A^{22} = A^{33}. \tag{10}$$

In this eventuality, we can set $\alpha = 0, \beta = 1$ and find a solution. Thus we can always find a linear combination of these matrices, corresponding to $\beta = 1$ and with 3 distinct eigenvalues, so they will determine separable coordinates. We could have carried through this same construction for the second order symmetries with matrices $\mathcal{A}^{(22)}$ and $\gamma\mathcal{A}^{(11)} + \delta\mathcal{A}^{(13)}$ and for the second order symmetries with matrices $\mathcal{A}^{(11)}$ and $\mu\mathcal{A}^{(22)} + \xi\mathcal{A}^{(23)}$ and shown that we could always find solutions with $\delta = \xi = 1$. Thus the system is multiseparable (in at least 3 coordinate systems). Q.E.D.

Corollary 1 *Let V be a superintegrable nondegenerate potential in a 3D conformally flat space. Then there is a continuous 1-parameter (or multi-parameter) family of separable systems for V, spanning at least a 5-dimensional subspace of symmetries.*

Proof: We follow the method of proof of the theorem.

Case I: Suppose $A^{23}(\mathbf{x}_0) \neq 0$. From (9) we can verify that the symmetries with matrices

$$\mathcal{A} = \begin{pmatrix} 0, & 0, & 0 \\ 0, & f^2, & -f \\ 0, & -f, & 1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} g, & 1/2, & f/2 \\ 1/2, & 0, & 0 \\ f/2, & 0, & 0 \end{pmatrix}, \tag{11}$$

are in involution provided

$$-4f^2B^{23} + 2f^2C^{33} + 3fB^{33} + 4fA^{12} - 2fB^{22} + fB^{23} - 4f^2gA^{23} - 4fgA^{22}$$
$$+4fgA^{33} + 4gA^{23} + 2B^{23} - 2A^{13} - 2f^3B^{33} = 0.$$

Since $A^{23}(\mathbf{x}_0) \neq 0$ this equation can be solved for g as a function of f for f in some open set. The resulting symmetries $\mathcal{A}, \mathcal{B}$ are in involution and have eigenvalues $(0, 0, f^2 + 1)$ and $(0, \frac{1}{2}[g + \sqrt{f^2 + g^2 + 1}], \frac{1}{2}[g - \sqrt{f^2 + g^2 + 1}])$, respectively. Thus they determine a 1-parameter family of separable coordinates. Moreover, as f varies in an open set, the space spanned by the symmetries (including the Hamiltonian) has dimension 6.

Case II: If $A^{23}(\mathbf{x}_0) = 0$, we can assume that equations (10) hold. Then the problem breaks up into a series of special cases. Suppose first that $C^{33} - 2A^{13} = \ell \neq 0$. Then we can verify that the symmetries with matrices

$$\mathcal{A} = \begin{pmatrix} 0, & 2gk/K, & 0 \\ 2gk/K, & 1, & -g/K \\ 0, & -g/K, & g^2/K \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} f, & g/2, & 1/2 \\ g/2, & 0, & 0 \\ 1/2, & 0, & k \end{pmatrix}, \tag{12}$$

are in involution provided $K = 1 - 4fk \neq 0$ and g satisfies

$$-g(2A^{12} - B^{22}) + 2k(A^{22} + 2B^{12}) = \ell.$$

If $2A^{12} - B^{22} \neq 0$ then there is a nonzero solution expression g as a function of k. Since f, k are essentially arbitrary, they determine a 5-dimensional space spanned by the symmetries and a 2-parameter family of separable coordinates. If $2A^{12} - B^{22} = 0$, $A^{22} + 2B^{12} \neq 0$ then k is a nonzero constant and f, g are essentially arbitrary, so they again determine a 5-dimensional space spanned by the symmetries and a 2-parameter family of separable coordinates. If $2A^{12} - B^{22} = 0$, $A^{22} + 2B^{12} = 0$, then the symmetries with matrices

$$\mathcal{A} = \begin{pmatrix} 0, & H/K, & 0 \\ H/K, & 1, & 0 \\ 0, & 0, & L/K \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} f, & g/2, & 1/2 \\ g/2, & 0, & h/2 \\ 1/2, & h/2, & k \end{pmatrix}, \tag{13}$$

where $K = 1 - 4fk - h^2 - 2hgf \neq 0$, and

$$H = -h + 2gk + hg^2, \; G = -g + 2fh + gh^2, \; L = g^2 - h^2 + 2hg(k - f),$$

are in involution provided $f = (g/2h - h/2g)$. (This implies $G = 0$ and $L = Hh$.) They determine a 6-dimensional space spanned by the symmetries and a 3-parameter family of separable coordinates. This covers all cases where $\ell \neq 0$.

Now suppose $\ell = 0$, i.e., $C^{33} = 2A^{13}$. Then the symmetries with matrices (13) are in involution provided

$$(2A^{12} - B^{22})(h^2 - g^2 + 2hgf - 2hkg) + (A^{22} + 2B^{12})(-h + 2kg + hg^2) = 0.$$

If $2A^{12} - B^{22} \neq 0$ then we can solve this equation to express f as a nonzero function of g, h, k. This yields at least a 5-dimensional space spanned by the symmetries and a 3-parameter family of separable coordinates. Finally, suppose in addition that $2A^{12} - B^{22} = 0$. Then we can verify that the symmetries with matrices

$$\mathcal{A} = \begin{pmatrix} 0, & 0, & 0 \\ 0, & f^2, & -f \\ 0 & -f, & 1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} g, & 1/2, & f/2 \\ 1/2, & 0, & 0 \\ f/2, & 0, & 0 \end{pmatrix}, \tag{14}$$

are in involution with no conditions on f, g. Again, as f, g vary in an open set, the space spanned by the symmetries (including the Hamiltonian) has dimension 6. Q.E.D.

In [17] the following result was obtained.

Theorem 3 *Let u_1, u_2, u_3 be an orthogonal separable coordinate system for a 3D conformally flat space with metric $d\tilde{s}^2$. Then there is a function f such that $fd\tilde{s}^2 = ds^2$ where ds^2 is a constant curvature space metric and ds^2 is orthogonally separable in exactly these same coordinates u_1, u_2, u_3. The function f is called a Stäckel multiplier with respect to this coordinate system.*

Thus the possible separable coordinate systems for a conformally flat space are all obtained, modulo a Stäckel multiplier, from separable systems on 3D flat space or on the 3-sphere.

Theorem 4 *Every superintegrable system with nondegenerate potential on a 3D conformally flat space is Stäckel equivalent to a superintegrable system on either 3D flat space or the 3-sphere.*

PROOF: Suppose we have a superintegrable system with nondegenerate potential on a conformally flat space. Then by Corollary 1 this system separates in a 1- or multi-parameter parameter family of coordinate systems spanning a 5-dimensional subspace of symmetries. By Theorem 3 each of these three systems is conformal to a separable system in flat space or on the 3-sphere. Thus from [19], page 85, the metric for the space in standard Cartesian-like coordinates x, y, z is simultaneously conformal to three systems corresponding to the following possible choices for the metric function $\lambda(x, y, z)$, namely

$$1,\ 1/(x+iy)^2,\ 1/r^4,\ (\text{flatspace});\quad 1/x^2,\ 1/(1+r^2/4)^2,\ (3-\text{sphere}); \tag{15}$$

in the same coordinates, and each of the conformal factors is a Stäckel multiplier with respect to the corresponding separable coordinates. From the Corollary we see that we can find two separable systems such that the factor (15) is the same, i.e., the metric must take the form $d\tilde{s}^2 = fds^2$ where ds^2 is the metric on a single constant curvature space, either 3D flat space or the 3-sphere, and the constant curvature space separates in these same two coordinate systems. Further the space of symmetries spanned by the two sets is at least 5-dimensional.

Then we have $(\tilde{\mathcal{H}} + \tilde{V})/f = \mathcal{H} + V$ where $\tilde{\mathcal{H}} + \tilde{V}$ is the original superintegrable system, $\mathcal{H}$ is the Hamiltonian on a constant curvature space, and V is the induced multiparameter potential. Under the transform f each of the commuting second order symmetries $\mathcal{S}$ of the original system that defines a coordinate separation transforms to a symmetry of the form $\mathcal{S} + g_S\mathcal{H}$ for g_S a function. There are at least 5 such functionally linearly independent symmetries arising from separation in 2 coordinate systems, so the constant curvature space system admits 5 functionally linearly independent symmetries. Thus the potential V must satisfy the -Darboux equations for these symmetries. It follows that V is nondegenerate and by Theorem 2 of [3] that the system $\mathcal{H} + V$ is itself superintegrable with nondegenerate potential. The function f is simultaneously a Stäckel multiplier with respect to the two coordinate systems whose symmetries completely characterize the superintegrable system $\mathcal{H} + V$. That is, f satisfies the Bertrand-Darboux equations for 5 functionally linearly independent symmetries. Hence f itself satisfies the equations that determine the nondegenerate potential V. This means That the system $\tilde{\mathcal{H}} + \tilde{V}$ is Stäckel equivalent to the constant curvature space superintegrable system. Q.E.D.

4 Classification of nondegenerate systems

4.1 Separable systems in complex Euclidean space

It is a difficult task to list all 3D conformally flat superintegrable systems with nondegenerate potential and to show that the classification is complete. However, we now have tools to simplify the problem. First, since every such system is Stäckel equivalent to a system on Euclidean space or the complex sphere, we can restrict ourselves to those two spaces. Second, since every such system is multiseparable, we can bring to bear our knowledge of all orthogonal separable coordinates on these spaces. These results can be gleaned from the books [20, 16] and many papers of the authors, e.g., [17]. Thus in principle, we have enough information to accomplish our task, though the details are formidably complicated.

We begin by summarizing the full list of orthogonal separable systems in complex Euclidean space and the associated symmetry operators. Here, a "natural" basis for first order symmetries is given by $p_1 \equiv p_x$, $p_2 \equiv p_y$,$p_3 \equiv p_z$, $J_1 = yp_z - zp_y$,$J_2 = zp_x - xp_z$, $J_3 = xp_y - yp_x$ in the classical case and $p_1 = \partial_x$, $p_2 = \partial_y$, $p_3 = \partial_z$, $J_1 = y\partial_z - z\partial_y$, $J_2 = z\partial_x - x\partial_z$, $J_3 = x\partial_y - y\partial_x$ in the quantum case. (In the operator characterizations for the quantum case, the classical product of two constants of the motion is replaced by the symmetrized product of the corresponding operator symmetries.) The Hamiltonian is $H = p_1^2 + p_2^2 + p_3^2$. In each case below we list the coordinates followed by the constants of the motion that characterize them.

Note: The bracket notation used to describe generic coordinates in three dimensional Euclidean space is due to Bôcher and is an adaptation of the notation used to describe the elementary divisors of two quadratic forms one of which is the quadratic form associated with Euclidean space and the second with the quadratic form of the coordinate curves describing the coordinate system. In order to do this in three dimensions and also deal with separable solutions of Laplace's equation we use the symbol $[p_0, p_1, \cdots, p_r]$ where $\sum_{i=0}^{r} p_i = 5$ and $p_0 \geq 2$.(See [16] for further details). This determines a coordinate system whose infinitesimal distance is of the form

$$ds^2 = \frac{1}{(u-v)(u-w)}P(u)p_u^2 + \frac{1}{(v-u)(v-w)}P(v)p_v^2 + \frac{1}{(w-v)(w-u)}P(w)p_w^2$$

where $P(\lambda) = (\lambda - e_1)^{p_1} \cdots (\lambda - e_r)^{p_r}$. The index p_0 is associated with ∞ .

$$[2111] \quad x^2 = c^2 \frac{(u-e_1)(v-e_1)(w-e_1)}{(e_1-e_2)(e_1-e_3)}, \quad y^2 = c^2 \frac{(u-e_2)(v-e_2)(w-e_2)}{(e_2-e_1)(e_2-e_3)}$$

$$z^2 = c^2 \frac{(u-e_3)(v-e_3)(w-e_3)}{(e_3-e_1)(e_3-e_2)}$$

$$L_1 = J_1^2 + J_2^2 + J_3^2 + c^2((e_1+e_2)p_3^2 + (e_1+e_3)p_2^2 + (e_3+e_2)p_1^2),$$

$$L_2 = e_1 J_1^2 + e_2 J_2^2 + e_3 J_3^2 + c^2(e_1 e_2 p_3^2 + e_1 e_3 p_2^2 + e_3 e_2 p_1^2).$$

$$[221] \quad x^2 + y^2 = -c^2 \left[\frac{(u-e_1)(v-e_1)(w-e_1)}{(e_1-e_2)^2} \right]$$

$$-\frac{c^2}{e_1-e_2}\left[(u-e_1)(v-e_1) + (u-e_1)(w-e_1) + (v-e_1)(w-e_1)\right],$$

$$(x-iy)^2 = c^2 \frac{(u-e_1)(v-e_1)(w-e_1)}{e_1-e_2}, \quad z^2 = c^2 \frac{(u-e_2)(v-e_2)(w-e_2)}{(e_2-e_1)^2}.$$

$$L_1 = J_1^2 + J_2^2 + J_3^2 + c^2((e_1-e_2)(p_1+ip_2)^2 + 2e_2 p_3^2 + (e_1+e_2)(p_1^2+p_2^2)),$$

$$L_2 = e_2(J_1^2 + J_2^2) + (e_2 - e_1)(J_1 + iJ_2)^2 + e_1 J_3^2 +$$
$$c^2((e_1 e_2 (p_1^2 + p_2^2) + e_1(e_1-e_2)(p_1+ip_2)^2 + e_2^2 p_3^2).$$

$$[23] \quad x - iy = \frac{1}{2}c(\frac{u^2+v^2+w^2}{uvw} - \frac{1}{2}\frac{u^2v^2 + u^2w^2 + v^2w^2}{u^3v^3w^3}),$$

$$z = \frac{1}{2}c(\frac{uv}{w} + \frac{uw}{v} + \frac{vw}{u}), \quad x + iy = cuvw.$$

$$L_1 = J_1^2 + J_2^2 + J_3^2 + 2c^2(p_1+ip_2)p_3, \quad L_2 = -2J_3(J_1+iJ_2) + c^2(p_1+ip_2)^2.$$

$$[311] \quad x = \frac{c}{4}(u^2+v^2+w^2 + \frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{w^2}) + \frac{3}{2}c,$$

$$y = -\frac{c}{4}\frac{(u^2-1)(v^2-1)(w^2-1)}{uvw}, \quad z = i\frac{c}{4}\frac{(u^2+1)(v^2+1)(w^2+1)}{uvw}.$$

$$L_1 = c(J_3 p_2 - J_2 p_3) + c^2(p_1^2 - p_2^2), \quad L_2 = -\frac{1}{4}J_1^2 - cJ_2 p_3 - c^2 p_3^2.$$

[32] $x+iy=uvw,\quad x-iy=-(\frac{uv}{w}+\frac{uw}{v}+\frac{vw}{u}),\quad z=\frac{1}{2}(u^2+v^2+w^2).$

$$L_1=-c(J_2+iJ_1)(p_1+ip_2)-c(J_2-iJ_1)(p_1-ip_2)-c^2(p_1+ip_2)^2$$
$$L_2=J_3^2-2c(J_2-iJ_1)(p_1+ip_2)$$

[41] $x+iy=u^2v^2+u^2w^2+v^2w^2-\frac{1}{2}(u^4+v^4+w^4),\ x-iy=c^2(u^2+v^2+w^2),\ z=2icuvw.$

The symmetries that describe this system are

$$L_1=-iJ_3(p_1-ip_2)+(J_2+iJ_1)p_3+\frac{1}{4}c^4(p_1+ip_2)^2,\ L_2=-(J_1-iJ_2)^2-2ic^4(J_1+iJ_2)p_3.$$

[5] $x+iy=c(u+v+w),\quad x-iy=\frac{c}{4}(u-v-w)(u+v-w)(u+w-v),$

$$z=-\frac{c}{4}(u^2+v^2+w^2-2(uv+uw+vw)).$$
$$L_1=iJ_3(p_1+ip_2)+(J_2-iJ_1)p_3+cp_3(p_1-ip_2),$$
$$L_2=\frac{1}{4}(J_2-iJ_1)^2-c(2(J_2+iJ_1i(p_1+ip_2)+i(p_1-ip_2)(J_1+iJ_2))+\frac{c^2}{4}(p_1-ip_2)^2.$$

We summarize the remaining degenerate separable coordinates:

Euclidean coordinates. All of these have one symmetry in common: $L_1=p_3^2$. The 7 systems are, polar, Cartesian, light cone, elliptic, parabolic, hyperbolic and semihyperbolic.

Complex sphere coordinates. These all have the symmetry $L_1=J_1^2+J_2^2+J_3^2$ in common. The 5 systems are spherical, horospherical, elliptical, hyperbolic, and semi-circular parabolic.

Rotational types of coordinates. There are 3 of these systems, each of which is characterized by the fact that one defining symmetry is a perfect square.

Nonorthogonal heat type coordinates. Each of these nonorthogonal systems corresponds to one first order symmetry. Hence it cannot arise for systems with nondegenerate potentials.

Note that the first 7 separable systems are "generic," i.e., they occur in one, two or three - parameter families, whereas the remaining systems are special limiting cases of the generic ones. We shall show that each of the generic separable systems uniquely determines a nondegenerate superintegrable system.

4.2 "Generic" 3D Euclidean superintegrable systems

Each of the 7 "generic" Euclidean separable systems depends on a scaling parameter c and up to three parameters e_1, e_2, e_3. For each such set of coordinates we shall show that there is exactly one nondegenerate superintegrable system that admits separation in these coordinates *simultaneously for all values of the parameters* c, e_j.

Consider the system [23], for example. If a nondegenerate superintegrable system separates in these coordinates for all values of the parameter c, then the space of second order symmetries must contain the 5 symmetries

$$\mathcal{H} = p_x^2 + p_y^2 + p_z^2 + V, \quad \mathcal{S}_1 = J_1^2 + J_2^2 + J_3^2 + f_1, \quad \mathcal{S}_2 = J_3(J_1 + iJ_2) + f_2,$$

$$\mathcal{S}_3 = (p_x + ip_y)^2 + f_3, \quad \mathcal{S}_4 = p_z(p_x + ip_y) + f_4.$$

It is straightforward to check that the 12×5 matrix of coefficients of the second derivative terms in the 12 Bertrand-Darboux equations associated with symmetries $\mathcal{S}_1, \cdots, \mathcal{S}_4$ has rank 5 in general. Thus, there is at most one nondegenerate superintegrable system admitting these symmetries. Solving the Bertrand-Darboux equations for the potential we find the unique solution

$$V(\mathbf{x}) := \alpha(x^2 + y^2 + z^2) + \frac{\beta}{(x+iy)^2} + \frac{\gamma z}{(x+iy)^3} + \frac{\delta(x^2 + y^2 - 3z^2)}{(x+iy)^4}.$$

Finally, we can use the symmetry conditions for this potential to obtain the full 6-dimensional space of second order symmetries. This is the superintegrable system III on the following table. The other six cases yield corresponding results.

Theorem 5 *Each of the* 7 *"generic" Euclidean separable systems determines a unique nondegenerate superintegrable system that permits separation simultaneously for all values of the scaling parameter* c *and any other defining parameters* e_j*. For each of these systems there is a basis of* 5 *(strongly) functionally independent and* 6 *linearly independent second order symmetries. The corresponding nondegenerate potentials and basis of symmetries are (the* f_j *are functions of* x_1, x_2, x_3*):*

$$\text{I [2111]} \qquad V = \frac{\alpha_1}{x^2} + \frac{\alpha_2}{y^2} + \frac{\alpha_3}{z^2} + \delta(x^2 + y^2 + z^2), \tag{16}$$

$$\mathcal{P}_i = \partial_{x_i}^2 + \delta x_i^2 + \frac{\alpha_i}{x_i^2}, \qquad \mathcal{J}_{ij} = (x_i p_{x_j} - x_j p_{x_i})^2 + \alpha_i^2 \frac{x_j^2}{x_i^2} + \alpha_j^2 \frac{x_i^2}{x_j^2}, \quad i \geq j.$$

$$\text{II [221]} \qquad V = \alpha(x^2 + y^2 + z^2) + \beta\frac{x - iy}{(x+iy)^3} + \frac{\gamma}{(x+iy)^2} + \frac{\delta}{z^2}, \tag{17}$$

$$\mathcal{S}_1 = J\cdot J + f_1,\quad \mathcal{S}_2 = p_z^2 + f_2,\quad \mathcal{S}_3 = J_3^2 + f_3,$$
$$\mathcal{S}_4 = (p_x+ip_y)^2 + f_4,\quad L_5 = (J_2 - iJ_1)^2 + f_5.$$

III [23] $$V = \alpha(x^2+y^2+z^2) + \frac{\beta}{(x+iy)^2} + \frac{\gamma z}{(x+iy)^3} + \frac{\delta(x^2+y^2-3z^2)}{(x+iy)^4}, \tag{18}$$

$$\mathcal{S}_1 = J\cdot J + f_1,\quad \mathcal{S}_2 = (J_2 - iJ_1)^2 + f_2,\quad \mathcal{S}_3 = J_3(J_2 - iJ_1) + f_3,$$
$$\mathcal{S}_4 = (p_x+ip_y)^2 + f_4,\quad \mathcal{S}_5 = p_z(p_x+ip_y) + f_5.$$

IV [311] $$V = \alpha(4x^2+y^2+z^2) + \beta x + \frac{\gamma}{y^2} + \frac{\delta}{z^2}, \tag{19}$$

$$\mathcal{S}_1 = p_x^2 + f_1,\quad \mathcal{S}_2 = p_y^2 + f_2,\quad \mathcal{S}_3 = p_z J_2 + f_3,$$
$$\mathcal{S}_4 = p_y J_3 + f_4,\quad \mathcal{S}_5 = J_1^2 + f_5.$$

V [32] $$V = \alpha(4x^2+y^2+z^2) + \beta x + \frac{\gamma}{(y+iz)^2} + \frac{\delta(y-iz)}{(y+iz)^3}, \tag{20}$$

$$\mathcal{S}_1 = p_x^2 + f_1,\quad \mathcal{S}_2 = J_1^2 + f_2,\quad \mathcal{S}_3 = (p_z - ip_y)(J_2 + iJ_3) + f_3,$$
$$\mathcal{S}_4 = p_z J_2 - p_y J_3 + f_4,\quad \mathcal{S}_5 = (p_z - ip_y)^2 + f_5.$$

VI [41] $$V = \alpha\left(z^2 - 2(x-iy)^3 + 4(x^2+y^2)\right) + \beta\left(2(x+iy) - 3(x-iy)^2\right) + \gamma(x-iy) + \frac{\delta}{z^2}, \tag{21}$$

$$\mathcal{S}_1 = (p_x - ip_y)^2 + f_1,\quad \mathcal{S}_2 = p_z^2 + f_2,\quad \mathcal{S}_3 = p_z(J_2 + iJ_1) + f_3,$$
$$\mathcal{S}_4 = J_3(p_x - ip_y) - \frac{i}{4}(p_x+ip_y)^2 + f_4,\quad \mathcal{S}_5 = (J_2 + iJ_1)^2 + 4ip_z J_1 + f_5.$$

VII [5] $$V = \alpha(x+iy) + \beta(\frac{3}{4}(x+iy)^2 + \frac{1}{4}z) + \gamma((x+iy)^3 + \frac{1}{16}(x-iy) + \frac{3}{4}(x+iy)z) \tag{22}$$

$$+\delta(\frac{5}{16}(x+iy)^4 + \frac{1}{16}(x^2+y^2+z^2) + \frac{3}{8}(x+iy)^2 z),$$

$$\mathcal{S}_1 = (J_1 + iJ_2)^2 + 2iJ_1(p_x+ip_y) - J_2(p_x+ip_y) + \frac{1}{4}(p_y^2 - p_z^2) - iJ_3 p_z + f_1,$$
$$\mathcal{S}_2 = J_2 p_z - J_3 p_y + i(J_3 p_x - J_1 p_z) - \frac{i}{2}p_y p_z + f_2,\quad \mathcal{S}_3 = (p_x+ip_y)^2 + f_4,$$
$$\mathcal{S}_4 = J_3 p_z + iJ_1 p_y + iJ_2 p_x + 2J_1 p_x + \frac{i}{4}p_z^2 + f_3,\quad \mathcal{S}_5 = p_z(p_x+ip_y) + f_5.$$

Note that in the complete list of orthogonal separable coordinate systems for complex 3D Euclidean space there are some other systems besides the first 7 that have parameter dependence, e.g., cylindrical elliptic coordinates $L_1 = p_3^2$, $L_2 = J_3^2 + c^2 p_1^2$. However, for all of these other coordinates the corresponding Bertrand-Darboux equations have only rank 4, hence they do not uniquely determine a possible superintegrable system.

4.3 Interbasis expansions for Euclidean systems

To proceed with the classification of nondegenerate Euclidean superintegrable systems we need to look more closely at the relationship between a standard basis of symmetries for such a system and the "natural" basis written in terms of the linear and angular momentum generators $p_k, J_k,\ k = 1, \cdots, 3$.

Let us denote our preferred Cartesian coordinate system by $\mathbf{x} = (u, v, w)$ and let $\mathbf{x}_0 = (x, y, z)$, be a fixed regular point. We define the translated Cartesian coordinates (X, Y, Z) by $u = x + X, v = y + Y,\ w = z + Z$. Then, near the regular point (x, y, z) we have a basis of "natural symmetries" $p_1 = p_X, p_2 = p_Y, p_3 = p_Z, J_1 = Yp_Z - Zp_Y, J_2 = Zp_X - Xp_Z, J_3 = Xp_Y - Yp_X$. Now suppose we have a Euclidean superintegrable system with nondegenerate potential. Then there will exist 15 rational functions $A^{ij}(x, y, z)$, $B^{ij}(x, y, z)$, $C^{ij}(x, y, z)$, that completely characterize the superintegrable system. In particular, only 10 of these are linearly independent, see relations (52),

$$A^{22},\ A^{33},\ B^{22},\ B^{33},\ C^{33},\ A^{12},\ B^{12},\ A^{13},\ A^{23},\ B^{23}, \tag{23}$$

and they are subject to the five quadratic conditions (53) with $G \equiv 0$. These functions are related to the symmetries $\mathcal{S} = \sum a^{ij} p_i p_j + W$ via the conditions (51). Recall that the second order basis symmetries at the regular point $\mathcal{S}^{(\ell m)}_{\mathbf{x}_0}(\mathbf{x}) = \sum a^{ij}_{(\ell m)}(\mathbf{x}) p_i p_j + f_{(\ell m)}$ take the form $\mathcal{S}^{(\ell m)}_{\mathbf{x}_0}(\mathbf{x}_0) = p_i p_j + f_{(\ell m)}(\mathbf{x}_0)$ when evaluated at the point. Thus we can expand each standard basis symmetry in terms of the natural basis at the point via

$$\begin{aligned}
\mathcal{S}^{(\ell m)}_{\mathbf{x}_0} &= p_\ell p_m + \alpha_3^{(\ell m)} J_1^2 + \alpha_4^{(\ell m)} J_2^2 + \alpha_5^{(\ell m)} J_3^2 \\
&+ \alpha_6^{(\ell m)} p_1 J_1 + \alpha_7^{(\ell m)} p_2 J_2 + \alpha_8^{(\ell m)} p_1 J_2 + \alpha_9^{(\ell m)} p_1 J_3 + \alpha_{10}^{(\ell m)} p_2 J_1 \\
&+ \alpha_{11}^{(\ell m)} p_2 J_3 + \alpha_{12}^{(\ell m)} p_3 J_1 + \alpha_{13}^{(\ell m)} p_3 J_2 + \alpha_{14}^{(\ell m)} J_1 J_2 + \alpha_{15}^{(\ell m)} J_1 J_3 \\
&+ \alpha_{16}^{(\ell m)} J_2 J_3 + W^{(\ell m)}(\mathbf{x}),
\end{aligned} \tag{24}$$

where the $\alpha_k^{(\ell m)}$ are constants in X, Y, Z but rational functions of the parameters x, y, z of the regular point. (This notation for the expansion coefficients α_s is not completely logical, but since all of our software programs use the same notation we continue to use it to avoid (our) confusion.)

We conclude that all of the expansion constants $\alpha_k^{(\ell m)}$ can be expressed in terms of the 10 numbers (23). However, we shall not embark on this straightforward task but instead restrict ourselves to expanding the two symmetries

$$\begin{aligned}
\mathcal{S}^{(12)}_{\mathbf{x}_0} &= p_1 p_2 + \alpha_3 J_1^2 + \alpha_4 J_2^2 + \alpha_5 J_3^2 \\
&+ \alpha_6 p_1 J_1 + \alpha_7 p_2 J_2 + \alpha_8 p_1 J_2 + \alpha_9 p_1 J_3 + \alpha_{10} p_2 J_1 \\
&+ \alpha_{11} p_2 J_3 + \alpha_{12} p_3 J_1 + \alpha_{13} p_3 J_2 + \alpha_{14} J_1 J_2 + \alpha_{15} J_1 J_3 \\
&+ \alpha_{16} J_2 J_3 + W^{(12)}(\mathbf{x}),
\end{aligned} \tag{25}$$

$$
\begin{aligned}
\mathcal{S}^{(13)}_{\mathbf{x}_0} &= p_1p_3+\alpha'_3J_1^2+\alpha'_4J_2^2+\alpha'_5J_3^2 \\
&+ \alpha'_6p_1J_1+\alpha'_7p_2J_2+\alpha'_8p_1J_2+\alpha'_9p_1J_3+\alpha'_{10}p_2J_1 \\
&+ \alpha'_{11}p_2J_3+\alpha'_{12}p_3J_1+\alpha'_{13}p_3J_2+\alpha'_{14}J_1J_2+\alpha'_{15}J_1J_3 \\
&+ \alpha'_{16}J_2J_3+W^{(13)}(\mathbf{x}).
\end{aligned} \tag{26}
$$

(Here, $\alpha_s=\alpha_s^{(12)},\ \alpha'_s=\alpha_s^{(13)}$.) Indeed it is easy to verify that the 6 Bertrand-Darboux equations for these two symmetries have rank 5 (an illustration of Lemma 1 of [3].) Thus these two symmetries completely determine the $A^{ij},\ B^{ij},\ C^{ij}$, hence the superintegrable system.

If $a^{ij}(\mathbf{x})$ is the quadratic form associated with $\mathcal{S}^{(12)}(\mathbf{x})$ it is straightforward to verify that

$$
\begin{aligned}
a_1^{11}(\mathbf{x}_0) &= 0,\quad a_2^{11}(\mathbf{x}_0)=-\alpha_9,\quad a_3^{11}(\mathbf{x}_0)=\alpha_8, \\
a_1^{22}(\mathbf{x}_0) &= \alpha_{11},\quad a_2^{22}(\mathbf{x}_0)=0,\quad a_3^{22}(\mathbf{x}_0)=-\alpha_{10}, \\
a_1^{33}(\mathbf{x}_0) &= -\alpha_{13},\quad a_2^{33}(\mathbf{x}_0)=\alpha_{12},\quad a_3^{33}(\mathbf{x}_0)=0, \\
a_1^{12}(\mathbf{x}_0) &= \frac{1}{2}\alpha_9,\quad a_2^{12}(\mathbf{x}_0)=-\frac{1}{2}\alpha_{11},\quad a_3^{12}(\mathbf{x}_0)=\frac{1}{2}(\alpha_7-\alpha_6), \\
a_1^{13}(\mathbf{x}_0) &= -\frac{1}{2}\alpha_8,\quad a_2^{13}(\mathbf{x}_0)=\frac{1}{2}\alpha_6,\quad a_3^{13}(\mathbf{x}_0)=\frac{1}{2}\alpha_{13}, \\
a_1^{23}(\mathbf{x}_0) &= -\frac{1}{2}\alpha_7,\quad a_2^{23}(\mathbf{x}_0)=\frac{1}{2}\alpha_{10},\quad a_3^{23}(\mathbf{x}_0)=-\frac{1}{2}\alpha_{12},
\end{aligned} \tag{27}
$$

where $a_k^{ij}(\mathbf{x}_0)=\partial_k a^{ij}(\mathbf{x})|_{\mathbf{x}_0}$. There are identical relations for the other symmetries $\mathcal{S}^{(\ell m)}(\mathbf{x})$. Using the table (27) and the identities (52), (51) we can express the expansion coefficients $\alpha_6,\cdots,\alpha_{13}$ in terms of the 10 numbers (23) at $\mathbf{x}_0$:

$$
\begin{aligned}
\alpha_6 &= \frac{1}{3}(2A^{13}-B^{23}),\quad \alpha_7=\frac{1}{3}(A^{13}-2B^{23}),\quad \alpha_8=-\frac{1}{3}A^{23}, \\
\alpha_9 &= \frac{1}{3}A^{22},\quad \alpha_{10}=\frac{1}{3}A^{23},\quad \alpha_{11}=\frac{1}{3}B^{22}, \\
\alpha_{12} &= \frac{1}{3}(B^{12}-A^{22}+A^{33}),\quad \alpha_{13}=-\frac{1}{3}(B^{33}+A^{12}).
\end{aligned} \tag{28}
$$

The corresponding results for the expansion coefficients $\alpha'_6,\cdots,\alpha'_{13}$ of $\mathcal{S}^{(13)}$ are

$$
\begin{aligned}
\alpha'_6 &= -\frac{1}{3}(2A^{12}+B^{33}),\quad \alpha'_7=-\frac{1}{3}(A^{12}+2B^{23}),\quad \alpha'_8=-\frac{1}{3}A^{33}, \\
\alpha'_9 &= \frac{1}{3}A^{23},\quad \alpha'_{10}=-\frac{1}{3}B^{12},\quad \alpha'_{11}=\frac{1}{3}B^{23}, \\
\alpha'_{12} &= -\frac{1}{3}A^{23},\quad \alpha'_{13}=-\frac{1}{3}C^{33}.
\end{aligned} \tag{29}
$$

The expansion coefficients of the terms of the form $J_\ell J_m$, i.e., $\alpha_3, \alpha_4, \alpha_5, \alpha_{14}, \alpha_{15}, \alpha_{16}$ can be expressed in terms of second derivatives of the associated quadratic form, evaluated at the regular point $\mathbf{x}_0$. For example, $\alpha_{14} = 2a^{13}_{23}(\mathbf{x}_0) = -a^{12}_{33}(\mathbf{x}_0) = -a^{33}_{12}(\mathbf{x}_0)$. For a superintegrable system the integrability conditions for the symmetry relations (51) are satisfied identically, so these equations can be differentiated to compute the second derivatives $a^{ij}_{k\ell}(\mathbf{x}_0)$ as a quadratic expression in the 10 basic constants (subject to the 5 quadratic identities (53)). Though straightforward, these computations are tedious. The only relations that we will use here are those for the expansion coefficients $\alpha^{(\ell m)}_{14}$. We have

$$
\begin{aligned}
\alpha^{(11)}_{14} &= \frac{1}{9}\Big(4A^{23}(B^{33}-B^{22}) - 4B^{23}(A^{33}-A^{22}) - 2A^{13}B^{12} + 2A^{12}A^{23}\Big)\\
\alpha^{(22)}_{14} &= \frac{1}{9}\Big(4A^{12}A^{23} + 2B^{12}B^{23} + 2A^{23}B^{33} - 2A^{13}B^{12} - 2B^{23}A^{33}\\
&+ 2B^{23}A^{22} - 4B^{12}B^{23}\ \Big) \qquad (30)\\
\alpha^{(33)}_{14} &= \frac{1}{9}\Big(2B^{23}(A^{22}-A^{33}+B^{12}) - 4A^{13}B^{12} + 2A^{23}(A^{12}-2B^{22}+B^{33})\Big)\\
\alpha^{(12)}_{14} &= \frac{1}{9}\Big((2B^{23}-A^{13})(B^{33}-B^{22}) - 2(B^{33}+A^{12}-B^{22})B^{23}\\
&- A^{23}B^{12} + (2B23 + A^{13})A^{12} + 2B^{33}A^{13} - A^{23}A^{33}\ \Big)\\
\alpha^{(13)}_{14} &= \frac{1}{18}\Big(7(B^{33})^2 + (A^{33})^2 - 2A^{22}B^{12} - (A^{23})^2 + 4A^{12}B^{33}\\
&- A^{13}C^{33} - 3(A^{12})^2 - 5(B^{12})^2 - 4A^{33}B^{12} - 7B^{22}B^{33} - 7B^{23}C^{33}\\
&+ 2A^{13}B^{23} + 7(B^{23})^2 - A^{22}A^{33}\ \Big)\\
\alpha^{(23)}_{14} &= \frac{1}{9}\Big(A^{23}(-B^{23}+C^{33}) + (A^{23}-A^{22}+B^{12})(A^{12}+B^{22}-B^{33})\Big)
\end{aligned}
$$

Note that since the Hamiltonian is $\mathcal{S}^{(11)} + \mathcal{S}^{(22)} + \mathcal{S}^{(33)}$ and the coefficient of J_1J_2 in the Hamiltonian is 0, we must have $\alpha^{(11)}_{14} + \alpha^{(22)}_{14} + \alpha^{(33)}_{14} = 0$, which can be verified directly from the above expressions.

As a result of the previous discussion we have the result

Theorem 6 *For a nondegenerate superintegrable system the expansion coefficients $\alpha^{(\ell m)}_k$ expressing the standard basis $\mathcal{S}^{\ell m}$ in terms of the natural basis p_hp_k, p_hJ_k, J_hJ_k are explicit linear and quadratic expressions in the 10 terms (23).*

4.4 The significance of "generic" Euclidean systems

Suppose we have a nondegenerate Euclidean superintegrable system with potential V that is separable with respect to some orthogonal coordinates.

(Since every superintegrable system is multiseparable, we know that such coordinates exist.) By performing an Euclidean transformation, if necessary, we can assume that the separable coordinates are in a standard form determined by two constants of the motion in involution,

$$L_1 = \sum a^{ij} p_i p_j + f_1, \quad L_2 = \sum b^{ij} p_i p_j + f_2.$$

Clearly, L_1 and L_2 lie in the 6-dimensional space of second order symmetries for the superintegrable system. Thus, the quadratic form a^{ij}, for example, satisfies the three Bertrand-Darboux equations for potential V. Since V is nondegenerate we can express the second derivatives $V_{jj} - V_{kk}$ and V_{jk} with $j \neq k$ in the Bertrand-Darboux equations as linear combinations of the first derivatives V_h. Equating coefficients of V_1, V_2, V_3 separately in each of the three equations, we end up with nine linear conditions for the 10 constants $A^{22}, \cdots, B^{23}$ at each regular point. A typical example of one of these conditions is

$$\begin{aligned} A^{13}(3a^{11} - 3a^{33}) + B^{23}(0) + A^{23}(-3a^{12}) + A^{22}(0) + B^{22}(0) & + \\ B^{12}(0) + A^{33}(-3a^{13}) + B^{33}(0) + A^{12}(-3a^{23}) + C^{33}(0) & = -a^{13}_1 + a^{11}_3. \end{aligned}$$

Here, $B^{23}(0) = 0$, etc. For the second symmetry there will be nine more linear conditions with a^{ij} replaced by b^{ij}. Thus we will have 18 linear equations (not linearly independent) for the 10 quantities $A^{22}, \cdots, B^{23}$. Another source of conditions is obtained by writing the symmetry L_1 in terms of the standard basis:

$$a^{ij}(\mathbf{x}) = \sum_{\ell \leq m} a^{\ell m}(\mathbf{x}_0) \mathcal{A}^{ij}_{(\ell,m)}(\mathbf{x}),$$

where $\mathcal{A}^{ij}_{(\ell,m)}$ is the quadratic form associated with the standard basis symmetry $\mathcal{S}^{(\ell,m)}$ at $\mathbf{x}_0$. Expanding both sides of this equation in terms of the natural basis we obtain linear and quadratic conditions on the 10 basic quantities. For example if we equate coefficients of the natural basis element $J_1 J_2$ we find the quadratic conditions for L_1 and L_2:

$$2a^{13}_{23}(\mathbf{x}_0) = \sum_{\ell \leq m} a^{\ell m}(\mathbf{x}_0) \alpha^{(\ell m)}_{14} \quad 2b^{13}_{23}(\mathbf{x}_0) = \sum_{\ell \leq m} b^{\ell m}(\mathbf{x}_0) \alpha^{(\ell m)}_{14}. \tag{31}$$

Though there are many other quadratic conditions for L_1, L_2 to belong to the symmetry algebra, we shall use only these 2 and the 5 fundamental quadratic identities (53) that hold independent of any choice of L_1, L_2. Note that by equating coefficients of natural basis elements of the form $p_i J_k$ we could obtain linear identities. However, these are equivalent to the linear conditions for a^{ij}, b^{ij} already discussed above.

We give an example to show how this works. Suppose we have a nondegenerate superintegrable system that admits separation for some special choice of ellipsoidal coordinates [2111]. (Here we do *not* assume that the system separates for all values of the parameters c, e_1, e_2, e_3, but only for one value.) By performing an Euclidean transformation and a change of scale we can assume that the coordinates are in the standard form [2111] in our table and that $c = 1, e_1 = 0, e_2 = 1$ and $e_3 = a$ where a is any fixed complex number such that $a(a-1) \neq 0$. It follows that

$$a^{11} = y^2 + z^2 + a + 1, \quad a^{22} = x^2 + z^2 + a, \quad a^{33} = x^2 + y^2 + 1,$$

$$a^{12} = -xy, \quad a^{13} = -xz, \quad a^{23} = -yz, \quad b^{11} = ay^2 + z^2 + a,$$

$$b^{22} = ax^2, \quad b^{33} = x^2, \quad b^{12} = -axy, \quad b^{13} = -xz, \quad b^{23} = 0,$$

at any regular point with coordinates (x, y, z). Substituting these expressions into the 18 linear conditions discussed above, with the help of Maple, we find that there are exactly 7 independent linear conditions. Thus the 10 quantities $A^{22}, \cdots, B^{23}$ can be expressed linearly in terms of 3 of these quantities. Substituting this result into the 5 fundamental quadratic identities (53) we find that these identities yield a single *linear* relation for the remaining 3 unknowns. Finally we substitute our expressions in terms of the 3 unknowns and (30) into (50) and obtain (with the help of Maple) 2 more independent linear conditions. Thus we end up with 10 independent linear conditions for our 10 unknowns, and we obtain the unique solution

$$A^{12} = B^{12} = A^{23} = B^{23} = A^{13} = B^{33} = 0, \ A^{33} = A^{22} = \frac{3}{x}, \ C^{33} = -\frac{3}{z}, \ B^{22} = -\frac{3}{y},$$

which corresponds to the nondegenerate potential [I],

$$V = \frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{z^2} + \delta(x^2 + y^2 + z^2).$$

Note that it was obvious that our conditions would have solutions, since we already knew that system [I] separated simultaneously for all choices of the parameters c, e_1, e_2, e_3. What was far from obvious is the fact that *no other* nondegenerate superintegrable system separates for *any* special case of ellipsoidal coordinates.

Theorem 7 *A 3D Euclidean nondegenerate superintegrable system admits separation in a special case of the generic coordinates [2111], [221], [23], [311], [32], [41] or [5], respectively, if and only if it is equivalent via a Euclidean transformation to system [I], [II], [III], [IV], [V], [VI] or [VII], respectively.*

The proof (complicated but straightforward) proceeds exactly as the case [2111] described above. For each case [221]-[5] we use the symmetries a^{ij}, b^{ij} from the table. The 18 linear conditions discussed above reduce to exactly 7 independent linear conditions. Thus always the 10 quantities $A^{22}, \cdots, B^{23}$ can be expressed linearly in terms of 3 of these quantities. Substituting into the 5 fundamental quadratic identities (53) we find that these identities yield a single *linear* relation for the remaining 3 unknowns. Substituting our expressions in terms of the 3 unknowns and (30) into (50) we obtain 2 more independent linear conditions. Thus we end up with 10 independent linear conditions for our 10 unknowns, and a unique solution, the corresponding generic superintegrable system.

This does not settle the problem of classifying all 3D nondegenerate superintegrable systems in complex Euclidean space, for we have not excluded the possibility of such systems that separate only in degenerate separable coordinates. In fact we have already studied two such systems in [3]:

$$[O] \quad V(x,y,z) = \alpha x + \beta y + \gamma z + \delta(x^2+y^2+z^2).$$

$$[OO] \quad V(x,y,z) = \frac{\alpha}{2}(x^2+y^2+\frac{1}{4}z^2) + \beta x + \gamma y + \frac{\delta}{z^2}. \tag{32}$$

However, both of these nondegenerate syperintegrable systems are only weakly functionally independent, in contrast to systems [I]-[VII]. Thus we consider [O] and [OO] as associate members of the superintegrable family, not regular members. An investigation of other possible Euclidean systems is in progress.

4.5 "Generic" superintegrable systems on the 3-sphere

An important task remaining is to classify the possible systems on the 3-sphere (particularly those 3-sphere systems not Stäckel equivalent to a flat space system). We choose a standardized Cartesian-like coordinate system $\{x, y, z\}$ on the 3-sphere such that the metric and Hamiltonian are

$$ds^2 = \frac{1}{(1+\frac{r^2}{4})^2}(dx^2+dy^2+dz^2), \quad \mathcal{H} = (1+\frac{r^2}{4})^2(p_x^2+p_y^2+p_z^2)+V, \tag{33}$$

where $r^2 = x^2+y^2+z^2$. These coordinates can be related to the standard realization of the sphere via complex coordinates $\mathbf{s} = (s_1, s_2, s_3, s_4)$ such that $\sum_{j=1}^4 s_j^2 = 1$ and $ds^2 = \sum_j ds_j^2$ via

$$s_1 = \frac{4x}{4+r^2}, \quad s_2 = \frac{4y}{4+r^2}, \quad s_3 = \frac{4z}{4+r^2}, \quad s_4 = \frac{4-r^2}{4+r^2} \tag{34}$$

with inverse $x = 2s_1/(1+s_4)$,$y = 2s_2/(1+s_4)$, $z = 2s_3/(1+s_4)$. Here, x, y, z are local coordinates in a neighborhood of the pole $\mathbf{P} = (0, 0, 0, 1)$ on the 3-sphere. A basis of Killing vectors for the zero potential system is J_h, K_h, $h = 1, 2, 3$ where

$$J_1 = yp_z - zp_y, \quad J_2 = zp_x - xp_z, \quad J_3 = xp_y - yp_x, \tag{35}$$

$$K_1 = (1+\frac{x^2 - y^2 - z^2}{4})p_x + \frac{xy}{2}p_y + \frac{xz}{2}p_z,\ K_2 = (1+\frac{y^2 - x^2 - z^2}{4})p_y + \frac{xy}{2}p_x + \frac{yz}{2}p_z,$$

$$K_3 = (1 + \frac{z^2 - x^2 - y^2}{4})p_z + \frac{xz}{2}p_x + \frac{yz}{2}p_y.$$

The commutation relations are

$$\{J_1, J_2\} = J_3, \quad \{K_1, K_2\} = J_3, \quad \{K_1, J_2\} = K_3 \tag{36}$$

and their cyclic permutations. The relation between this basis and the standard basis of rotation generators on the sphere $I_{\ell m} = s_\ell p_m - s_m p_\ell = -I_{m\ell}$ is

$$J_1 = I_{23},\ J_2 = I_{31},\ J_3 = I_{12},\ K_1 = I_{41},\ K_2 = I_{42},\ K_3 = I_{43}. \tag{37}$$

We shall use the x, y, z coordinates as standard but we also need to see how these coordinates relate to analogous Cartesian-like coordinates centered at any point $\mathbf{T}$ on the sphere. We can always find a complex orthogonal matrix O, not unique, such that $\mathbf{T} = O\mathbf{P}$. If X, Y, Z, (34), define local Cartesian-like coordinates near $\mathbf{P}$ then via $\mathbf{t} = O\mathbf{s}(X, Y, Z)$ they also define local coordinates in a neighborhood of $\mathbf{T} = (T_1, T_2, T_3, T_4)$. Moreover, since O is orthogonal we have

$$ds^2 = d\mathbf{t}\cdot d\mathbf{t} = d\,O\mathbf{s}\cdot d\,O\mathbf{s} = d\mathbf{s}\cdot d\mathbf{s} = \frac{1}{(1+\frac{R^2}{4})^2}(dX^2 + dY^2 + dZ^2),$$

so we can consider X, Y, Z as Cartesian-like coordinates in a neighborhood of $\mathbf{T}$. We can also require that the coordinate axes line up so that differentiation of $\mathbf{s}$ by X, Y, Z, respectively, at $\mathbf{P}$ corresponds to (normalized) differentiation of $\mathbf{t}$ by x, y, z, respectively, at $\mathbf{T}$, i.e., so that p_X corresponds to $(1+r^2/4)p_x$, etc. Thus,

$$(\frac{2}{1+t_4}\partial_x\mathbf{t})\Big|_{\mathbf{t}=\mathbf{T}} = O\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad (\frac{2}{1+t_4}\partial_y\mathbf{t})\Big|_{\mathbf{t}=T} = O\begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

$$\left(\frac{2}{1+t_4}\partial_z \mathbf{t}\right)\Big|_{\mathbf{t}=T} = O\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \mathbf{T} = O\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

This determines O uniquely, since the column vectors on the left hand sides of these expressions are mutually orthogonal unit vectors. We find

$$O_T = \begin{pmatrix} -\frac{T_1^2-T_4-1}{T_4+1}, & -\frac{T_2T_1}{T_4+1}, & -\frac{T_3T_1}{T_4+1}, & T_1 \\ -\frac{T_1T_2}{T_4+1}, & -\frac{T_2^2-T_4-1}{T_4+1}, & -\frac{T_3T_2}{T_4+1}, & T_2 \\ -\frac{T_1T_3}{T_4+1}, & -\frac{T_2T_3}{T_4+1}, & -\frac{T_3^2-T_4-1}{T_4+1}, & T_3 \\ -T_1, & -T_2, & -T_3, & T_4 \end{pmatrix}. \tag{38}$$

In the $\mathbf{P}$-based coordinate system the coordinates of $\mathbf{t}$ are u, v, w where $u = 2t_1/(1+t_4)$,$v = 2t_2/(1+t_4)$, $w = 2t_3/(1+t_4)$. From the equation $\mathbf{t} = O_T\mathbf{s}$ we can solve for u, v, w to obtain

$$\begin{aligned} u &= \frac{4[r^2X - 2x(xX+yY+zZ) + 4(x+X) - xR^2]}{16 - 8(xX+yY+zZ) + r^2R^2} \qquad (39)\\ v &= \frac{4[r^2Y - 2y(xX+yY+zZ) + 4(y+Y) - yR^2]}{16 - 8(xX+yY+zZ) + r^2R^2} \\ w &= \frac{4[r^2Z - 2z(xX+yY+zZ) + 4(z+Z) - zR^2]}{16 - 8(xX+yY+zZ) + r^2R^2}. \end{aligned}$$

To recapitulate: $\mathbf{t}$ is a point on the complex unit sphere, (x, y, z) are the coordinates of $\mathbf{T}$ in the $\mathbf{P}$-based system, (u, v, w) are the coordinates of $\mathbf{t}$ in the $\mathbf{P}$-based system, and (X, Y, Z) are the coordinates of $\mathbf{t}$ in the $\mathbf{T}$-based system. Thus, for fixed $\mathbf{T}$, equations (39) define the coordinate transformation between (u, v, w) and (X, Y, Z). We can write equations (39) in a simpler form by introducing the supplementary variables

$$U = \frac{u-x}{1+\frac{r^2}{4}}, \quad V = \frac{v-y}{1+\frac{r^2}{4}}, \quad W = \frac{w-z}{1+\frac{r^2}{4}}, \qquad Q^2 = U^2+V^2+W^2.$$

Then

$$U = \frac{1-\frac{1}{2}xX}{1-\frac{1}{2}(xX+yY+zZ)+\frac{r^2R^2}{16}}, \tag{40}$$

$$V = \frac{1-\frac{1}{2}yY}{1-\frac{1}{2}(xX+yY+zZ)+\frac{r^2R^2}{16}}, \; W = \frac{1-\frac{1}{2}zZ}{1-\frac{1}{2}(xX+yY+zZ)+\frac{r^2R^2}{16}},$$

with inverse

$$X = \frac{U+\frac{x}{4}Q^2}{1+\frac{1}{2}(xU+yV+zW)+\frac{r^2Q^2}{16}}, \tag{41}$$

$$Y=\frac{V+\frac{y}{4}Q^2}{1+\frac{1}{2}(xU+yV+zW)+\frac{r^2Q^2}{16}},\ Z=\frac{W+\frac{z}{4}Q^2}{1+\frac{1}{2}(xU+yV+zW)+\frac{r^2Q^2}{16}}.$$

In reference [22] we have determined all orthogonal separable coordinate systems on the complex unit 3-sphere. Of the 21 systems listed those that are "generic", in the sense we used for Euclidean separable systems, are given as follows with coordinates followed by defining constants of the motion. (Here we take the Hamiltonian as $\mathcal{L}_0=I_{12}^2+I_{13}^2+I_{14}^2+I_{23}^2+I_{24}^2+I_{34}^2$, and we recall the identity $I_{23}I_{41}+I_{31}I_{42}+I_{12}I_{43}=0$.)

[**1111**] (system (17) in [22])

$$s_1^2=\frac{(x_1-e_1)(x_2-e_1)(x_3-e_1)}{(e_1-e_2)(e_1-e_3)(e_1-e_4)},\quad s_2^2=\frac{(x_1-e_2)(x_2-e_2)(x_3-e_2)}{(e_2-e_1)(e_2-e_3)(e_2-e_4)},$$

$$s_3^2=\frac{(x_1-e_3)(x_2-e_3)(x_3-e_3)}{(e_3-e_1)(e_3-e_2)(e_3-e_4)},\quad s_4^2=\frac{(x_1-e_4)(x_2-e_4)(x_3-e_4)}{(e_4-e_2)(e_4-e_3)(e_4-e_1)}.$$

$$\mathcal{L}_1=(e_1+e_2)I_{12}^2+(e_1+e_3)I_{13}^2+(e_1+e_4)I_{14}^2+(e_2+e_3)I_{23}^2+(e_2+e_4)I_{24}^2+(e_3+e_4)I_{34}^2,$$

$$\mathcal{L}_2=e_1e_2I_{12}^2+e_1e_3I_{13}^2+e_1e_4I_{14}^2+e_2e_3I_{23}^2+e_2e_4I_{24}^2+e_3e_4I_{34}^2.$$

[211] (system (18) in [22])

$$(is_1+s_2)^2=-2\frac{(x_1-e_1)(x_2-e_1)(x_3-e_1)}{(e_1-e_3)(e_1-e_4)},$$

$$s_1^2+s_2^2=-\partial_{e_1}\left(\frac{(x_1-e_1)(x_2-e_1)(x_3-e_1)}{(e_1-e_3)(e_1-e_4)}\right),$$

$$s_3^2=-\frac{(x_1-e_3)(x_2-e_3)(x_3-e_3)}{(e_3-e_1)^2(e_3-e_4)},\quad s_4^2=-\frac{(x_1-e_4)(x_2-e_4)(x_3-e_4)}{(e_4-e_1)^2(e_4-e_3)}.$$

$$\mathcal{L}_1=(I_{14}+iI_{24})^2+(I_{13}+iI_{23})^2+2e_1\left(2I_{12}^2+I_{14}^2+I_{24}^2+I_{13}^2+I_{23}^2\right)$$
$$+2e_3\left(I_{34}^2+I_{13}^2+I_{23}^2\right)+2e_4\left(I_{14}^2+I_{24}^2+I_{34}^2\right),$$

$$\mathcal{L}_2=e_1^2I_{12}^2+e_1e_3\left(I_{13}^2+I_{23}^2\right)+e_1e_4\left(I_{14}^2+I_{24}^2\right)+e_3e_4I_{34}^2$$
$$+\frac{e_3}{2}\left(I_{13}+iI_{23}\right)^2+\frac{e_4}{2}\left(I_{14}+iI_{24}\right)^2.$$

[22] (system (19) in [22])

$$(s_1+is_2)^2=-2\frac{(x_1-e_1)(x_2-e_1)(x_3-e_1)}{(e_1-e_3)^2},\ (s_3+is_4)^2=$$

$$-2\frac{(x_1-e_3)(x_2-e_3)(x_3-e_3)}{(e_1-e_3)^2},\ s_1^2+s_2^2=-\frac{\partial}{\partial e_1}\left(\frac{(x_1-e_1)(x_2-e_1)(x_3-e_1)}{(e_1-e_3)^2}\right),$$

$$\mathcal{L}_1=-I_{24}^2+I_{13}^2+iI_{13}I_{23}+iI_{14}I_{24}+iI_{23}I_{24}+iI_{13}I_{14}$$
$$+(e_1-e_3)\left[I_{12}^2-I_{34}^2\right]+e_2,$$
$$\mathcal{L}_2=e_1^2I_{12}^2+e_3^2I_{34}^2+e_1e_3\left[I_{13}^2+I_{24}^2+I_{14}^2+I_{23}^2\right]$$
$$+\frac{1}{4}\left[I_{13}^2+I_{24}^2-I_{14}^2-I_{23}^2+2iI_{13}I_{23}+2iI_{13}I_{14}-2iI_{14}I_{24}-2iI_{23}I_{24}\right.$$
$$\left.-4I_{13}I_{24}-2I_{12}I_{34}\right]+\frac{e_1}{2}\left[I_{13}^2-I_{24}^2-I_{14}^2+I_{23}^2+2iI_{13}I_{14}+2iI_{23}I_{24}\right]$$
$$+\frac{e_3}{2}\left[-I_{24}^2+I_{13}^2+I_{14}^2-I_{23}^2+2iI_{13}I_{23}+2iI_{14}I_{24}\right].$$

[**31**] (system (20) in [22])

$$(s_1+is_2)^2=-2\frac{(x_1-e_1)(x_2-e_1)(x_3-e_1)}{(e_1-e_4)},\ s_4^2=-\frac{(x_1-e_4)(x_2-e_4)(x_3-e_4)}{(e_4-e_1)^3},$$

$$\sqrt{2}s_3(s_1+is_2)=-\frac{\partial}{\partial e_1}\left(\frac{(x_1-e_1)(x_2-e_1)(x_3-e_1)}{(e_1-e_4)}\right),$$
$$s_1^2+s_2^2+s_3^2=-\frac{1}{2}\frac{\partial^2}{\partial e_1^2}\frac{(x_1-e_1)(x_2-e_1)(x_3-e_1)}{(e_1-e_4)}.$$
$$\mathcal{L}_1=\sqrt{2}\left(I_{14}I_{34}-I_{12}I_{23}+iI_{24}I_{34}+iI_{12}I_{13}\right)$$
$$+e_1\left(I_{12}^2+I_{13}^2+I_{23}^2\right)+e_4\left(I_{34}^2+I_{14}^2+I_{24}^2\right).$$
$$\mathcal{L}_2=-\frac{1}{2}I_{13}^2+\frac{1}{2}I_{23}^2-iI_{13}I_{23}+e_1e_4\left(I_{34}^2+I_{14}^2+I_{24}^2\right)$$
$$=e_1^2\left(I_{12}^2+I_{13}^2+I_{23}^2\right)-\frac{e_1}{\sqrt{2}}\left(-2iI_{12}I_{13}+2I_{12}I_{23}\right)$$
$$+\sqrt{2}e_4\left(I_{14}I_{34}+iI_{24}I_{34}\right).$$

[**4**] (system (21) in [22])

$$(s_1+is_2)^2=-2(x_1-e_1)(x_2-e_2)(x_3-e_3),$$

$$(s_1+is_2)(s_3+is_4)=-\frac{\partial}{\partial e_1}\left((x_1-e_1)(x_2-e_1)(x_3-e_1)\right),$$

$$2(s_1+is_2)(s_3-is_4)+(s_3+is_4)^2=-\frac{\partial^2}{\partial e_1^2}\left((x_1-e_1)(x_2-e_1)(x_3-e_1)\right).$$

$$\mathcal{L}_1 = \frac{1}{2}\Big(2I_{34}I_{14} + 2I_{12}I_{14} - 2I_{23}I_{34} - 2I_{12}I_{23} - I_{24}^2 + I_{23}^2 - I_{14}^2$$
$$+I_{13}^2 + 2iI_{13}I_{14} + 2iI_{23}I_{24} + 2iI_{12}I_{24} + 2iI_{13}I_{34} + 2iI_{24}I_{34} + 2iI_{12}I_{13}\Big),$$
$$\mathcal{L}_2 = -I_{14}^2 - I_{23}I_{34} - \frac{1}{4}I_{24}^2 + \frac{1}{4}I_{23}^2 + \frac{1}{4}I_{14}^2 - \frac{1}{4}I_{13}^2 - \frac{i}{2}I_{13}I_{23} - \frac{i}{2}I_{23}I_{24}$$
$$\frac{i}{2}I_{14}I_{24} + \frac{i}{2}I_{13}I_{14} - iI_{24}I_{34} + iI_{13}I_{34} - \frac{1}{2}I_{13}I_{24} - \frac{1}{2}I_{14}I_{23}.$$

We now show that each generic separable system on the 3-sphere uniquely determines a superintegrable system with nondegenerate potential. The proof is, in most part, analogous to that for the Euclidean case. Consider system (1111). If we have a superintegrable system that admits the symmetries $\mathcal{L}_1, \mathcal{L}_2$ for all values of the parameters $e_1, \cdots, e_4$ then it must have the basis of symmetries

$$\text{VIII} \qquad \mathcal{S}_0 = I_{12}^2 + f_0, \quad \mathcal{S}_1 = I_{13}^2 + f_1, \quad \mathcal{S}_2 = I_{14}^2 + f_2, \quad \mathcal{S}_3 = I_{32}^2 + f_3,$$
$$\mathcal{S}_4 = I_{24}^2 + f_4, \quad \mathcal{S}_5 = I_{34}^2 + f_5.$$

The system of Bertrand-Darboux equations associated with these symmetries has rank 5 so the potential is uniquely determined. Solving the Bertrand-Darboux equations we obtain the nondegenerate potential on the 3-sphere

$$V(\mathbf{s}) = \frac{\alpha}{s_1^2} + \frac{\beta}{s_2^2} + \frac{\gamma}{s_3^2} + \frac{\delta}{s_4^2}. \tag{42}$$

This potential is not Stäckel equivalent to a potential on Euclidean space.

Three of the four remaining systems can be obtained in the same way. However there is an alternative approach which enables us to obtain the systems 2,3, and 4 from 1 via well defined limiting processes. These are discussed elsewhere, e.g. [16, 17], but we content ourselves with an example of how to obtain [211] from [1111]. If we make the transformations

$$s_1 \to \frac{1}{\sqrt{-\epsilon}}y_1, \ s_2 \to \frac{1}{\sqrt{\epsilon}}(y_1 + \epsilon y_2), \ \alpha \to -\sqrt{\beta}2\epsilon^2, \ \beta \to \frac{\alpha}{\epsilon} - \frac{\beta}{2\epsilon^2}$$

then we deduce the relations

$$y_1^2 = -\frac{(u-e_1)(v-e_1)(w-e_1)}{(e_1-e_2)(e_1-e_3)}, \quad 2y_1y_2 = \frac{\partial}{\partial e_1}y_1^2.$$

The coordinates on the sphere can be represented using the identifications $y_1 = (s_1 + is_2)/\sqrt{2}$,$y_2 = (s_1 - is_2)/\sqrt{2}$, $y_3 = s_3$, $y_4 = s_4$ where $2y_1y_2 + y_3^2 +$

$y_4^2 = s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$. We then transform the potential according to $\alpha/s_1^2 + \beta/s_2^2 + \gamma/s_3^2 + \delta/s_4^2 \to \alpha/y_1^2 + \beta y_2/y_1^3 + \gamma/y_3^2 + \delta/y_4^2$.

An exactly similar approach leads to the coordinates, constants of the motion and nondegenerate potential for the system [22]. Here the limit is taken in the form $e_2 = e_1 + \epsilon$,$e_4 = e_3 + \epsilon'$ where $\epsilon, \epsilon' \to 0$. For the system [31] we set $e_2 = e_1 + \epsilon$, $e_3 = e_1 + \epsilon'$ and allow $\epsilon, \epsilon' \to 0$, whereas for system [4] we set $e_2 = e_1 + \epsilon$, $e_3 = e_1 + \epsilon_1, e_4 = e_1 + \epsilon_2$ and allow $\epsilon, \epsilon_1, \epsilon_2 \to 0$. In all cases except [4] the requirement that we have separation for all values of the parameters e_j yields a set of 6 linearly independent second-order constants of the motion that can be verified to correspond to a nondegenerate superintegrable system. In the case [4] the constants of the motion don't depend on e_1 and we have only 3 independent symmetries. However, there is a unique potential that is obtained as the limit of the nondegenerate potential for case [1111]. By writing down the Bertrand-Darboux equations for this limit potential we can directly verify that it admits 6 linearly independent symmetries and is nondegenerate.

Theorem 8 *Each of the* 5 *"generic" 3-sphere separable systems determines a unique nondegenerate superintegrable system that permits separation simultaneously for all values of the parameters e_j. For each of these systems there is a basis of* 5 *(strongly) functionally independent and* 6 *linearly independent second order symmetries. In addition to system [VIII] above there are the following superintegrable systems (nondegenerate potential, followed by a basis of constants of the motion):*

I' [211] (Stäckel equivalent to the Euclidean superintegrable system [I])

$$V = \frac{\alpha_1}{(s_1 + is_2)^2} + \frac{\alpha_2(s_1 - is_2)}{(s_1 + is_2)^3} + \frac{\alpha_3}{s_3^2} + \frac{\alpha_4}{s_4^2}. \tag{43}$$

$$\mathcal{S}_0 = I_{12}^2 + f_0, \quad \mathcal{S}_1 = I_{34}^2 + f_1, \quad \mathcal{S}_2 = I_{13}^2 + I_{23}^2 + f_2,$$

$$\mathcal{S}_3 = I_{14}^2 + I_{24}^2 + f_3, \quad \mathcal{S}_4 = I_{13}(I_{13} + iI_{23}) + f_4, \quad \mathcal{S}_5 = I_{14}(I_{14} + iI_{24}) + f_5.$$

II' [22] (Stäckel equivalent to the Euclidean superintegrable system [II])

$$V = \frac{\alpha_1}{(s_1 + is_2)^2} + \frac{\alpha_2(s_1 - is_2)}{(s_1 + is_2)^3} + \frac{\alpha_3}{(s_3 + is_4)^2} + \frac{\alpha_4(s_3 - is_4)}{(s_3 + is_4)^3}. \tag{44}$$

$$\mathcal{S}_0 = I_{12}^2 + f_0, \quad \mathcal{S}_1 = I_{34}^2 + f_1, \quad \mathcal{S}_2 = I_{13}^2 + I_{14}^2 + I_{23}^2 + I_{24}^2 + f_2,$$

$$\mathcal{S}_3 = I_{13}^2 + I_{14}^2 + i(I_{13}I_{23} + I_{14}I_{24}) + f_3, \quad \mathcal{S}_4 = I_{13}^2 + I_{23}^2 + i(I_{13}I_{14} + I_{23}I_{24}) + f_4,$$

$$\mathcal{S}_5 = I_{13}^2 + I_{24}^2 + i(I_{13}I_{14} + I_{13}I_{23} - I_{14}I_{24} - I_{23}I_{24}) - 2I_{13}I_{24} - I_{12}I_{34} + f_5.$$

IV' [31] (Stäckel equivalent to the Euclidean superintegrable system [IV])

$$V = \frac{\alpha_1}{(s_1+is_2)^2} + \frac{\alpha_2 s_3}{(s_1+is_2)^3} + \frac{\alpha_3(s_1^2+s_2^2-3s_3^2)}{(s_1+is_2)^4} + \frac{\alpha_4}{s_4^2}. \quad (45)$$

$$\mathcal{S}_0 = I_{12}^2 + I_{13}^2 + I_{23}^2 + f_0, \quad \mathcal{S}_1 = I_{14}^2 + I_{24}^2 + I_{34}^2 + f_1, \quad \mathcal{S}_2 = (I_{23} - iI_{13})^2 + f_2,$$
$$\mathcal{S}_3 = I_{12}(I_{23} - iI_{13}) + f_3, \quad \mathcal{S}_4 = I_{34}(I_{14} + iI_{24}) + f_4,$$
$$\mathcal{S}_5 = I_{14}I_{34} - I_{12}I_{23} + i(I_{24}I_{34} + I_{12}I_{13}) + f_5.$$

VI' [4] (Stäckel equivalent to the Euclidean superintegrable system [VI])

$$V = \frac{\alpha_1}{(s_1+is_2)^2} + \frac{\alpha_2(s_3+is_4)}{(s_1+is_2)^3} + \frac{\alpha_3[(s_1+is_2)(s_3-is_4) - \frac{3}{2}(s_3+is_4)^2]}{(s_1+is_2)^4}$$
$$+ \frac{\alpha_4[(s_1+is_2)(s_1^2+s_2^2-\frac{3}{2}(s_3^2+s_4^2)) + (s_3+is_4)^3]}{(s_1+is_2)^5}. \quad (46)$$

$$\mathcal{S}_0 = I_{12}^2 + I_{13}^2 + I_{14}^2 + I_{23}^2 + I_{24}^2 + I_{34}^2 + V,$$
$$\mathcal{S}_1 = (I_{13} - I_{24} + iI_{23} + iI_{14})^2 + f_1,$$
$$\mathcal{S}_2 = 4(I_{23}I_{34} + I_{14}I_{34} + I_{13}I_{24}) + 4i(I_{24}I_{34} - I_{13}I_{34})$$
$$+2i(I_{13}I_{23} - I_{14}I_{24} - I_{13}I_{14} + I_{23}I_{24}) - 2I_{12}I_{34} + I_{13}^2 + I_{24}^2 - I_{14}^2 - I_{23}^2 + f_2,$$
$$\mathcal{S}_3 = 2(I_{12}I_{23} + I_{23}I_{34} - I_{12}I_{14} - I_{14}I_{34})$$
$$-2i(I_{23}I_{24} + I_{13}I_{14} + I_{13}I_{34} + I_{24}I_{34} + I_{12}I_{24} + I_{12}I_{13})$$
$$-I_{13}^2 + I_{24}^2 + I_{14}^2 - I_{23}^2 + f_3,$$
$$\mathcal{S}_4 = (I_{13} - I_{24} + iI_{23} + iI_{14})(I_{13} + I_{24} + iI_{23} - iI_{14}) + f_4.$$
$$\mathcal{S}_5 = (I_{13} - I_{24} + iI_{23} + iI_{14})(I_{34} - I_{12}) + f_5.$$

We also mention that the nongeneric superintegrable system on the 3-sphere with potential

$$\mathbf{00'} \qquad V = \frac{\alpha}{(s_1+is_2)^2} + \frac{\beta s_3}{(s_1+is_2)^3} + \frac{\gamma s_4}{(s_1+is_2)^3} + \frac{\delta(1-4s_3^2-4s_4^2)}{(s_1+is_2)^4}$$

is Stäckel equivalent to the Euclidean superintegrable system [00].

4.6 Interbasis expansions for 3-sphere systems

In analogy with our treatment of Euclidean systems, to proceed with the classification of nondegenerate superintegrable systems on the 3-sphere we need to look more closely at the relationship between a standard basis of symmetries and the "natural" basis written in terms of the angular momentum generators J_ℓ, K_ℓ $\ell = 1, \cdots, 3$. Then, near the regular point $\mathbf{T}$, i.e., (x, y, z), we have a basis of "natural symmetries" $J_1 = Yp_Z - Zp_Y$, $J_2 = Zp_X - Xp_Z$, $J_3 = Xp_Y - Yp_X$, $K_1 = K_X$,$K_2 = K_Y$,$K_3 = K_Z$. At the point itself we have $(1+r^2/4)p_u = p_X$, $(1+r^2/4)p_v = p_Y$, $(1+r^2/4)p_w = p_Z$. Now suppose we have a 3-sphere superintegrable system with nondegenerate potential. Then there will exist 15 rational functions $A^{ij}[x, y, z]$, $B^{ij}[x, y, z]$, $C^{ij}[x, y, z]$, (with respect to the (X, Y, Z) coordinates and restricted to the point $(X, Y, Z) = (0, 0, 0)$, that completely characterize the superintegrable system. In particular, only 10 of these, (23), are linearly independent, see relations (52), and they are subject to the five quadratic conditions (53) with $G(X, Y, Z) = \ln\lambda = -2\ln(1 + R^2/4)$. These functions are related to the symmetries $\mathcal{S} = \sum a^{ij}p_ip_j + W$ via the conditions (51). The second order basis symmetries at the regular point $\mathcal{S}^{(\ell m)}_{\mathbf{x}_0}(\mathbf{X}) = \sum a^{ij}_{(\ell m)}(\mathbf{X})p_ip_j + f_{(\ell m)}(\mathbf{X})$ take the form $\mathcal{S}^{(\ell m)}_{\mathbf{x}_0}(0, 0, 0) = p_ip_j + f_{(\ell m)}(0, 0, 0)$ when evaluated at the point. Thus we can expand each standard basis symmetry in a neighborhood of the point (x, y, z) in terms of the natural basis at the point via

$$\begin{aligned}
\mathcal{S}^{(\ell m)}_{\mathbf{x}_0} &= K_\ell K_m + \alpha^{(\ell m)}_3 J_1^2 + \alpha^{(\ell m)}_4 J_2^2 + \alpha^{(\ell m)}_5 J_3^2 \\
&+ \alpha^{(\ell m)}_6 K_1J_1 + \alpha^{(\ell m)}_7 K_2J_2 + \alpha^{(\ell m)}_8 K_1J_2 + \alpha^{(\ell m)}_9 K_1J_3 + \alpha^{(\ell m)}_{10} K_2J_1 \\
&+ \alpha^{(\ell m)}_{11} K_2J_3 + \alpha^{(\ell m)}_{12} K_3J_1 + \alpha^{(\ell m)}_{13} K_3J_2 + \alpha^{(\ell m)}_{14} J_1J_2 + \alpha^{(\ell m)}_{15} J_1J_3 \\
&+ \alpha^{(\ell m)}_{16} J_2J_3 + W^{(\ell m)}(\mathbf{X}), \qquad (47)
\end{aligned}$$

where the $\alpha^{(\ell m)}_k$ are constants in X, Y, Z but rational functions of the parameters x, y, z of the regular point. Here we are taking into account the identity $\sum_{h=1}^3 K_hJ_h = 0$ and the fact that $K_h = p_h$ at the point $(X, Y, Z) = (0, 0, 0)$. Again, nondegenerate superintegrable system is uniquely determined by the 10 numbers (23), and these numbers themselves are subject to 5 quadratic identities (53). (Note that G and all of its first and second derivatives vanish when $X = Y = Z = 0$, except that $G_{ii} = -1$, $i = 1, 2, 3$. Furthermore, we can use relations (40) to express the derivatives of V at the regular point with respect to the (X, Y, Z) coordinates in terms of derivatives with respect to (u, v, w). Thus the numbers (23) can be expressed as linear combinations of the corresponding numbers with respect to the (u, v, w) coordinates.)

Although all of the expansion constants $\alpha^{(\ell m)}_k$ can be expressed in terms of these 10 numbers, we shall restrict ourselves to expanding the two symmetries

$$\begin{aligned}
\mathcal{S}^{(12)}_{\mathbf{x}_0} &= K_1K_2 + \alpha_3 J_1^2 + \alpha_4 J_2^2 + \alpha_5 J_3^2 \\
&+ \alpha_6 K_1J_1 + \alpha_7 K_2J_2 + \alpha_8 K_1J_2 + \alpha_9 K_1J_3 + \alpha_{10} K_2J_1 \\
&+ \alpha_{11} K_2J_3 + \alpha_{12} K_3J_1 + \alpha_{13} K_3J_2 + \alpha_{14} J_1J_2 + \alpha_{15} J_1J_3 \\
&+ \alpha_{16} J_2J_3 + W^{(12)}(\mathbf{X}),
\end{aligned} \tag{48}$$

$$\begin{aligned}
\mathcal{S}^{(13)}_{\mathbf{x}_0} &= K_1K_3 + \alpha'_3 J_1^2 + \alpha'_4 J_2^2 + \alpha'_5 J_3^2 \\
&+ \alpha'_6 K_1J_1 + \alpha'_7 K_2J_2 + \alpha'_8 K_1J_2 + \alpha'_9 K_1J_3 + \alpha'_{10} K_2J_1 \\
&+ \alpha'_{11} K_2J_3 + \alpha'_{12} K_3J_1 + \alpha'_{13} K_3J_2 + \alpha'_{14} J_1J_2 + \alpha'_{15} J_1J_3 \\
&+ \alpha'_{16} J_2J_3 + W^{(13)}(\mathbf{X}).
\end{aligned} \tag{49}$$

(Here, $\alpha_s = \alpha_s^{(12)},\ \alpha'_s = \alpha_s^{(13)}$.) Since the 6 Bertrand-Darboux equations for these two symmetries have rank 5, the symmetries completely determine the $A^{ij},\ B^{ij},\ C^{ij}$, hence the superintegrable system.

From (39) - (41) we have (with $J_w = up_v - vp_u$ and cyclic permutations)

$$\begin{aligned}
J_1 &= \frac{1}{1+\frac{r^2}{4}}\left((1+\frac{x^2}{4}-\frac{y^2}{4}-\frac{z^2}{4})J_u + \frac{zx}{2}J_w + \frac{yx}{2}J_v - yK_w + zK_v\right), \\
J_2 &= \frac{1}{1+\frac{r^2}{4}}\left((1-\frac{x^2}{4}+\frac{y^2}{4}-\frac{z^2}{4})J_v + \frac{xy}{2}J_u + \frac{zy}{2}J_w - zK_u + xK_w\right), \\
J_3 &= \frac{1}{1+\frac{r^2}{4}}\left((1-\frac{x^2}{4}-\frac{y^2}{4}+\frac{z^2}{4})J_w + \frac{yz}{2}J_v + \frac{xz}{2}J_u - xK_v + yK_u\right), \\
K_1 &= \frac{1}{1+\frac{r^2}{4}}\left((1+\frac{x^2}{4}-\frac{y^2}{4}-\frac{z^2}{4})K_u + \frac{yx}{2}K_v + \frac{zx}{2}K_w - yJ_w + zJ_v\right), \\
K_2 &= \frac{1}{1+\frac{r^2}{4}}\left((1-\frac{x^2}{4}+\frac{y^2}{4}-\frac{z^2}{4})K_v + \frac{xy}{2}K_u + \frac{zy}{2}K_w - zJ_u + xJ_w\right), \\
K_3 &= \frac{1}{1+\frac{r^2}{4}}\left((1-\frac{x^2}{4}-\frac{y^2}{4}+\frac{z^2}{4})K_w + \frac{xz}{2}K_u + \frac{yz}{2}K_v - xJ_v + yJ_u\right).
\end{aligned}$$

The inverse of these relations takes almost exactly the same form. Now, suppose we have a nondegenerate 3-sphere superintegrable system with potential V, that is separable with respect to some orthogonal coordinates. (Since every superintegrable system is multiseparable, we know that such coordinates exist.) By performing an Euclidean transformation, if necessary, we can assume that the separable coordinates are in some standard form determined by two constants of the motion in involution, $L_1 = \sum a^{ij} p_i p_j + f_1$,

$L_2 = \sum b^{ij} p_i p_j + f_2$. Clearly, L_1 and L_2 lie in the 6-dimensional space of second order symmetries for the superintegrable system. Thus, the quadratic form a^{ij}, for example, satisfies the three Bertrand-Darboux equations for potential V. Since V is nondegenerate we can express the second derivatives $V_{jj} - V_{kk}$ and V_{jk} with $j \neq k$ in the Bertrand-Darboux equations as linear combinations of the first derivatives V_h. Equating coefficients of V_1, V_2, V_3 separately in each of the three equations, we end up with nine linear conditions for the 10 constants $A^{22}, \cdots, B^{23}$ at each regular point. If we choose the Cartesian-like coordinates X, Y, Z that vanish at the regular point, then we obtain the same 18 conditions as in the Euclidean case. Indeed, the first derivatives G_i all vanish at the regular point.

For the second symmetry there will be nine more such linear conditions with a^{ij} replaced by b^{ij}. Thus we will have 18 linear equations (not linearly independent) for the 10 quantities $A^{22}, \cdots, B^{23}$.

The 5 fundamental quadratic identities (53) are identical to those for the Euclidean case. This is because the only nonzero terms in the metric for the 3-sphere are $G_{ii} = -1$ and all such terms occur in the form $G_{ii} - G_{jj} = 0$ in the 5 quadratic conditions.

Another source of conditions is obtained by writing the symmetry L_1 in terms of the standard basis: $a^{ij}(\mathbf{x}) = \sum_{\ell \leq m} a^{\ell m}(\mathbf{x}_0) \mathcal{A}^{ij}_{(\ell,m)}(\mathbf{x})$, where $\mathcal{A}^{ij}_{(\ell,m)}$ is the quadratic form associated with the standard basis symmetry $\mathcal{S}^{(\ell,m)}$ at $\mathbf{x}_0$. Expanding both sides of this equation in terms of the natural basis we obtain linear and quadratic conditions on the 10 basic quantities. In this case there is a difference between the Euclidean and 3-sphere expressions. For example if we equate coefficients of the natural basis element $J_1 J_2$ we find the quadratic conditions for L_1 and L_2

$$-a^{33}_{12}(\mathbf{x}_0) = \sum_{\ell \leq m} a^{\ell m}(\mathbf{x}_0) \alpha^{(\ell m)}_{14}, \quad -b^{33}_{12}(\mathbf{x}_0) = \sum_{\ell \leq m} b^{\ell m}(\mathbf{x}_0) \alpha^{(\ell m)}_{14}. \tag{50}$$

It is no longer true that $-a^{33}_{12} = 2a^{13}_{23}$ as in the Euclidean case. The expressions for the terms $\alpha^{(\ell m)}_{14}$ can be computed from the basic formulas (51). They involve the terms G_{ii} and differ from the Euclidean case. For example, from (51) and formulas for the derivatives $\partial_i A^{jk}$, $\partial_i B^{jk}$, $\partial_i C^{jk}$ we can calculate $-a^{33}_{12}(\mathbf{x}_0)$ corresponding to the basis symmetry $\mathcal{S}^{(12)}$ and obtain

$$-3\alpha^{(12)}_{14} = -\frac{1}{3}B^{22}C^{23} - \frac{2}{3}(C^{13})^2 - \frac{2}{3}C^{13}A^{22} + \frac{1}{6}(A^{23})^2 + \frac{1}{3}C^{33}(\frac{7}{2}B^{23} - 2C^{22})$$
$$+\frac{3}{2} - \frac{1}{2}(B^{33})^2 + \frac{5}{6}B^{33}B^{22} + \frac{5}{6}(B^{12})^2 + A^{33}B^{12}$$
$$-\frac{1}{2}(B^{23})^2 + \frac{1}{6}(A^{12})^2 - \frac{1}{3}B^{33}A^{12} - \frac{1}{6}A^{22}A^{33} + \frac{1}{6}(A^{33})^2 - \frac{1}{6}C^{33}A^{13}.$$

Though there are many other quadratic conditions for L_1, L_2 to belong to the symmetry algebra, we shall use only these 2.

4.7 Significance of "generic" 3-sphere systems

Suppose we have a nondegenerate superintegrable system that admits separation for some special choice of ellipsoidal coordinates [1111]. (Here we do *not* assume that the system separates for all values of the parameters c, e_1, e_2, e_3, e_4, but only for one value.) By performing an Euclidean transformation and a change of scale we can assume that the coordinates are in the standard form [1111] in our table and that $e_1 = 0,\ e_2 = 1$, $e_3 = a$, and $e_4 = b$ where a, b are any fixed complex numbers such that $ab(a-1)(b-1)(b-a) \neq 0$. We follow the same method given before in the Euclidean case. We evaluate the a^{ij}, b^{ij} at any regular point with coordinates (x, y, z). Substituting these expressions into the 18 linear conditions, with the help of Maple, we find that there are exactly 7 independent linear conditions. Thus the 10 quantities $A^{22}, \cdots, B^{23}$ can be expressed linearly in terms of 3 of these quantities. Substituting this result into the 5 fundamental quadratic identities (53) we find that these identities yield exactly two solutions. Finally we substitute each of these solutions into (50) and find conditions that rule out one of these solutions. Thus only one solution exists and it must be the one that we already knew: system [VIII] that separates simultaneously for all choices of the parameters $e_1, \cdots, e_4$. What was far from obvious is the fact that *no other* nondegenerate superintegrable system separates for *any* special case of ellipsoidal coordinates on the 3-sphere.

Theorem 9 *A 3-sphere nondegenerate superintegrable system admits separation in a special case of the generic coordinates [1111], [211], [22], [31] or [4], respectively, if and only if it is equivalent via a complex rotation to system [VII], [I'], [II'], [IV'], or [VI'], respectively.*

We have indicated the proof for coordinates [1111]. The other generic coordinates are Stäckel transforms of generic coordinates in Euclidean space so the proof for them follows immediately from Theorem 7.

5 Appendix

This is a list of some important results from reference [3]. Using the nondegenerate potential condition and the Bertrand-Darboux equations we can solve for all of the first partial derivatives a_i^{jk} of a quadratic symmetry to obtain the defining conditions (with $\lambda = \exp(G)$)

$$\begin{aligned}
a_1^{11} &= -G_1a^{11} - G_2a^{12} - G_3a^{13} \\
a_2^{22} &= -G_1a^{12} - G_2a^{22} - G_3a^{23}, \\
a_3^{33} &= -G_1a^{13} - G_2a^{23} - G_3a^{33}, \\
3a_1^{12} &= a^{12}A^{22} - (a^{22} - a^{11})A^{12} - a^{23}A^{13} + a^{13}A^{23} \\
&\quad +G_2a^{11} - 2G_1a^{12} - G_2a^{22} - G_3a^{23}, \\
3a_2^{11} &= -2a^{12}A^{22} + 2(a^{22} - a^{11})A^{12} + 2a^{23}A^{13} - 2a^{13}A^{23} \\
&\quad -2G_2a^{11} + G_1a^{12} - G_2a^{22} - G_3a^{23}, \\
3a_3^{13} &= -a^{12}C^{23} + (a^{33} - a^{11})C^{13} + a^{23}C^{12} - a^{13}C^{33} \\
&\quad -G_1a^{11} - G_2a^{12} - 2G_3a^{13} + G_1a^{33}, \\
3a_1^{33} &= 2a^{12}C^{23} - 2(a^{33} - a^{11})C^{13} - 2a^{23}C^{12} + 2a^{13}C^{33} \\
&\quad -G_1a^{11} - G_2a^{12} + G_3a^{13} - 2G_1a^{33}, \\
3a_2^{23} &= a^{23}(B^{33} - B^{22}) - (a^{33} - a^{22})B^{23} - a^{13}B^{12} + a^{12}B^{13} \\
&\quad -G_1a^{13} - 2G_2a^{23} - G_3a^{33} + G_3a^{22}, \\
3a_3^{22} &= -2a^{23}(B^{33} - B^{22}) + 2(a^{33} - a^{22})B^{23} + 2a^{13}B^{12} - 2a^{12}B^{13} \\
&\quad -G_1a^{13} + G_2a^{23} - G_3a^{33} - 2G_3a^{22}, \\
3a_1^{13} &= -a^{23}A^{12} + (a^{11} - a^{33})A^{13} + a^{13}A^{33} + a^{12}A^{23} \\
&\quad -2G_1a^{13} - G_2a^{23} - G_3a^{33} + G_3a^{11}, \\
3a_3^{11} &= 2a^{23}A^{12} + 2(a^{33} - a^{11})A^{13} - 2a^{13}A^{33} - 2a^{12}A^{23} \\
&\quad +G_1a^{13} - G_2a^{23} - G_3a^{33} - 2G_3a^{11}, \\
3a_2^{33} &= -2a^{13}C^{12} + 2(a^{22} - a^{33})C^{23} + 2a^{12}C^{13} - 2a^{23}(C^{22} - C^{33}) \\
&\quad -G_1a^{12} - G_2a^{22} + G_3a^{23} - 2G_2a^{33}, \\
3a_3^{23} &= a^{13}C^{12} - (a^{22} - a^{33})C^{23} - a^{12}C^{13} - a^{23}(C^{33} - C^{22}) \\
&\quad -G_1a^{12} - G_2a^{22} - 2G_3a^{23} + G_2a^{33}, \\
3a_2^{12} &= -a^{13}B^{23} + (a^{22} - a^{11})B^{12} - a^{12}B^{22} + a^{23}B^{13} \\
&\quad -G_1a^{11} - 2G_2a^{12} - G_3a^{13} + G_1a^{22}, \\
3a_1^{22} &= 2a^{13}B^{23} - 2(a^{22} - a^{11})B^{12} + 2a^{12}B^{22} - 2a^{23}B^{13} \\
&\quad -G_1a^{11} + G_2a^{12} - G_3a^{13} - 2G_1a^{22}, \\
3a_1^{23} &= a^{12}(B^{23} + C^{22}) + a^{11}(B^{13} + C^{12}) - a^{22}C^{12} - a^{33}B^{13} \\
&\quad +a^{13}(B^{33} + C^{23}) - a^{23}(C^{13} + B^{12}) - 2G_1a^{23} + G_2a^{13} + G_3a^{12}. \\
3a_3^{12} &= a^{12}(-2B^{23} + C^{22}) + a^{11}(C^{12} - 2B^{13}) - a^{22}C^{12} + 2a^{33}B^{13} \\
&\quad +a^{13}(-2B^{33} + C^{23}) + a^{23}(-C^{13} + 2B^{12}) - 2G_3a^{12} + G_2a^{13} + G_1a^{23}. \\
3a_2^{13} &= a^{12}(B^{23} - 2C^{22}) + a^{11}(B^{13} - 2C^{12}) + 2a^{22}C^{12} - a^{33}B^{13}
\end{aligned} \tag{51}$$

$$+a^{13}(B^{33}-2C^{23})+a^{23}(2C^{13}-B^{12})-2G_2a^{13}+G_1a^{23}+G_3a^{12},$$

plus the linear relations

$$\begin{gathered} A^{23}=B^{13}=C^{12}, \quad B^{23}-A^{31}-C^{22}=0, \\ B^{12}-A^{22}+A^{33}-C^{13}=0, \quad B^{33}+A^{12}-C^{23}=0. \end{gathered} \tag{52}$$

Using the linear relations we can express $C^{12}, C^{13}, C^{22}, C^{23}$ and B^{13} in terms of the remaining 10 functions. Finally, requiring that the integrability conditions for (51) hold identically, we obtain exactly 5 quadratic identities for the 10 independent functions:

$$\begin{array}{lrl}
a) & -A^{23}B^{33}-A^{12}A^{23}+A^{13}B^{12}+B^{22}A^{23}+B^{23}A^{33} & \\
 & +\frac{1}{2}A^{22}G_3-\frac{1}{2}A^{33}G_3-\frac{1}{2}B^{12}G_3-\frac{1}{2}G_1G_3 & \\
 & -\frac{1}{2}A^{13}G_1+\frac{3}{2}G_{13}-\frac{1}{2}A^{23}G_2-A^{22}B^{23} & =0, \\
b) & \left(A^{33}\right)^2+B^{12}A^{33}-A^{33}A^{22}-B^{33}A^{12}-C^{33}A^{13}+B^{22}A^{12} & \\
 & -B^{12}A^{22}+A^{13}B^{23}-\left(A^{12}\right)^2+ & \\
 & \frac{3}{2}G_{22}-\frac{1}{2}G_y^2-\frac{3}{2}G_{33}+\frac{1}{2}A^{13}G_3+\frac{1}{2}B^{33}G_2+ & \\
 & -\frac{1}{2}A^{22}G_1+\frac{1}{2}A^{33}G_1-\frac{1}{2}B^{23}G_3-\frac{1}{2}B^{22}G_2+\frac{1}{2}C^{33}G_3+\frac{1}{2}(G_3)^2 & =0, \\
c) & -\left(B^{33}\right)^2-B^{33}A^{12}+B^{33}B^{22}+B^{12}A^{33}+B^{23}C^{33}-\left(B^{23}\right)^2 & \\
 & +\left(B^{12}\right)^2+\frac{1}{2}(G_1)^2-\frac{3}{2}G_{11}+\frac{3}{2}G_{33} & \\
 & -\frac{1}{2}B^{33}G_2-\frac{1}{2}A^{33}G_1-\frac{1}{2}(G_3)^2-\frac{1}{2}C^{33}G_3 & =0,
\end{array}$$

$$\begin{array}{lrl}
d) & -B^{12}A^{23}-A^{33}A^{23}+A^{13}B^{33}+A^{12}B^{23} & \\
 & +\frac{3}{2}G_{23}-\frac{1}{2}A^{23}G_1-\frac{1}{2}A^{12}G_3 & \\
 & -\frac{1}{2}B^{23}G_2-\frac{1}{2}G_2G_3-\frac{1}{2}B^{33}G_3 & =0, \\
e) & A^{12}B^{12}+C^{33}A^{23}-A^{23}B^{23}+B^{33}A^{22}-B^{33}A^{33} & \\
 & +\frac{3}{2}G_{12}-\frac{1}{2}G_1G_2-\frac{1}{2}A^{12}G_1 & \\
 & -\frac{1}{2}B^{12}G_2-\frac{1}{2}A^{23}G_3 & =0
\end{array} \tag{53}$$

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