

# Second order superintegrable systems in two and three dimensions

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# Abstract

A classical (or quantum) superintegrable system is an integrable  $n$ -dimensional Hamiltonian system with potential that admits  $2n - 1$  functionally independent constants of the motion, polynomial in the momenta, the maximum possible. If the constants are all quadratic the system is second order superintegrable. Such systems have remarkable properties: multi-integrability and multi-separability, a quadratic algebra of symmetries whose representation theory yields spectral information about the Schrödinger operator, deep connections with special functions and with QES systems.

# Abstract continued

For  $n=2$  (and  $n=3$  on conformally flat spaces with nondegenerate potentials) we have worked out the structure and classified the possible spaces and potentials. The quadratic algebra closes at order 6 and there is a 1-1 classical-quantum relationship. All such systems are Stäckel transforms of systems on complex Euclidean space or the complex 3-sphere.

# 2nd order superintegrability (classical)

Classical superintegrable system on an  $n$ -dimensional local Riemannian manifold:

$$\mathcal{H} = \sum_{ij} g^{ij} p_i p_j + V(\mathbf{x}).$$

Require that Hamiltonian admits  $2n - 1$  functionally independent 2nd-order symmetries  $\mathcal{S}_k = \sum a_{(k)}^{ij}(\mathbf{x}) p_i p_j + W_{(k)}(\mathbf{x})$ ,  
That is,  $\{\mathcal{H}, \mathcal{S}_k\} = 0$  where  $\{f, g\} = \sum_{j=1}^n (\partial_{x_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{x_j} g)$   
is the Poisson bracket. Note that  $2n - 1$  is the maximum possible number of functionally independent symmetries.

# Significance

Generically, every trajectory  $\mathbf{p}(t), \mathbf{x}(t)$ , i.e., solution of the Hamilton equations of motion, is characterized (and parametrized) as a common intersection of the (constants of the motion) hypersurfaces

$$\mathcal{S}_k(\mathbf{p}, \mathbf{x}) = c_k, \quad k = 0, \dots, 2n - 2.$$

The trajectories can be obtained without solving the equations of motion. This is better than integrability.

# 2nd order superintegrability (quantum)

Schrödinger operator

$$H = \Delta + V(\mathbf{x})$$

where  $\Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} g^{ij}) \partial_{x_j}$  is the Laplace-Beltrami operator on a Riemannian manifold, expressed in local coordinates  $x_j$  and  $S_1, \dots, S_n$ . Here there are  $2n - 1$  second-order symmetry operators

$$S_k = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} a_{(k)}^{ij}) \partial_{x_j} + W_{(k)}, \quad k = 1, \dots, 2n - 1$$

with  $S_1 = H$  and  $[H, S_k] \equiv HS_k - S_kH = 0$ .

# Why second order?

This is the most tractable case due to the association with separation of variables. Special function theory can be applied and is relevant for the same reason.

# Integrability and superintegrability

1. An integrable system has  $n$  functionally independent constants of the motion in involution. A superintegrable system has  $2n - 1$  functionally independent constants of the motion. (Sometimes the definition of superintegrability also requires integrability. In this talk we prove it.)
2. Multiseparable systems yield many examples of superintegrability.
3. Superintegrable systems can be solved explicitly in multiple ways. It is the information gleaned from comparing the distinct solutions and expressing one solution set in terms of another that is a primary reason for their interest.



# 3D example:

The generalized anisotropic oscillator: Schrödinger equation  $H\Psi = E\Psi$  or

$$H\Psi = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Psi + V(x, y, z)\Psi = E\Psi.$$

The 4-parameter “nondegenerate” potential

$$V(x, y, z) = \frac{\omega^2}{2} (x^2 + y^2 + 4(z + \rho)^2) + \frac{1}{2} \left[ \frac{\lambda_1}{x^2} + \frac{\lambda_2}{y^2} \right]$$

# Separable coordinates

The corresponding Schrödinger equation has separable solutions in five coordinate systems: Cartesian coordinates, cylindrical polar coordinates, cylindrical elliptic coordinates, cylindrical parabolic coordinates and parabolic coordinates. The energy eigenstates for this equation are degenerate and important special function identities arise by expanding one basis of separable eigenfunctions in terms of another.

# Basis for 2nd order symmetries

$$M_1 = \partial_x^2 - \omega^2 x^2 + \frac{\lambda_1}{x^2}, \quad M_2 = \partial_y^2 - \omega^2 y^2 - \frac{\lambda_2}{y^2},$$

$$P = \partial_z^2 - 4\omega^2(z + \rho)^2, \quad L = L_{12}^2 - \lambda_1 \frac{y^2}{x^2} - \lambda_2 \frac{x^2}{y^2} - \frac{1}{2},$$

$$S_1 = -\frac{1}{2}(\partial_x L_{13} + L_{13} \partial_x) + \rho \partial_x^2 + (z + \rho)(\omega^2 x^2 - \lambda_1 x^2),$$

$$S_2 = -\frac{1}{2}(\partial_y L_{23} + L_{23} \partial_y) + \rho \partial_y^2 + (z + \rho)(\omega^2 y^2 - \lambda_2 y^2),$$

where  $L_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i}$ .

# Quadratic algebra closing at level 6

The nonzero commutators are

$$[M_1, L] = [L, M_2] = Q, \quad [L, S_1] = [S_2, L] = B,$$

$$[M_i, S_i] = A_i, \quad [P, S_i] = -A_i.$$

Nonzero commutators of the basis symmetries with  $Q$  (4th order symmetries) are expressible in terms of the second order symmetries, e.g.,

$$[M_i, Q] = [Q, M_2] = 4\{M_1, M_2\} + 16\omega^2 L,$$

$$[S_1, Q] = [Q, S_2] = 4\{M_1, M_2\},$$

$$[L, Q] = 4\{M_1, L\} - 4\{M_2, L\} - 16\lambda_1 M_1 + 16\lambda_2 M_2.$$

# Level 6 closure

The squares of  $Q$ ,  $B$ ,  $A_i$  and products such as  $\{Q, B\}$ , (all 6th order symmetries) are all expressible in terms of 2nd order symmetries, e.g.,

$$Q^2 = \frac{8}{3}\{L, M_1, M_2\} + 8\omega^2\{L, L\} + 16\lambda_1 M_1^2 + 16\lambda_2 M_2^2$$

$$+ \frac{64}{3}\{M_1, M_2\} - \frac{128}{3}\omega^2 L - 128\omega^2 \lambda_1 \lambda_2,$$

$$\{Q, B\} = -\frac{8}{3}\{M_2, L, S_1\} - \frac{8}{3}\{M_1, L, S_2\} - 16\lambda_1\{M_2, S_2\}$$

$$- 16\lambda_2\{M_1, S_1\} - \frac{64}{3}\{M_1, S_2\} - \frac{64}{3}\{M_2, S_1\}.$$

Here  $\{C_1, \dots, C_j\}$  is the completely symmetrized product of operators  $C_1, \dots, C_j$ .

# Important properties-1

The algebra generated by the second order symmetries is *closed under commutation* in both the classical and operator cases. This is a remarkable, but typical of superintegrable systems with nondegenerate potentials.

Closure is at level 6, since we have to express the products of the 3rd order operators in terms of the basis of 2nd order operators.

The eigenfunctions of one separable system can be expanded in terms of the eigenfunctions of another, and this is the source of nontrivial special function expansion theorems.

The quadratic algebra identities allow us to relate eigenbases and eigenvalues of one symmetry operator to those of another.

# Important properties-2

The representation theory of the abstract quadratic algebra can be used to derive spectral properties of the generators  $S_j$ , in a manner analogous to the use of Lie algebra representation theory to derive spectral properties of quantum systems that admit Lie symmetry algebras,

A common feature of quantum superintegrable systems is that after splitting off a gauge factor, the Schrödinger and symmetry operators are acting on a space of polynomials: MULTIVARIABLE ORTHOGONAL POLYNOMIALS.

# Important properties-3

Closely related to the theory of QUASI-EXACTLY SOLVABLE SYSTEMS (QES). In many 2D and 3D examples the one-dimensional ODEs are quasi-exactly solvable and the eigenvalues that give polynomial solutions are easily obtained from the PDE superintegrable systems. Generalizes results of Ushveridze. Leads to new examples.



# Use of the quadratic algebra-1

Consider the nondegenerate superintegrable potential

$$V(x, y) = -\frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{B_1}{4} \frac{\sqrt{\sqrt{x^2 + y^2} + x}}{\sqrt{x^2 + y^2}} + \frac{B_2}{4} \frac{\sqrt{\sqrt{x^2 + y^2} - x}}{\sqrt{x^2 + y^2}},$$

separable in parabolic coordinates, and in parabolic coordinates of the second type  $x = \mu\nu$ ,  $y = \frac{1}{2}(\mu^2 - \nu^2)$ .

# Use of the quadratic algebra-2

A basis for the quadratic algebra is  $L_1, L_2$  and  $H$  with defining relations

$$[R, L_1] = -4L_2H + B_1B_2, \quad [R, L_2] = 4L_1H + \frac{1}{2}(B_1^2 - B_2^2)$$

$$R^2 = 4L_1^2H + 4L_2^2H - 16\alpha^2H + (B_2^2 - B_1^2)L_1 - 2B_1B_2L_2 - 2\alpha^2(B_1^2 + B_2^2)$$

with  $R = [L_1, L_2]$ .

# Use of the quadratic algebra-3

If we look for eigenfunctions of the operators  $L_1, L_2$  respectively, we have  $L_1\varphi_m = \lambda_m\varphi_m$ ,  $L_2\psi_n = \rho_n\psi_n$ . If we write  $L_1\psi_n = \sum_{\tau} C_{n\tau}\psi_{\tau}$  then the quadratic algebra relations imply

$$[(\rho_n - \rho_{\tau})^2 + 8E]C_{n\tau} = -\left[\frac{1}{2}(B_1^2 - B_2^2) - 16\alpha E\right]\delta_{n\tau}$$

$$\sum_{\tau} C_{n\tau}C_{\tau\sigma}(2\rho_{\tau} - \rho_n - \rho_{\sigma}) = (8E\rho_n + B_1B_2 + 16\alpha E)\delta_{n\sigma}.$$

# Use of the quadratic algebra-4

These relations in turn imply that

$C_{nn} = -[\frac{1}{2}(B_1^2 - B_2^2) + 16\alpha E]/8E$  and  $C_{nn+1} = C_{n+1n}^*$  are the only nonzero coefficients. They can essentially be determined by the relation

$$4\sqrt{-2E} (|C_{n,n+1}|^2 - |C_{n-1,n}|^2) = 8E\rho_n + B_1B_2 + 16\alpha E.$$

# Use of the quadratic algebra-5

We see that the eigenvalues  $\lambda_m$  and  $\rho_n$  are

$$\lambda_m = 2\alpha - \frac{B_1^2}{8E} - (2m + 1)\sqrt{-2E},$$

$$\rho_n = 2\alpha - \frac{(B_1 + B_2)^2}{16E} - (2n + 1)\sqrt{-2E}$$

and the quantization condition for  $E$  is  $4\alpha - \frac{B_1^2 + B_2^2}{8E} = -(q + 2)\sqrt{-2E}$  for integer  $q$ . Thus the bound state spectral resolution of  $H, L_1, L_2$  has been obtained from the structure of the quadratic algebra alone.

# 2D classical structure theory-1

Can assume Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{\lambda(x, y)}(p_1^2 + p_2^2) + V(x, y), \quad (x, y) = (x_1, x_2),$$

i.e., the complex metric is  $ds^2 = \lambda(x, y)(dx^2 + dy^2)$ .

Necessary and sufficient conditions that

$\mathcal{S} = \sum a^{ji}(x, y)p_j p_i + W(x, y)$  be a symmetry of  $\mathcal{H}$  are the Killing equations

$$(1) \quad a_i^{ii} = -\frac{\lambda_1}{\lambda} a^{i1} - \frac{\lambda_2}{\lambda} a^{i2}, \quad i = 1, 2$$

$$2a_i^{ij} + a_j^{ii} = -\frac{\lambda_1}{\lambda} a^{j1} - \frac{\lambda_2}{\lambda} a^{j2}, \quad i, j = 1, 2, \quad i \neq j,$$

# 2D classical structure theory-2

and the Bertrand-Darboux conditions on the potential  
 $\partial_i W_j = \partial_j W_i$  or

$$(V_{22} - V_{11})a^{12} + V_{12}(a^{11} - a^{22}) =$$
$$\left[ \frac{(\lambda a^{12})_1 - (\lambda a^{11})_2}{\lambda} \right] V_1 + \left[ \frac{(\lambda a^{22})_1 - (\lambda a^{12})_2}{\lambda} \right] V_2.$$

Similar but more complicated conditions for the higher order symmetries.

# Nondegenerate potentials-1

>From the 3 second order constants of the motion we get 3 Bertrand-Darboux equations and can solve them to obtain fundamental PDEs for the potential of the form

$$V_{22} - V_{11} = A^{(22)}(\mathbf{x})V_1 + B^{22}(\mathbf{x})V_2, \quad V_{12} = A^{12}(\mathbf{x})V_1 + B^{12}(\mathbf{x})V_2.$$

If the B-D equations provide no further conditions on the potential and if the integrability conditions for the PDEs are satisfied identically, we say that the potential is *nondegenerate*. That means, at each regular point  $\mathbf{x}_0$  where the  $A^{ij}, B^{ij}$  are defined and analytic, we can prescribe the values of  $V, V_1, V_2$  and  $V_{11}$  arbitrarily and there will exist a unique potential  $V(\mathbf{x})$  with these values at  $\mathbf{x}_0$ .



# Nondegenerate potentials-2

Nondegenerate potentials depend on 3 parameters, in addition to the trival additive parameter. Degenerate potentials depend on  $< 3$  parameters.

# Spaces of polynomial constants-1

It is important to compute the dimensions of the spaces of symmetries of these nondegenerate systems that are of orders 2,3,4 and 6. These symmetries are necessarily of a special type.

The highest order terms in the momenta are independent of the parameters in the potential.

The terms of order 2 in the momenta less are linear in these parameters.

those of order 4 less are quadratic.

Those of order 6 less are cubic.

# Spaces of polynomial constants-2

The system is **2nd order superintegrable with nondegenerate potential** if

it admits 3 functionally independent second-order symmetries (here  $2N - 1 = 3$ )

the potential is 3-parameter (in addition to the usual additive parameter) in the sense given above:

$$V(x, y) = \alpha_1 V^{(1)}(x, y) + \alpha_2 V^{(2)}(x, y) + \alpha_3 V^{(3)}(x, y).$$

# Functional linear independence

The 3 functionally independent symmetries are **functionally linearly independent** if at each regular point  $(x_0, y_0)$  the 3 matrices  $a_{(1)}^{ij}(x_0, y_0)$ ,  $a_{(1)}^{ij}(x_0, y_0)$ ,  $a_{(1)}^{ij}(x_0, y_0)$  are linearly independent.

There is essentially only one functionally linearly dependent superintegrable system in 2D:

$$\mathcal{H} = p_z p_{\bar{z}} + V(z),$$

where  $V(z)$  is an arbitrary function of  $z$  alone. This system separates in only one set of coordinates  $z, \bar{z}$ .

# Spaces of polynomial constants-3

**THEOREM:** Let  $\mathcal{H}$  be the Hamiltonian of a 2D superintegrable (functionally linearly independent) system with nondegenerate potential.

The space of second order constants of the motion is exactly 3-dimensional.

The space of third order constants of the motion is at most 1-dimensional.

The space of fourth order constants of the motion is at most 6-dimensional.

The space of sixth order constants is at most 10-dimensional.

# Spaces of polynomial constants-4

**THEOREM:** Let  $\mathcal{K}$  be a third order constant of the motion for a superintegrable system with nondegenerate potential  $V$ :

$$\mathcal{K} = \sum_{k,j,i=1}^2 a^{kji}(x, y) p_k p_j p_i + \sum_{\ell=1}^2 b^{\ell}(x, y) p_{\ell}.$$

Then  $b^{\ell}(x, y) = \sum_{j=1}^2 f^{\ell,j}(x, y) \frac{\partial V}{\partial x_j}(x, y)$  with  $f^{\ell,j} + f^{j,\ell} = 0$ ,  $1 \leq \ell, j \leq 2$ . The  $a^{ijk}, b^{\ell}$  are uniquely determined by the number  $f^{1,2}(x_0, y_0)$  at some regular point  $(x_0, y_0)$  of  $V$ .

# Structure theory -1

Let

$$\mathcal{S}_1 = \sum a_{(1)}^{kj} p_k p_j + W_{(1)}, \quad \mathcal{S}_2 = \sum a_{(2)}^{kj} p_k p_j + W_{(2)}$$

be second order constants of the the motion and let  $\mathcal{A}_{(i)}(x, y) = \{a_{(i)}^{kj}(x, y)\}$ ,  $i = 1, 2$  be  $2 \times 2$  matrix functions. Then the Poisson bracket of these symmetries is given by

$$\{\mathcal{S}_1, \mathcal{S}_2\} = \sum_{k,j,i=1}^2 a^{kji}(x, y) p_k p_j p_i + b^\ell(x, y) p_\ell$$

where

$$f^{k,\ell} = 2\lambda \sum_j (a_{(2)}^{kj} a_{(1)}^{j\ell} - a_{(1)}^{kj} a_{(2)}^{j\ell}).$$

# Structure theory -2

Thus  $\{\mathcal{S}_1, \mathcal{S}_2\}$  is uniquely determined by the skew-symmetric matrix

$$[\mathcal{A}_{(2)}, \mathcal{A}_{(1)}] \equiv \mathcal{A}_{(2)}\mathcal{A}_{(1)} - \mathcal{A}_{(1)}\mathcal{A}_{(2)},$$

hence by the constant matrix  $[\mathcal{A}_{(2)}(x_0, y_0), \mathcal{A}_{(1)}(x_0, y_0)]$  evaluated at a regular point.



# Standard 2D form-1

For superintegrable nondegenerate potentials there is a standard structure that

allows the identification of the space of second order constants of the motion with the space of  $2 \times 2$  symmetric matrices

allows identification of the space of third order constants of the motion with the space of  $2 \times 2$  skew-symmetric matrices.

# Standard 2D form-2

If  $\mathbf{x}_0$  is a regular point then there is a 1 – 1 linear correspondence between second order operators  $S$  and their associated symmetric matrices  $\mathcal{A}_{(\mathbf{x}_0)}$ . Let  $\{\mathcal{S}_1, \mathcal{S}_2\}' = \{\mathcal{S}_2, \mathcal{S}_1\}$  be the reversed Poisson bracket. the map  $\{\mathcal{S}_1, \mathcal{S}_2\}' \iff [\mathcal{A}_{(1)}(\mathbf{x}_0), \mathcal{A}_{(2)}(\mathbf{x}_0)]$  is an algebraic isomorphism.

Let  $\mathcal{E}^{ij}$  be the  $2 \times 2$  matrix with a 1 in row  $i$ , column  $j$  and 0 for every other matrix element. Then the symmetric matrices

$$\mathcal{A}^{(ij)} = \frac{1}{2}(\mathcal{E}^{ij} + \mathcal{E}^{ji}) = \mathcal{A}^{(ji)}, \quad i, j = 1, 2$$

form a basis for the 3-dimensional space of symmetric matrices.

# Standard 2D form-3

Moreover,

$$[\mathcal{A}^{(ij)}, \mathcal{A}^{(kl)}] = \frac{1}{2} \left( \delta_{jk} \mathcal{B}^{(il)} + \delta_{jl} \mathcal{B}^{(ik)} + \delta_{ik} \mathcal{B}^{(jl)} + \delta_{il} \mathcal{B}^{(jk)} \right)$$

where

$$\mathcal{B}^{(ij)} = \frac{1}{2} (\mathcal{E}^{ij} - \mathcal{E}^{ji}) = -\mathcal{B}^{(ji)}, \quad i, j = 1, 2.$$

Here  $\mathcal{B}^{(ii)} = 0$  and  $\mathcal{B}^{(12)}$  forms a basis for the space of skew-symmetric matrices. This gives the commutation relations for the second order symmetries.

# Standard 2D form-4

We define a standard set of basis symmetries  $\mathcal{S}_{(k\ell)} = \sum a^{ij}(\mathbf{x})p_i p_j + W_{(k\ell)}(\mathbf{x})$  corresponding to a regular point  $\mathbf{x}_0$  by

$$\begin{pmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{pmatrix}_{\mathbf{x}_0} = \lambda(\mathbf{x}_0) \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}_{\mathbf{x}_0} = \lambda(\mathbf{x}_0) \mathcal{A}^{(k\ell)},$$

$W_{(k\ell)}(\mathbf{x}_0) = 0$ . The condition on  $W_{(k\ell)}$  is actually 3 conditions since  $W^{(k\ell)}$  depends on 3 parameters.

# 2D multiseparability

COROLLARY: Let  $V$  be a superintegrable nondegenerate potential and  $L$  be a second order constant of the motion with matrix function  $\mathcal{A}(\mathbf{x})$ . If at some regular point  $\mathbf{x}_0$  the matrix  $\mathcal{A}(\mathbf{x}_0)$  has 2 distinct eigenvalues, then  $H, L$  characterize an orthogonal separable coordinate system.

Note: Since a generic  $2 \times 2$  symmetric matrix has distinct roots, it follows that any superintegrable nondegenerate potential is multiseparable.

# The quadratic algebra

THEOREM:

The 6 distinct monomials

$$(\mathcal{S}_{(11)})^2, (\mathcal{S}_{(22)})^2, (\mathcal{S}_{(12)})^2, \mathcal{S}_{(11)}\mathcal{S}_{(22)}, \mathcal{S}_{(11)}\mathcal{S}_{(12)}, \mathcal{S}_{(12)}\mathcal{S}_{(22)},$$

form a basis for the space of fourth order symmetries.

The 10 distinct monomials

$$(\mathcal{S}_{(ii)})^3, (\mathcal{S}_{(ij)})^3, (\mathcal{S}_{(ii)})^2\mathcal{S}_{(jj)}, (\mathcal{S}_{(ii)})^2\mathcal{S}_{(ij)}, (\mathcal{S}_{(ij)})^2\mathcal{S}_{(ii)},$$

$\mathcal{S}_{(11)}\mathcal{S}_{(12)}\mathcal{S}_{(22)}$ , for  $i, j = 1, 2, i \neq j$  form a basis for the space of sixth order symmetries.

One can expand explicitly any 4th or 6th order symmetry in terms of the standard basis.

# The Stäckel transform-1

ALL 2D SUPERINTEGRABLE NONDEGENERATE POTENTIALS IN EUCLIDEAN SPACE AND ON THE 2-SPHERE HAVE BEEN CLASSIFIED.

There are 11 families of nondegenerate potentials in flat space (4 in real Euclidean space).

There are 6 families of nondegenerate potentials on the complex 2-sphere (2 on the real sphere).

HOW TO GET ALL SUCH POTENTIALS ON GENERAL 2D MANIFOLDS?

# The Stäckel transform-2

Suppose we have a (classical or quantum) superintegrable system

$$\mathcal{H} = \frac{1}{\lambda(x, y)}(p_1^2 + p_2^2) + V(x, y), \quad H = \frac{1}{\lambda(x, y)}(\partial_{11} + \partial_{22}) + V(x, y)$$

in local orthogonal coordinates, with nondegenerate potential  $V(x, y)$  and suppose  $U(x, y)$  is a particular case of the 3-parameter potential  $V$ , nonzero in an open set. Then the transformed systems

$$\tilde{\mathcal{H}} = \frac{1}{\tilde{\lambda}(x, y)}(p_1^2 + p_2^2) + \tilde{V}(x, y), \quad \tilde{H} = \frac{1}{\tilde{\lambda}(x, y)}(\partial_{11} + \partial_{22}) + \tilde{V}(x, y)$$

are also superintegrable, where  $\tilde{\lambda} = \lambda U$ ,  $\tilde{V} = \frac{V}{U}$ .



# The Stäckel transform-3

THEOREM:

1.

$$\{\tilde{\mathcal{H}}, \tilde{\mathcal{S}}\} = 0 \iff \{\mathcal{H}, \mathcal{S}\} = 0.$$

2.

$$\tilde{\mathcal{S}} = \sum_{ij} \frac{1}{\lambda U} p_i \left( (a^{ij} + \delta^{ij} \frac{1 - W_U}{\lambda U}) \lambda U \right) p_j + \left( W - \frac{W_U V}{U} + \frac{V}{U} \right)$$

3.

$$[\tilde{H}, \tilde{S}] = 0 \iff [H, S] = 0.$$

4.

$$\tilde{S} = \sum_{ij} \frac{1}{\lambda U} \partial_i \left( (a^{ij} + \delta^{ij} \frac{1 - W_U}{\lambda U}) \lambda U \right) \partial_j + \left( W - \frac{W_U V}{U} + \frac{V}{U} \right)$$

# The Stäckel transform-4

COROLLARY: If  $\mathcal{S}^{(1)}, \mathcal{S}^{(2)}$  are second order constants of the motion for  $\mathcal{H}$ , then

$$\{\tilde{\mathcal{S}}^{(1)}, \tilde{\mathcal{S}}^{(2)}\} = 0 \iff \{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}\} = 0.$$

If  $S^{(1)}, S^{(2)}$  are second order symmetry operators for  $H$ , then

$$[\tilde{S}^{(1)}, \tilde{S}^{(2)}] = 0 \iff [S^{(1)}, S^{(2)}] = 0.$$

# The Stäckel transform-5

This transform of one (classical or quantum) superintegrable system into another on a different manifold is called the **Stäckel transform**. Two such systems related by a Stäckel transform are called **Stäckel equivalent**.

**THEOREM:** Every nondegenerate second-order classical or quantum superintegrable system in two variables is Stäckel equivalent to a superintegrable system on a constant curvature space.

# Classification results-1

**BASIC RESULT:** If  $ds^2 = \lambda(dx^2 + dy^2)$  is the metric of a nondegenerate superintegrable system (expressed in coordinates  $x, y$  such that  $\lambda_{12} = 0$ ) then  $\lambda = \mu$  is a solution of the system

$$\mu_{12} = 0, \quad \mu_{22} - \mu_{11} = 3\mu_1(\ln a^{12})_1 - 3\mu_2(\ln a^{12})_2 + \left(\frac{a_{11}^{12} - a_{22}^{12}}{a^{12}}\right)\mu,$$

where either

$$I) \quad a^{12} = X(x)Y(y), \quad X'' = \alpha^2 X, \quad Y'' = -\alpha^2 Y,$$

or

# Classification results-2

$$II) \quad a^{12} = \frac{2X'(x)Y'(y)}{C(X(x) + Y(y))^2},$$

$$(X')^2 = F(X), \quad X'' = \frac{1}{2}F'(X), \quad (Y')^2 = G(Y), \quad Y'' = \frac{1}{2}G'(Y)$$

where

$$F(X) = \frac{\alpha}{24}X^4 + \frac{\gamma_1}{6}X^3 + \frac{\gamma_2}{2}X^2 + \gamma_3X + \gamma_4,$$

$$G(Y) = -\frac{\alpha}{24}Y^4 + \frac{\gamma_1}{6}Y^3 - \frac{\gamma_2}{2}Y^2 + \gamma_3Y - \gamma_4.$$

# Classification results-3

Conversely, every solution  $\lambda$  of one of these systems of equations defines a nondegenerate superintegrable system. If  $\lambda$  is a solution then the remaining solutions  $\mu$  are exactly the nondegenerate superintegrable systems that are Stäckel equivalent to  $\lambda$ .

# Classification results-4

In a tour de force, Koenigs has classified all 2D manifolds (i.e., no potential) that admit exactly 3 second-order Killing tensors and listed them in two tables: Tableau VI and Tableau VII. Our methods show that these are exactly the spaces that admit superintegrable systems with nondegenerate potentials.

# TABLEAU VI

$$[1] \quad ds^2 = \left[ \frac{c_1 \cos x + c_2}{\sin^2 x} + \frac{c_3 \cos y + c_4}{\sin^2 y} \right] (dx^2 - dy^2)$$

$$[2] \quad ds^2 = \left[ \frac{c_1 \cosh x + c_2}{\sinh^2 x} + \frac{c_3 e^y + c_4}{e^{2y}} \right] (dx^2 - dy^2)$$

$$[3] \quad ds^2 = \left[ \frac{c_1 e^x + c_2}{e^{2x}} + \frac{c_3 e^y + c_4}{e^{2y}} \right] (dx^2 - dy^2)$$

$$[4] \quad ds^2 = \left[ c_1(x^2 - y^2) + \frac{c_2}{x^2} + \frac{c_3}{y^2} + c_4 \right] (dx^2 - dy^2)$$

$$[5] \quad ds^2 = \left[ c_1(x^2 - y^2) + \frac{c_2}{x^2} + c_3 y + c_4 \right] (dx^2 - dy^2)$$

$$[6] \quad ds^2 = \left[ c_1(x^2 - y^2) + c_2 x + c_3 y + c_4 \right] (dx^2 - dy^2)$$



# TABLEAU VII-1

$$\begin{aligned}
 [1] \quad ds^2 &= \left[ c_1 \left( \frac{1}{\operatorname{sn}^2(x, k)} - \frac{1}{\operatorname{sn}^2(y, k)} \right) + c_2 \left( \frac{1}{\operatorname{cn}^2(x, k)} - \frac{1}{\operatorname{cn}^2(y, k)} \right) \right. \\
 &+ \left. c_3 \left( \frac{1}{\operatorname{dn}^2(x, k)} - \frac{1}{\operatorname{dn}^2(y, k)} \right) + c_4 (\operatorname{sn}^2(x, k) - \operatorname{sn}^2(y, k)) \right] \\
 &\times (dx^2 - dy^2)
 \end{aligned}$$

$$\begin{aligned}
 [2] \quad ds^2 &= \left[ c_1 \left( \frac{1}{\sin^2 x} - \frac{1}{\sin^2 y} \right) + c_2 \left( \frac{1}{\cos^2 x} - \frac{1}{\cos^2 y} \right) \right. \\
 &+ \left. c_3 (\cos 2x - \cos 2y) + c_4 (\cos 4x - \cos 4y) \right] (dx^2 - dy^2)
 \end{aligned}$$

$$\begin{aligned}
 [3] \quad ds^2 &= [c_1 (\sin 4x - \sin 4y) + c_2 (\cos 4x - \cos 4y) \\
 &+ c_3 (\sin 2x - \sin 2y) + c_4 (\cos 2x - \cos 2y)] (dx^2 - dy^2)
 \end{aligned}$$

# TABLEAU VII-2

$$[4] \quad ds^2 = \left[ c_1 \left( \frac{1}{x^2} - \frac{1}{y^2} \right) + c_2(x^2 - y^2) + c_3(x^4 - y^4) \right. \\ \left. + c_4(x^6 - y^6) \right] (dx^2 - dy^2)$$

$$[5] \quad ds^2 = \left[ c_1(x - y) + c_2(x^2 - y^2) + c_3(x^3 - y^3) + c_4(x^4 - y^4) \right] \\ \times (dx^2 - dy^2)$$

# 2D quantum systems-1

For a manifold with metric  $ds^2 = \lambda(x, y)(dx^2 + dy^2)$  the Hamiltonian system

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{\lambda(x, y)} + V(x, y)$$

is replaced by the Hamiltonian (Schrödinger) operator with potential

$$H = \frac{1}{\lambda(x, y)}(\partial_{11} + \partial_{22}) + V(x, y).$$

# 2D quantum systems-2

A second-order symmetry of the Hamiltonian system

$$S = \sum_{k,j=1}^2 a^{kj}(x,y)p_k p_j + W(x,y),$$

with  $a^{kj} = a^{jk}$ , corresponds to the operator

$$S = \frac{1}{\lambda(x,y)} \sum_{k,j=1}^2 \partial_k (a^{kj}(x,y)\lambda(x,y)\partial_j) + W(x,y), \quad a^{kj} = a^{jk}.$$

# 2D quantum systems-3

LEMMA:

$$\{\mathcal{H}, \mathcal{S}\} = 0 \iff [H, S] = 0.$$

**Not generally true for higher dimensional manifolds.**

It follows that the classical results for the space of second order symmetries corresponding to a nondegenerate potential can be taken over without change.

# The 2D quantum quadratic algebra-1

THEOREM: The 6 distinct monomials

$$\{S_{(11)}, S_{(11)}\}, \{S_{(22)}, S_{(22)}\}, \{S_{(12)}, S_{(12)}\}, \{S_{(11)}, S_{(22)}\}, \\ \{S_{(11)}, S_{(12)}\}, \{S_{(12)}, S_{(22)}\},$$

form a basis for the space of fourth order symmetry operators.

THEOREM The 10 distinct symmetrized monomials

$$\{S_{(ii)}, S_{(ii)}, S_{(ii)}\}, \{S_{(ij)}, S_{(ij)}, S_{(ij)}\}, \{S_{(ii)}, S_{(ii)}, S_{(jj)}\}, \\ \{S_{(ii)}, S_{(ii)}, S_{(ij)}\}, \{S_{(ij)}, S_{(ij)}, S_{(ii)}\}, \{S_{(11)}, S_{(12)}, S_{(22)}\},$$

for  $i, j = 1, 2$ ,  $i \neq j$  form a basis for the space of sixth order symmetries.

# The 2D quantum quadratic algebra-2

These theorems establish the closure of the quadratic algebra for 2D quantum superintegrable potentials: All fourth order and sixth order symmetry operators can be expressed as symmetric polynomials in the second order symmetry operators.

# 3D summary-1

We have classified all 2D superintegrable systems, including those with degenerate potentials.

The integrability condition approach that works for systems on 2D Riemannian manifolds extends to 3D conformally flat spaces ( $2n-1=5$  functionally independent constants of the motion), with complications.

$5 \implies 6$  Theorem. 5 functionally independent second order symmetries for a nondegenerate superintegrable 3D system imply 6 linearly independent second order symmetries.

For 3D systems with nondegenerate potential the maximum possible dimensions of the spaces of second, third fourth and sixth order symmetries are 6,4,26 and 56 respectively and these dimensions are achieved



# 3D summary-2

The 3D quadratic algebra always closes at level 6 and there is a standard structure for the algebra.

The passage from the 3D conformally flat classical superintegrable systems to quantum superintegrable systems is still straightforward, but requires modifying the quantum potential by an additive term that is the scalar curvature.

All 3D conformally flat superintegrable systems with nondegenerate potential are Stäckel equivalent to superintegrable systems in 3D Euclidean space and on the 3-sphere.

All 3D conformally flat superintegrable systems have “essentially” been classified.

# Conformally flat systems-1

An  $n$ -dimensional Riemannian space is conformally flat if and only if it admits a set of local coordinates  $x_1, \dots, x_n$  such that the contravariant metric tensor takes the form  $g^{ij} = \delta^{ij} / \lambda(\mathbf{x})$ , i.e., the metric is  $ds^2 = \lambda(\mathbf{x})(\sum_{i=1}^n dx_i^2)$ .

A classical superintegrable system  $\mathcal{H} = \sum_{ij} g^{ij} p_i p_j + V(\mathbf{x})$  on this manifold is one that admits  $2n - 1$  functionally independent generalized symmetries (or constants of the motion)  $\mathcal{S}_k$ ,  $k = 1, \dots, 2n - 1$  with  $\mathcal{S}_1 = \mathcal{H}$ . That is,  $\{\mathcal{H}, \mathcal{S}_k\} = 0$  where

$$\{f, g\} = \sum_{j=1}^n (\partial_{x_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{x_j} g)$$

is the Poisson bracket for functions  $f(\mathbf{x}, \mathbf{p}), g(\mathbf{x}, \mathbf{p})$  on phase space .

# Conformally flat systems-2

Note that  $2n - 1$  is the maximum possible number of functionally independent symmetries and, locally, such symmetries always exist. The main interest is in symmetries that are polynomials in the  $p_k$  and are globally defined, except for lower dimensional singularities such as poles and branch points. Here we concentrate on second-order superintegrable systems, that is those in which the symmetries take the form  $\mathcal{S} = \sum a^{ij}(\mathbf{x})p_i p_j + W(\mathbf{x})$ , quadratic in the momenta.

# Conformally flat systems-3

An analogous definition for second-order quantum superintegrable systems with Schrödinger operator

$$H = \Delta + V(\mathbf{x}), \quad \Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} g^{ij}) \partial_{x_j},$$

the Laplace-Beltrami operator plus a potential function. Here there are  $2n - 1$  second-order symmetry operators

$$S_k = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} a_{(k)}^{ij}) \partial_{x_j} + W^{(k)}(\mathbf{x}), \quad k = 1, \dots, 2n - 1$$

with  $S_1 = H$  and  $[H, S_k] \equiv HS_k - S_kH = 0$ .

# Conformally flat systems-4

**Motivation:** Superintegrable systems can 1) be solved explicitly, and 2) they can be solved in multiple ways. It is the information gleaned from comparing the distinct solutions and expressing one solution set in terms of another that is a primary reason for their interest.

# Example: No quadratic algebra-1

The Euclidean space Schrödinger equation with 3-parameter extended Kepler-Coulomb potential:

$$\left( \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) + \left[ 2E + \frac{2\alpha}{\sqrt{x^2 + y^2 + z^2}} - \left( \frac{\lambda_1}{x^2} + \frac{\lambda_2}{y^2} \right) \right] \Psi = 0$$

# Example: No quadratic algebra-2

This equation admits separable solutions in the four coordinates systems: spherical, sphero-conical, prolate spheroidal and parabolic coordinates. Again the bound states are degenerate and important special function identities arise by expanding one basis of separable eigenfunctions in terms of another.

# Example: No quadratic algebra-3

However, the space of second order symmetries is only 5 dimensional and, although there are useful identities among the generators and commutators that enable one to derive spectral properties algebraically, there is no finite quadratic algebra structure.

The key difference with our first example is that the 3-parameter Kepler-Coulomb potential is **degenerate** and it cannot be extended to a 4-parameter potential.



# Example not of constant curvature-1

Space with metric

$$ds^2 = \lambda(A, B, C, D, E, \mathbf{x})(dx^2 + dy^2 + dz^2),$$

$$\lambda = A(x + iy) + B \left( \frac{3}{4}(x + iy)^2 + \frac{z}{4} \right)$$

$$+ C \left( (x + iy)^3 + \frac{1}{16}(x - iy) + \frac{3z}{4}(x + iy) \right)$$

$$+ D \left( \frac{5}{16}(x + iy)^4 + \frac{z^2}{16} + \frac{1}{16}(x^2 + y^2) + \frac{3z}{8}(x + iy)^2 \right) + E.$$

The nondegenerate potential is

$$V = \lambda(\alpha, \beta, \gamma, \delta, \epsilon, \mathbf{x}) / \lambda(A, B, C, D, E, \mathbf{x}).$$

# Example not of constant curvature-2

If  $A = B = C = D = 0$  this is a classical superintegrable system on complex Euclidean space, and it extends to a quantum superintegrable system.

The quadratic algebra always closes, and for general values of  $A, B, C, D, E$  the space is not of constant curvature. This is an example of a superintegrable system that is Stäckel equivalent to a system on complex Euclidean space.

# 3D Functional linear independence-1

We shed some light on the relationship between functional independence and functional linear independence for the set  $\{\mathcal{H}, \mathcal{S}_1, \dots, \mathcal{S}_4\}$

**THEOREM:** The functionally independent set  $\{\mathcal{H}, \mathcal{S}_1, \dots, \mathcal{S}_4\}$  is also functionally linearly independent in the sense that if the relation  $\sum_{h=0}^4 c^{(h)}(x) \mathcal{L}_h \equiv 0$  holds in an open set, then  $c^{(h)}(x) \equiv 0$  for all  $h$ .

# 3D Nondegenerate Potentials-1

>From the Bertrand-Darboux equations for the 5 functionally independent constants of the motion we can derive conditions on the potential of the form

$$V_{22} = V_{11} + A^{22}V_1 + B^{22}V_2 + C^{22}V_3,$$

$$V_{33} = V_{11} + A^{33}V_1 + B^{33}V_2 + C^{33}V_3,$$

$$V_{12} = A^{12}V_1 + B^{12}V_2 + C^{12}V_3$$

$$V_{13} = A^{13}V_1 + B^{13}V_2 + C^{13}V_3$$

$$V_{23} = A^{23}V_1 + B^{23}V_2 + C^{23}V_3$$

# 3D Nondegenerate Potentials-2

It is possible that there might be additional conditions of the form  $D_{(s)}^1 V_1 + D_{(s)}^2 V_2 + D_{(s)}^3 V_3 = 0$ ,  $s = 1, \dots, r$ . Here the  $A^{ij}, B^{ij}, C^{ij}, D_{(s)}^i$  are functions of  $x$  that can be calculated explicitly.

Suppose there are no such additional conditions and the integrability conditions are satisfied identically. In this case we say that the potential is *nondegenerate*. Otherwise the potential is *degenerate*.

# 3D Nondegenerate Potentials-3

If  $V$  is nondegenerate then at any point  $\mathbf{x}_0$ , where the  $A^{ij}, B^{ij}, C^{ij}$  are defined and analytic, there is a unique solution  $V(\mathbf{x})$  arbitrarily prescribed values of  $V_1(\mathbf{x}_0), V_2(\mathbf{x}_0), V_3(\mathbf{x}_0), V_{11}(\mathbf{x}_0)$  (as well as the value of  $V(\mathbf{x}_0)$  itself). The points  $\mathbf{x}_0$  are called *regular*. The points of singularity for the  $A^{ij}, B^{ij}, C^{ij}$  form a manifold of dimension  $< 3$ .

# 3D Nondegenerate Potentials-4

*Degenerate* potentials depend on fewer parameters.

This occurs for the generalized Kepler-Coulomb potential where the integrability conditions lead to an additional equation of the form  $V_{11} = A^{11}V_1 + B^{11}V_2 + C^{11}V_3$  so that  $V_{11}$  cannot be prescribed arbitrarily.

# Observed common features-1

They are usually multiseparable.

The eigenfunctions of one separable system can be expanded in terms of the eigenfunctions of another. This is the source of nontrivial special function expansion theorems.

The symmetry operators are in formal self-adjoint form and suitable for spectral analysis.



# Observed common features-2

The quadratic algebra identities allow us to relate eigenbases and eigenvalues of one symmetry operator to those of another. Indeed the representation theory of the abstract quadratic algebra can be used to derive spectral properties of the second order generators in a manner analogous to the use of Lie algebra representation theory to derive spectral properties of quantum systems that admit Lie symmetry algebras. (Note however that for superintegrable systems with nondegenerate potential, there is no first order Lie symmetry.)

# Observed common features-3

Another common feature of quantum superintegrable systems is that they can be modified by a gauge transformation so that the Schrödinger and symmetry operators are acting on a space of polynomials. This is closely related to the theory of exactly and quasi-exactly solvable systems. The characterization of ODE quasi-exactly solvable systems as embedded in PDE superintegrable systems provides considerable insight into the nature of these phenomena.

Classical analogs are obtained by the replacements  $\partial_{x_i} \rightarrow p_{x_i}$ . Commutators go over to Poisson brackets. The operator symmetries become second order constants of the motion. Symmetrized operators become products. The highest order terms in the quadratic algebra relations agree with the operator case but there are fewer nonzero lower order terms

# 3D complication-1

For 2D nondegenerate superintegrable systems we can show that the 3 functionally independent constants of the motion are (with one exception) also linearly independent, so at each regular point we can find a unique constant of the motion that matches a quadratic expression in the momenta at that point.

However, for 3D systems we have only 5 functionally independent constants of the motion and the quadratic forms span a 6 dimensional space.

# 3D complication-2

This is a major problem. However, for functionally linearly independent systems with nondegenerate potentials we can prove the “5  $\implies$  6 Theorem” to show that the space of second order constants of the motion is in fact 6 dimensional: there is a symmetry that is functionally dependent on the symmetries that arise from superintegrability, but linearly independent of them.

With that result established, the treatment of the 3D case proceeds in analogy with the nondegenerate 2D case.

# 3D conformally flat system-1

Assume a coordinate system  $x, y, z$  and function  $\lambda(x, y, z) = \exp[G(x, y, z)]$  such that the Hamiltonian is

$$\mathcal{H} = \frac{p_1^2 + p_2^2 + p_3^2}{\lambda} + V(x, y, z).$$

A quadratic constant of the motion (or generalized symmetry)

$$\mathcal{S} = \sum_{k,j=1}^3 a^{kj}(x, y, z) p_k p_j + W(x, y, z) \equiv \mathcal{L} + W, \quad a^{jk} = a^{kj}$$

must satisfy  $\{\mathcal{H}, \mathcal{S}\} = 0$ .

# 3D conformally flat system-2

The symmetry conditions are

$$a_i^{ii} = -G_1 a^{1i} - G_2 a^{2i} - G_3 a^{3i}$$

$$2a_i^{ij} + a_j^{ii} = -G_1 a^{1j} - G_2 a^{2j} - G_3 a^{3j}, \quad i \neq j$$

$$(2) \quad a_k^{ij} + a_j^{ki} + a_i^{jk} = 0, \quad i, j, k \text{ distinct}$$

and

$$(3) \quad W_k = \lambda \sum_{s=1}^3 a^{sk} V_s, \quad k = 1, 2, 3.$$

(Here a subscript  $j$  denotes differentiation with respect to  $x_j$ .)

# 3D conformally flat system-3

The requirement  $\partial_{x_\ell} W_j = \partial_{x_j} W_\ell$ ,  $\ell \neq j$  leads to the second order Bertrand-Darboux partial differential equations for the potential.

$$(4) \quad \sum_{s=1}^3 \left[ V_{sj} \lambda a^{s\ell} - V_{s\ell} \lambda a^{sj} + V_s \left( (\lambda a^{s\ell})_j - (\lambda a^{sj})_\ell \right) \right] = 0.$$

# 5 functionally independent symmetries

Require the the Hamilton-Jacobi equation admits four additional constants of the motion:

$$(5) \mathcal{S}_h = \sum_{j,k=1}^3 \alpha_{(h)}^{jk} p_k p_j + W_{(h)} = \mathcal{L}_h + W_{(h)}, \quad h = 1, \dots, 4.$$

We assume that the four functions  $\mathcal{S}_h$  together with  $\mathcal{H}$  are functionally independent in the six-dimensional phase space. (This means that we require that the five quadratic forms  $\mathcal{L}_h, \mathcal{H}_0$  are functionally independent.)



# 3D Functional linear independence-1

THEOREM: The functionally independent set  $\{\mathcal{H}, \mathcal{S}_1, \dots, \mathcal{S}_4\}$  may, or may not, be functionally linearly independent. However, for systems with nondegenerate potentials, we have the following. Let

$$\mathcal{S}_h = \sum_{j,k=1}^3 a_{(h)}^{jk} p_k p_j + W_{(h)} \quad h = 1, \dots, 5, \quad \mathcal{H} = \mathcal{S}_1$$

be symmetries for a system with nondegenerate potential  $V = W_{(1)}$ . Then these symmetries are functionally independent if and only if they are functionally linearly independent.

# 3D Nondegenerate Potential

>From the Bertrand-Darboux equations for the 5 functionally independent constants of the motion we can derive conditions on the potential of the form

$$V_{22} = V_{11} + A^{22}V_1 + B^{22}V_2 + C^{22}V_3,$$

$$V_{33} = V_{11} + A^{33}V_1 + B^{33}V_2 + C^{33}V_3,$$

$$V_{12} = A^{12}V_1 + B^{12}V_2 + C^{12}V_3$$

$$V_{13} = A^{13}V_1 + B^{13}V_2 + C^{13}V_3$$

$$V_{23} = A^{23}V_1 + B^{23}V_2 + C^{23}V_3$$

# Potential integrability conditions -1

We can further clarify the situation by introducing the dependent variables  $W^{(1)} = V_1$ ,  $W^{(2)} = V_2$ ,  $W^{(3)} = V_3$ ,  $W^{(4)} = V_{11}$ , the vector

$$(6) \quad \mathbf{w} = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(3)} \\ W^{(4)} \end{pmatrix},$$

and the matrices

$$(7) \quad \mathbf{A}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ A^{12} & B^{12} & C^{12} & 0 \\ A^{13} & B^{13} & C^{13} & 0 \\ A^{14} & B^{14} & C^{14} & B^{12} - A^{22} \end{pmatrix},$$

# Potential integrability conditions -2

Then the integrability conditions for the system

$$(10) \quad \partial_{x_j} \mathbf{w} = \mathbf{A}^{(j)} \mathbf{w} \quad j = 1, 2, 3.$$

must hold:

$$(11) \quad A_i^{(j)} - A_j^{(i)} = A^{(i)} A^{(j)} - A^{(j)} A^{(i)} \equiv [A^{(i)}, A^{(j)}].$$

# Symmetry integrability conditions-1

Using the nondegenerate potential requirements and the 2nd order Killing equations we can express each of the 18 quantities  $a_k^{ij}$ ,  $i \leq j$ , in terms of the quadratic coefficients  $a^{\ell m}$ . A typical example is

$$a_3^{22} = \frac{1}{3} \left[ -2a^{23}(B^{33} - B^{22}) + 2(a^{33} - a^{22})B^{23} \right. \\ \left. + 2a^{13}B^{12} - 2a^{12}B^{13} \right]$$

# Symmetry integrability conditions-2

Since this system of first order partial differential equations is involutive the general solution for the 6 functions  $a^{jk}$  can depend on at most 6 parameters, the values  $a^{jk}(\mathbf{x}_0)$  at a fixed regular point  $\mathbf{x}_0$ .

For the integrability conditions we define the vector-valued function

$$\mathbf{h}(x, y, z) = (a^{11}, a^{12}, a^{13}, a^{22}, a^{23}, a^{33})$$

and directly compute the  $6 \times 6$  matrix functions  $\mathcal{A}^{(j)}$  to get the first-order system

$$(12) \quad \partial_{x_j} \mathbf{h} = \mathcal{A}^{(j)} \mathbf{h} \quad j = 1, 2, 3.$$

# Symmetry integrability conditions-2

In terms of the  $6 \times 6$  matrices

$$\mathcal{S}^{(1)} = \mathcal{A}_2^{(3)} - \mathcal{A}_3^{(2)} - [\mathcal{A}^{(2)}, \mathcal{A}^{(3)}], \quad \mathcal{S}^{(2)} = \mathcal{A}_3^{(1)} - \mathcal{A}_1^{(3)} - [\mathcal{A}^{(3)}, \mathcal{A}^{(1)}],$$

$$\mathcal{S}^{(3)} = \mathcal{A}_1^{(2)} - \mathcal{A}_2^{(1)} - [\mathcal{A}^{(1)}, \mathcal{A}^{(2)}],$$

the integrability conditions are

$$(13) \quad \mathcal{S}^{(1)} \mathbf{h} = \mathcal{S}^{(2)} \mathbf{h} = \mathcal{S}^{(3)} \mathbf{h} = 0$$

# (5 $\implies$ 6)-1

**THEOREM:** Let  $V$  be a nondegenerate potential corresponding to a conformally flat space in 3 dimensions that is superintegrable, and there are 5 functionally independent constants of the motion. Then the space of second order symmetries for the Hamiltonian  $\mathcal{H} = (p_x^2 + p_y^2 + p_z^2)/\lambda(x, y, z) + V(x, y, z)$  (excluding multiplication by a constant) is of dimension  $D = 6$ .



# $(5 \implies 6)$ -2

**COROLLARY:** If  $\mathcal{H} + V$  is a superintegrable conformally flat system with nondegenerate potential, then the dimension of the space of 2nd order symmetries

$$\mathcal{S} = \sum_{k,j=1}^3 a^{kj}(x, y, z) p_k p_j + W(x, y, z)$$

is 6. At any regular point  $(x_0, y_0, z_0)$  and given constants  $\alpha^{kj} = \alpha^{jk}$  there is exactly one symmetry  $\mathcal{S}$  such that  $a^{kj}(x_0, y_0, z_0) = \alpha^{kj}$ . Given a set of 5 functionally independent 2nd order symmetries  $\mathcal{L} = \{\mathcal{S}_\ell : \ell = 1, \dots, 5\}$ , there is always a 6th second order symmetry  $\mathcal{S}_6$  that is functionally dependent on  $\mathcal{L}$ , but linearly independent.

# 3D third order constants-1

**THEOREM:** Let  $\mathcal{K}$  be a third order constant of the motion for a system with nondegenerate potential  $V$ :

$$\mathcal{K} = \sum_{k,j,i=1}^3 a^{kji}(x, y, z) p_k p_j p_i + \sum_{\ell=1}^3 b^{\ell}(x, y, z) p_{\ell}.$$

Then  $b^{\ell}(x, y, z) = \sum_{j=1}^3 f^{\ell,j}(x, y, z) V_j(x, y, z)$  with  $f^{\ell,j} + f^{j,\ell} = 0$ ,  $1 \leq \ell, j \leq 3$ . The  $a^{ijk}, b^{\ell}$  are uniquely determined by the four numbers  $f^{1,2}, f^{1,3}, f^{2,3}, f_3^{1,2}$  at any regular point  $(x_0, y_0, z_0)$  of  $V$ .

# 3D third order constants-2

Let

$$\mathcal{S}_1 = \sum a_{(1)}^{kj} p_k p_j + W_{(1)}, \quad \mathcal{S}_2 = \sum a_{(2)}^{kj} p_k p_j + W_{(2)}$$

be second order constants of the the motion for a superintegrable system with nondegenerate potential and let  $\mathcal{A}_{(i)}(x, y, z) = \{a_{(i)}^{kj}(x, y, z)\}$ ,  $i = 1, 2$  be  $3 \times 3$  matrix functions. Then the Poisson bracket is given by

$$\{\mathcal{S}_1, \mathcal{S}_2\} = \sum_{k,j,i=1}^3 a^{kji}(x, y, z) p_k p_j p_i + b^\ell(x, y, z) p_\ell$$

$$(14) \quad f^{k,\ell} = 2\lambda \sum_j (a_{(2)}^{kj} a_{(1)}^{j\ell} - a_{(1)}^{kj} a_{(2)}^{j\ell}).$$

# 3D third order constants-3

$\{\mathcal{S}_1, \mathcal{S}_2\}$  is uniquely determined by the skew-symmetric matrix

$$[\mathcal{A}_{(2)}, \mathcal{A}_{(1)}] \equiv \mathcal{A}_{(2)}\mathcal{A}_{(1)} - \mathcal{A}_{(1)}\mathcal{A}_{(2)},$$

hence by the constant matrix

$[\mathcal{A}_{(2)}(x_0, y_0, z_0), \mathcal{A}_{(1)}(x_0, y_0, z_0)]$  evaluated at a regular point, and by the number  $\mathcal{F}(x_0, y_0, z_0) = f_3^{1,2}(x_0, y_0, z_0)$ .

# 3D standard structure-1

Allows identification of the space of second order constants of the motion with the space  $S_3$  of  $3 \times 3$  symmetric matrices, as well as identification of the space of third order constants of the motion with a subspace of the space  $K_3 \times F$  of  $3 \times 3$  skew-symmetric matrices  $K_3$ . crossed with the line.  $F = \{\mathcal{F}(\mathbf{x}_0)\}$ .

If  $\mathbf{x}_0$  is a regular point then there is a 1 – 1 linear correspondence between second order symmetries  $\mathcal{S}$  and their associated symmetric matrices  $\mathcal{A}_{(\mathbf{x}_0)}$ . Let  $\{\mathcal{S}_1, \mathcal{S}_2\}' = \{\mathcal{S}_2, \mathcal{S}_1\}$  be the reversed Poisson bracket. Then the map

$$\{\mathcal{S}_1, \mathcal{S}_2\}' \iff [\mathcal{A}_{(1)}(\mathbf{x}_0), \mathcal{A}_{(2)}(\mathbf{x}_0)]$$

is an algebraic homomorphism.

# 3D standard structure-2

$\mathcal{S}_1, \mathcal{S}_2$  are in involution if and only if matrices  $\mathcal{A}_{(1)}(\mathbf{x}_0), \mathcal{A}_{(2)}(\mathbf{x}_0)$  commute and  $\mathcal{F}(\mathbf{x}_0) = 0$ .

If  $\{\mathcal{S}_1, \mathcal{S}_2\} \neq 0$  then it is a third order symmetry and can be uniquely associated with the skew-symmetric matrix  $[\mathcal{A}_{(1)}(\mathbf{x}_0), \mathcal{A}_{(2)}(\mathbf{x}_0)]$  and the parameter  $\mathcal{F}(\mathbf{x}_0)$ .

# 3D standard structure-3

Let  $\mathcal{E}^{ij}$  be the  $3 \times 3$  matrix with a 1 in row  $i$ , column  $j$  and 0 for every other matrix element. Then the symmetric matrices

$$(15) \quad \mathcal{A}^{(ij)} = \frac{1}{2}(\mathcal{E}^{ij} + \mathcal{E}^{ji}) = \mathcal{A}^{(ji)}, \quad i, j = 1, 2, 3$$

form a basis for the 6-dimensional space of symmetric matrices. Moreover,

$$(16) \quad [\mathcal{A}^{(ij)}, \mathcal{A}^{(kl)}] = \frac{1}{2} \left( \delta_{jk} \mathcal{B}^{(il)} + \delta_{jl} \mathcal{B}^{(ik)} + \delta_{ik} \mathcal{B}^{(jl)} + \delta_{il} \mathcal{B}^{(jk)} \right)$$

where

$$\mathcal{B}^{(ij)} = \frac{1}{2}(\mathcal{E}^{ij} - \mathcal{E}^{ji}) = -\mathcal{B}^{(ji)}, \quad i, j = 1, 2, 3.$$

# 3D standard basis

We can define a standard set of basis symmetries

$\mathcal{S}^{(jk)} = \sum a^{ij}(\mathbf{x})p_i p_j + W^{(ij)}(\mathbf{x})$  corresponding to a regular point  $\mathbf{x}_0$  by

$$\frac{1}{\lambda} \begin{pmatrix} f_1^1 & f_2^1 & f_3^1 \\ f_1^2 & f_2^2 & f_3^2 \\ f_1^3 & f_2^3 & f_3^3 \end{pmatrix}_{\mathbf{x}_0} = \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix}_{\mathbf{x}_0} = \mathcal{A}^{(jk)},$$

$W^{(jk)}(\mathbf{x}_0) = 0$ . The condition on  $W^{(jk)}$  is actually 4 conditions since  $W^{(jk)}$  depends on 4 parameters.



# 3D maximum dimensions-1

In order to demonstrate the structure of the quadratic algebras for 3D superintegrable systems on conformally flat spaces, it is important to compute the dimensions of the spaces of symmetries of these systems that are of orders 4 and 6. These symmetries are necessarily of a special type. The highest order terms in the momenta are independent of the parameters in the potential, while the terms of order 2 less in the momenta are linear in these parameters, those of order 4 less are quadratic, and those of order 6 less are cubic. We obtain these dimensions exactly, but first we need to establish sharp upper bounds.

# 3D maximum dimensions-2

## THEOREM:

The maximum possible dimension of the space of purely fourth order symmetries for a nondegenerate potential is 21.

The maximal possible dimension of the space of purely sixth order symmetries is 56.

# Bases for constants of the motion-1

For a superintegrable system with nondegenerate potential, the dimension of the space of purely fourth order constants of the motion is at most 21. Note that at any regular point  $\mathbf{x}_0$ , we can define a standard basis of 6 second order constants of the motion  $\mathcal{S}^{(ij)} = A^{(ij)} + W^{(ij)}$  where the quadratic form  $A^{(ij)}$  has matrix  $\mathcal{A}^{(ij)}$  and  $W^{(ij)}$  is the potential term with  $W^{(ij)}(\mathbf{x}_0) \equiv 0$  identically in the parameters  $W^{(\alpha)}$ . By taking homogeneous polynomials of order two in the standard basis symmetries we can construct fourth order symmetries.

# Bases for constants of the motion-1

THEOREM: The 21 distinct standard monomials  $\mathcal{S}^{(ij)}\mathcal{S}^{(jk)}$ , defined with respect to a regular point  $\mathbf{x}_0$ , form a basis for the space of fourth order symmetries.

# Bases for constants of the motion-2

The dimension of the space of purely sixth order constants of the motion is at most 56. Again we shall show that the 56 independent homogeneous third order polynomials in the symmetries  $\mathcal{S}^{(ij)}$  form a basis for this space. At the sixth order level we have the symmetries

$$(\mathcal{S}^{(ii)})^3, (\mathcal{S}^{(ii)})^2 \mathcal{S}^{(ij)}, (\mathcal{S}^{(ii)})^2 \mathcal{S}^{(jj)}, (\mathcal{S}^{(ii)})^2 \mathcal{S}^{(jk)}$$

for  $i, j, k = 1, \dots, 3$   $i, j, k$  pairwise distinct. (18 possibilities)

$$\mathcal{S}^{(ii)} \mathcal{S}^{(ij)} \mathcal{S}^{(jj)}, \mathcal{S}^{(ii)} \mathcal{S}^{(ij)} \mathcal{S}^{(jk)}, \mathcal{S}^{(ii)} \mathcal{S}^{(jj)} \mathcal{S}^{(kk)},$$

for  $i, j, k = 1, \dots, 3$   $i, j, k$  pairwise distinct. (10 possibilities)

# Bases for constants of the motion-3

$$\mathcal{S}^{(\ell m)} \left( \mathcal{S}^{(ii)} \mathcal{S}^{(jj)} - (\mathcal{S}^{(ij)})^2 \right)$$

for  $i, j = 1, \dots, 3$   $i, j$  pairwise distinct. (10 possibilities)

$$\mathcal{S}^{(\ell m)} \left( \mathcal{S}^{(ij)} \mathcal{S}^{(ik)} - \mathcal{S}^{(ii)} \mathcal{S}^{(jk)} \right)$$

for  $i, j, k = 1, \dots, 3$   $i, j, k$  pairwise distinct. (18 possibilities)

# Bases for constants of the motion-4

**THEOREM:** The 56 distinct standard monomials  $\mathcal{S}^{(hi)} \mathcal{S}^{(jk)} \mathcal{S}^{(\ell m)}$ , defined with respect to a regular  $\mathbf{x}_0$ , form a basis for the space of sixth order symmetries.

We conclude that the quadratic algebra closes.

# Classical 3D Stäckel transform-1

Suppose we have a superintegrable system

$$H = \frac{p_1^2 + p_2^2 + p_3^2}{\lambda(x, y, z)} + V(x, y, z)$$

in local orthogonal coordinates, with nondegenerate potential  $V(x, y, z)$ :

$$\begin{aligned} V_{33} &= V_{11} + A^{33}V_1 + B^{33}V_2 + C^{33}V_3, \\ V_{22} &= V_{11} + A^{22}V_1 + B^{22}V_2 + C^{22}V_3, \\ V_{23} &= A^{23}V_1 + B^{23}V_2 + C^{23}V_3, \\ V_{13} &= A^{13}V_1 + B^{13}V_2 + C^{13}V_3, \\ V_{12} &= A^{12}V_1 + B^{12}V_2 + C^{12}V_3 \end{aligned}$$

and  $U(x, y, z)$  is a particular solution of the potential eqns.



# Classical 3D Stäckel transform-2

Then the transformed system

$$(17) \quad \tilde{H} = \frac{p_1^2 + p_2^2 + p_3^2}{\tilde{\lambda}(x, y, z)} + \tilde{V}(x, y, z)$$

with nondegenerate potential  $\tilde{V}(x, y, z)$ :

$$(18) \quad \begin{aligned} \tilde{V}_{33} &= \tilde{V}_{11} + \tilde{A}^{33}\tilde{V}_1 + \tilde{B}^{33}\tilde{V}_2 + \tilde{C}^{33}\tilde{V}_3, \\ \tilde{V}_{22} &= \tilde{V}_{11} + \tilde{A}^{22}\tilde{V}_1 + \tilde{B}^{22}\tilde{V}_2 + \tilde{C}^{22}\tilde{V}_3, \\ \tilde{V}_{23} &= \tilde{A}^{23}\tilde{V}_1 + \tilde{B}^{23}\tilde{V}_2 + \tilde{C}^{23}\tilde{V}_3, \\ \tilde{V}_{13} &= \tilde{A}^{13}\tilde{V}_1 + \tilde{B}^{13}\tilde{V}_2 + \tilde{C}^{13}\tilde{V}_3, \\ \tilde{V}_{12} &= \tilde{A}^{12}\tilde{V}_1 + \tilde{B}^{12}\tilde{V}_2 + \tilde{C}^{12}\tilde{V}_3, \end{aligned}$$

is also superintegrable,

# Classical 3D Stäckel transform-3

where

$$\tilde{\lambda} = \lambda U, \quad \tilde{V} = \frac{V}{U},$$

$$\tilde{A}^{33} = A^{33} + 2\frac{U_1}{U}, \quad \tilde{B}^{33} = B^{33}, \quad \tilde{C}^{33} = C^{33} - 2\frac{U_3}{U},$$

etc.

Let  $S = \sum a^{ij} p_i p_j + W = S_0 + W$  be a second order symmetry of  $H$  and  $S_U = \sum a^{ij} p_i p_j + W_U = S_0 + W_U$  be the special case that is in involution with  $\frac{p_1^2 + p_2^2 + p_3^2}{\lambda} + U$ . Then

$$\tilde{S} = S_0 - \frac{W_U}{U} H + \frac{1}{U} H$$

is the corresponding symmetry of  $\tilde{H}$ .

# 3D multiseparability-1

From the general theory of variable separation for Hamilton-Jacobi equations we know that second order symmetries  $\mathcal{S}_1, \mathcal{S}_2$  define a separable system for the equation

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{\lambda(x, y, z)} + V(x, y, z) = E$$

if and only if

The symmetries  $\mathcal{H}, \mathcal{S}_1, \mathcal{S}_2$  form a linearly independent set as quadratic forms.

$$\{\mathcal{S}_1, \mathcal{S}_2\} = 0.$$

The three quadratic forms have a common eigenbasis of differential forms.

# 3D multiseparability-2

For nondegenerate superintegrable potentials in a conformally flat space we see that

$$\{\mathcal{S}_1, \mathcal{S}_2\} = 0 \iff [\mathcal{A}_{(2)}(\mathbf{x}), \mathcal{A}_{(1)}(\mathbf{x})] = 0$$

so that the intrinsic conditions for the existence of a separable coordinate system are simplified.

**THEOREM:** Let  $V$  be a superintegrable nondegenerate potential in a 3D conformally flat space. Then  $V$  defines a multiseparable system.

# 3D Stäckel-1

In separation of variables theory we obtained the result.  
THEOREM: Let  $u_1, u_2, u_3$  be an orthogonal separable coordinate system for a 3D flat space with metric  $d\tilde{s}^2$ . Then there is a function  $f$  such that

$$f d\tilde{s}^2 = ds^2$$

where  $ds^2$  is a constant curvature space metric and  $d\tilde{s}^2$  is orthogonally separable in exactly these same coordinates  $u_1, u_2, u_3$ . The function  $f$  is a Stäckel multiplier with respect to this coordinate system.

Thus the possible conformally flat separable coordinate systems are obtained, modulo a Stäckel multiplier, from separable systems on 3D flat space or on the 3-sphere.

# 3D Stäckel-2

COROLLARY: Let  $V$  be a superintegrable nondegenerate potential in a 3D conformally flat space. Then there is a continuous 1-parameter (or multi-parameter) family of separable systems for  $V$ , spanning at least a 5-dimensional subspace of symmetries.

# 3D Stäckel-3

**THEOREM:** Every superintegrable system with nondegenerate potential on a 3D conformally flat space is Stäckel equivalent to a superintegrable system on either 3D flat space or the 3-sphere.

# Generic Euclidean separable systems

There are 7 families of separable complex Euclidean coordinate systems depending on nontrivial parameters. Example: Jacobian elliptical coordinates.



# 3D Euclidean classification

THEOREM: Each of the 7 “generic” Euclidean separable systems determines a unique nondegenerate superintegrable system that permits separation simultaneously for all values of the scaling parameter  $c$  and any other defining parameters  $e_j$ . For each of these systems there is a basis of 5 (strongly) functionally independent and 6 linearly independent second order symmetries. The corresponding nondegenerate potentials and basis of symmetries are:

# I [2111]

$$V = \frac{\alpha_1}{x^2} + \frac{\alpha_2}{y^2} + \frac{\alpha_3}{z^2} + \delta(x^2 + y^2 + z^2),$$

$$\mathcal{P}_i = \partial_{x_i}^2 + \delta x_i^2 + \frac{\alpha_i}{x_i^2}, \quad \mathcal{J}_{ij} = (x_i p_{x_j} - x_j p_{x_i})^2 + \alpha_i^2 \frac{x_j^2}{x_i^2} + \alpha_j^2 \frac{x_i^2}{x_j^2},$$

$$i \geq j.$$

# II [221]

$$V = \alpha(x^2 + y^2 + z^2) + \beta \frac{x - iy}{(x + iy)^3} + \frac{\gamma}{(x + iy)^2} + \frac{\delta}{z^2},$$

$$\mathcal{S}_1 = J \cdot J + f_1, \quad \mathcal{S}_2 = p_z^2 + f_2, \quad \mathcal{S}_3 = J_3^2 + f_3,$$

$$\mathcal{S}_4 = (p_x + ip_y)^2 + f_4, \quad L_5 = (J_2 - iJ_1)^2 + f_5.$$

# III [23]

$$V = \alpha(x^2 + y^2 + z^2) + \frac{\beta}{(x + iy)^2} + \frac{\gamma z}{(x + iy)^3} + \frac{\delta(x^2 + y^2 - 3z^2)}{(x + iy)^4},$$

$$\mathcal{S}_1 = J \cdot J + f_1, \quad \mathcal{S}_2 = (J_2 - iJ_1)^2 + f_2, \quad \mathcal{S}_3 = J_3(J_2 - iJ_1) + f_3,$$

$$\mathcal{S}_4 = (p_x + ip_y)^2 + f_4, \quad \mathcal{S}_5 = p_z(p_x + ip_y) + f_5.$$

# IV [311]

$$V = \alpha(4x^2 + y^2 + z^2) + \beta x + \frac{\gamma}{y^2} + \frac{\delta}{z^2},$$

$$\mathcal{S}_1 = p_x^2 + f_1, \quad \mathcal{S}_2 = p_y^2 + f_2, \quad \mathcal{S}_3 = p_z J_2 + f_3,$$

$$\mathcal{S}_4 = p_y J_3 + f_4, \quad \mathcal{S}_5 = J_1^2 + f_5.$$

# V [32]

$$V = \alpha(4x^2 + y^2 + z^2) + \beta x + \frac{\gamma}{(y + iz)^2} + \frac{\delta(y - iz)}{(y + iz)^3},$$

$$\mathcal{S}_1 = p_x^2 + f_1, \quad \mathcal{S}_2 = J_1^2 + f_2, \quad \mathcal{S}_3 = (p_z - ip_y)(J_2 + iJ_3) + f_3,$$

$$\mathcal{S}_4 = p_z J_2 - p_y J_3 + f_4, \quad \mathcal{S}_5 = (p_z - ip_y)^2 + f_5.$$

# VI [41]

$$V = \alpha (z^2 - 2(x - iy)^3 + 4(x^2 + y^2)) + \beta (2(x + iy) - 3(x - iy)^2) \\ + \gamma(x - iy) + \frac{\delta}{z^2},$$

$$\mathcal{S}_1 = (p_x - ip_y)^2 + f_1, \quad \mathcal{S}_2 = p_z^2 + f_2, \quad \mathcal{S}_3 = p_z(J_2 + iJ_1) + f_3,$$

$$\mathcal{S}_4 = J_3(p_x - ip_y) - \frac{i}{4}(p_x + ip_y)^2 + f_4, \quad \mathcal{S}_5 = (J_2 + iJ_1)^2 + 4ip_z J_1 + f_5.$$

# VII [5]

$$V = \alpha(x + iy) + \beta\left(\frac{3}{4}(x + iy)^2 + \frac{1}{4}z\right) +$$

$$\gamma((x + iy)^3 + \frac{1}{16}(x - iy) + \frac{3}{4}(x + iy)z)$$

$$+ \delta\left(\frac{5}{16}(x + iy)^4 + \frac{1}{16}(x^2 + y^2 + z^2) + \frac{3}{8}(x + iy)^2 z\right),$$

$$\mathcal{S}_1 = (J_1 + iJ_2)^2 + 2iJ_1(p_x + ip_y) - J_2(p_x + ip_y) + \frac{1}{4}(p_y^2 - p_z^2) - iJ_3p_z + f_1,$$

$$\mathcal{S}_2 = J_2p_z - J_3p_y + i(J_3p_x - J_1p_z) - \frac{i}{2}p_y p_z + f_2, \quad \mathcal{S}_3 = (p_x + ip_y)^2 + f_4,$$

$$\mathcal{S}_4 = J_3p_z + iJ_1p_y + iJ_2p_x + 2J_1p_x + \frac{i}{4}p_z^2 + f_3, \quad \mathcal{S}_5 = p_z(p_x + ip_y) + f_5.$$



# Generic systems are unique

THEOREM: A 3D Euclidean nondegenerate superintegrable system admits separation in a special case of the generic coordinates [2111], [221], [23], [311], [32], [41] or [5], respectively, if and only if it is equivalent via a Euclidean transformation to system [I], [II], [III], [IV], [V], [VI] or [VII], respectively.

# Other 3D Euclidean systems

$$[O] \quad V(x, y, z) = \alpha x + \beta y + \gamma z + \delta(x^2 + y^2 + z^2).$$

$$[OO] \quad V(x, y, z) = \frac{\alpha}{2}(x^2 + y^2 + \frac{1}{4}z^2) + \beta x + \gamma y + \frac{\delta}{z^2}.$$

Both of these nondegenerate superintegrable systems are only weakly functionally independent, in contrast to systems [I]-[VII]. Thus we consider [O] and [OO] as associate members of the superintegrable family, not regular members. An investigation of other possible Euclidean systems is in progress.

# 3-sphere generic coordinate systems

There are 5 such systems. Each determines a superintegrable system on the 3-sphere, but 4 are Stäckel equivalent to a system on Euclidean space. The only exception is Jacobi elliptic coordinates on the sphere, that give rise to the nondegenerate potential [VIII]

$$V(\mathbf{s}) = \frac{\alpha}{s_1^2} + \frac{\beta}{s_2^2} + \frac{\gamma}{s_3^2} + \frac{\delta}{s_4^2}.$$

where  $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$ .

# 3-sphere classification

Each of the 5 “generic” 3-sphere separable systems determines a unique nondegenerate superintegrable system that permits separation simultaneously for all values of the parameters  $e_j$ . For each of these systems there is a basis of 5 (strongly) functionally independent and 6 linearly independent second order symmetries.

# 2D quantum superintegrable systems-1

Here we discuss how our analysis of classical 2D superintegrable systems with nondegenerate potentials carries over to the quantum case. The quantization is much simpler in the 2D case than for dimensions greater than 2. For a manifold with metric  $ds^2 = \lambda(x, y)(dx^2 + dy^2)$  the classical Hamiltonian system  $\mathcal{H} = \frac{p_1^2 + p_2^2}{\lambda(x, y)} + V(x, y)$  is replaced by the Hamiltonian (Schrödinger) operator with potential

$$(19) \quad H = \frac{1}{\lambda(x, y)} (\partial_{11} + \partial_{22}) + V(x, y)$$

in local orthogonal coordinates.

# 2D quantum superintegrable systems-2

A second-order symmetry of the Hamiltonian system

$$\mathcal{S} = \sum_{k,j=1}^2 a^{kj}(x,y)p_k p_j + W(x,y), \text{ with } a^{kj} = a^{jk},$$

corresponds to the operator

$$S = \frac{1}{\lambda(x,y)} \sum_{k,j=1}^2 \partial_k (a^{kj}(x,y)\lambda(x,y)\partial_j) + W(x,y), \quad a^{kj} = a^{jk}.$$

The operator is formally self-adjoint with respect to the bilinear product

$$\langle f, g \rangle = \int f(x,y)g(x,y)\lambda(x,y) dx dy$$

on the manifold, i.e.,

$$\langle f, Hg \rangle = \langle Hf, g \rangle, \quad \langle f, Sg \rangle = \langle Sf, g \rangle.$$

# Commutator $[A, B] = AB - BA$

The following results relate the operator commutator and the Poisson bracket.

$$\{\mathcal{H}, \mathcal{S}\} = 0 \iff [H, S] = 0.$$

This result is not generally true for higher dimensional manifolds.

$$\{\mathcal{H}, \mathcal{L}\} = 0 \iff [H, L] = 0.$$

# Carry-over of classical results

The classical results for the space of second order symmetries corresponding to a nondegenerate potential can be taken over without change. The space is 3 dimensional and at any regular point  $\mathbf{x}_0$  there exists exactly one symmetry, up to an additive constant, such that  $a^{jk}(\mathbf{x}_0) = \alpha^{jk}$  for any constant symmetric matrix  $\alpha$ .



# 3rd order quantum symmetries-1

Consider third order differential operators  $K$  that commute with the Hamiltonian:  $[H, K] = 0$ . These equations are much more complicated than their classical analogs.

Simplifications make the problem tractable:

Since the second order symmetries are formally self-adjoint, the commutators will be skew-adjoint. Thus we can limit ourselves to  $K$  that are skew adjoint.

The self-adjoint part of a third order symmetry must be at most a second order symmetry, i.e., the third order terms vanish. For a nondegenerate superintegrable system we already know the 3-dimensional space of these second order symmetries.

# 3rd order quantum symmetries-2

Since  $H$  encodes a 3-parameter family of potentials, the symmetry  $K$  must also be a function of the parameters. The highest order terms  $a^{kji} \partial_{kji}$  in  $K$  (symmetric in  $k, j, i$ ) will be independent of the parameters but lower order terms may have linear parameter dependence.

The skew-adjoint requirement uniquely determines the coefficients of the second order terms in  $K$  and gives all the parameter-independent terms.

The final dimension results are identical to the classical case.

# 3rd order quantum symmetries-3

THEOREM: Let  $V$  be a superintegrable nondegenerate potential, Then the space of third order skew-adjoint symmetries is one-dimensional and is spanned by commutators of the second order self-adjoint symmetries.

# 3rd order quantum symmetries-4

COROLLARY: Let  $V$  be a superintegrable nondegenerate potential and  $S_1, S_2$  be second order formally self-adjoint symmetries with matrices  $\mathcal{A}_{(1)}, \mathcal{A}_{(2)}$ , respectively. Then

$$[S_1, S_2] \equiv 0 \iff [\mathcal{A}_{(1)}, \mathcal{A}_{(2)}] \equiv 0 \iff [\mathcal{A}_{(1)}(\mathbf{x}_0), \mathcal{A}_{(2)}(\mathbf{x}_0)] = 0$$

at a regular point  $\mathbf{x}_0$ .

# 4th order quantum symmetries-1

Similarly, the use of the formal self-adjoint property of the 4th order symmetries allows us to carry over the classical dimensional result. The subspace of purely fourth order polynomial symmetries is at most 6.

# 4th order quantum symmetries-2

If  $A, B$  are linear operators, we define their symmetrized product by

$$\{A, B\} \equiv \frac{1}{2}(AB + BA).$$

**THEOREM:** The 6 distinct monomials

$$\{S^{(11)}, S^{(11)}\}, \{S^{(22)}, S^{(22)}\}, \{S^{(12)}, S^{(12)}\}, \{S^{(11)}, S^{(22)}\}, \\ \{S^{(11)}, S^{(12)}\}, \{S^{(12)}, S^{(22)}\},$$

form a basis for the space of fourth order symmetry operators.

# 6th order quantum symmetries-1

Using an approach very similar to the above we can easily show that the space of truly sixth order formally self-adjoint operator symmetries of  $H$  cannot exceed the classical maximal dimension of 10. It remains to show that the maximum possible dimension is actually achieved. If  $A, B, C$  are linear operators, we define their symmetrized product by

$$\{A, B, C\} \equiv \frac{1}{6}(ABC + BAC + CAB + ACB + BCA + CBA).$$

# 6th order quantum symmetries-2

THEOREM: The 10 distinct monomials

$$\{S^{(ii)}, S^{(ii)}, S^{(ii)}\}, \{S^{(ij)}, S^{(ij)}, S^{(ij)}\}, \{S^{(ii)}, S^{(ii)}, S^{(ij)}\},$$

$$\{S^{(ij)}, S^{(ij)}, S^{(ii)}\},$$

$$\{S^{(11)}, S^{(12)}, S^{(22)}\},$$

for  $i, j = 1, 2$ ,  $i \neq j$  form a basis for the space of sixth order symmetries.



# 6th order quantum symmetries-2

These theorems establish the closure of the quadratic algebra for 2D quantum superintegrable potentials: All fourth order and sixth order symmetry operators can be expressed as symmetric polynomials in the second order symmetry operators.

# 2D quantum Stäckel transform

THEOREM: Every nondegenerate second-order quantum superintegrable system in two variables is Stäckel equivalent to a superintegrable system on a constant curvature space.

# Nondegenerate 3D quantum systems-1

For a manifold with metric  $ds^2 = \lambda(x, y, z)(dx^2 + dy^2 + dz^2)$  we replace the Hamiltonian

$$\mathcal{H} = (p_1^2 + p_2^2 + p_3^2)/\lambda(x, y, z) + V(x, y, z)$$

by a formally self-adjoint operator

$$\hat{H} = \frac{1}{\mu(x, y, z)} \sum_{k,j=1}^3 \partial_k \left( \frac{\delta^{kj}}{\lambda(x, y, z)} \mu(x, y, z) \partial_j \right) + V(x, y, z)$$

in local orthogonal coordinates. Here  $\delta^{kj}$  is the Kronecker delta and the weight function  $\mu$  is to be determined.

# Nondegenerate 3D quantum systems-2

Replace a second-order symmetry of the Hamiltonian system  $\mathcal{S} = \sum_{k,j=1}^3 a^{kj}(x, y, z)p_k p_j + W(x, y, z)$ , with  $a^{kj} = a^{jk}$ , by the formally self-adjoint operator

$$\hat{S} = \frac{1}{\mu} \sum_{k,j=1}^3 \partial_k (a^{kj} \mu \partial_j) + W + \hat{W}, \quad a^{kj} = a^{jk}$$

where  $\hat{W}(x, y, z)$  is to be determined. These operators are formally self-adjoint with respect to the bilinear product

$$\langle f, g \rangle = \int f(x, y, z) g(x, y, z) \mu(x, y, z) dx dy dz$$

# Nondegenerate 3D quantum systems-3

Assume that  $\{\mathcal{H}, \mathcal{S}\} = 0$  and require  $[\hat{H}, \hat{S}] = 0$ . Thus there are functions  $b_i$  such that

$$[\hat{H}, \hat{S}] = \frac{1}{\mu} \sum_{i=1}^3 \partial_i (b^i \mu) = \sum_{i=1}^3 \left( b^i \partial_i + \frac{(b^i \mu)_i}{\mu} \right).$$

Using  $\{\mathcal{H}, \mathcal{S}\} = 0$ , we see that

$$b^j = \sum_{i=1}^3 \left( \frac{1}{\lambda} \partial_{ii} + \frac{1}{\mu} \left( \frac{\mu}{\lambda} \right)_i \partial_i \right) \left( \frac{1}{\mu} \sum_{k=1}^3 (a^{kj} \mu)_k \right) -$$

$$\sum_{i,\ell=1}^3 \left( a^{i\ell} \partial_{i\ell} + \frac{1}{\mu} (a^{i\ell} \mu)_i \partial_\ell \right) \left( \frac{1}{\mu} \left( \frac{\mu}{\lambda} \right)_j \right) + \frac{2}{\lambda} \hat{W}_j.$$

# Nondegenerate 3D quantum systems-4

THEOREM:

$$\{\mathcal{H}, \mathcal{S}\} = 0 \iff [\hat{H}, \hat{S}] = 0.$$

where  $\mu = \lambda$  and  $\hat{W}_j = \frac{1}{2} \partial_{jik} a^{ik}$  (for  $i, j, k$  pairwise distinct).

# Spaces of higher order symmetries-1

Just as for the 2D quantum case we can use the formal skew-adjoint and self-adjoint properties of the higher order symmetry operators, to make tractable the solution of the symmetry equations and determination of the maximum dimensions of the solution spaces. The details are complicated but the results agree with the classical 3D case.

# Spaces of higher order symmetries-2

THEOREM: Let  $V$  be a superintegrable nondegenerate potential, Then the space of third order skew-adjoint symmetries is four-dimensional and is spanned by commutators of the second order self-adjoint symmetries.



# Spaces of higher order symmetries-3

THEOREM: Let  $V$  be a quantum superintegrable nondegenerate potential Then the associated system is multiseparable.

# Spaces of higher order symmetries-4

THEOREM: The subspace of truly fourth order self-adjoint symmetry operators is at most 21.

# Spaces of higher order symmetries-5

If  $A, B$  are linear operators, we define their symmetrized product by

$$\{A, B\} \equiv \frac{1}{2}(AB + BA).$$

**THEOREM:** The 21 distinct monomials  $\{S^{(ij)}, S^{(jk)}\}$  form a basis for the space of fourth order self-adjoint symmetry operators.

# Spaces of higher order symmetries-6

Similarly, the space of truly sixth order formally self-adjoint operator symmetries of  $H$  cannot exceed the classical maximal dimension of 56. If  $A, B, C$  are linear operators, we define their symmetrized product by

$$\{A, B, C\} \equiv \frac{1}{6}(ABC + BAC + CAB + ACB + BCA + CBA).$$

**THEOREM:** The 56 distinct standard monomials  $\{S^{(hi)}, S^{(jk)}, S^{(\ell m)}\}$  form a basis for the space of sixth order self-adjoint symmetry operators.

# Spaces of higher order symmetries-7

These results establish the closure of the quadratic algebra for 3D quantum superintegrable potentials: All fourth order and sixth order symmetry operators can be expressed as symmetric polynomials in the second order symmetry operators.

# Covariant 3D quantum case-1

We have achieved an operator realization of the classical commutator brackets for second-order symmetries but the differential operator Hamiltonian, though formally self-adjoint with respect to the weight function  $\lambda$ , is not the Laplace-Beltrami operator on the manifold. We can achieve this, at the expense of altering the potential  $V$ , by means of an appropriate gauge transformation. We now turn to this construction.

# Covariant 3D quantum case-2

Set

$$H = e^{-\mathcal{R}} \hat{H} e^{\mathcal{R}}, \quad S = e^{-\mathcal{R}} \hat{S} e^{\mathcal{R}}$$

where  $R(x, y, z)$  is a function to be determined. Then  $[H, S] = 0$  if and only if  $[\hat{H}, \hat{S}] = 0$ . We will choose  $R$  such that the differential operator part of  $H$  is the Laplace-Beltrami operator on the manifold with metric  $ds^2 = \lambda(dx^2 + dy^2 + dz^2)$ .

# Covariant 3D quantum case-2

It is straightforward to show that

$$H = e^{-\mathcal{R}} \hat{H} e^{\mathcal{R}} = \frac{1}{\lambda} \sum_{i=1}^3 (\partial_{ii} + 2\mathcal{R}_i \partial_i + \mathcal{R}_{ii} + \mathcal{R}_i^2) + V$$

so, if we set  $\mathcal{R} = \frac{1}{4} \ln \lambda$ , we have

$$H = \sum_{i=1}^3 \left( \frac{1}{\lambda^{\frac{3}{2}}} \partial_i (\lambda^{\frac{1}{2}} \partial_i) + \frac{\mathcal{R}_{ii} + \mathcal{R}_i^2}{\lambda} \right) + V.$$



# Covariant 3D quantum case-3

Similarly

$$S = \sum_{i,j=1}^3 \left( \frac{1}{\lambda^{\frac{3}{2}}} \partial_i (a^{ij} \lambda^{\frac{3}{2}} \partial_j) + a^{ij} (\mathcal{R}_{ij} + 5\mathcal{R}_i \mathcal{R}_j) + a_i^{ij} \mathcal{R}_j \right) + W + \hat{W}.$$

The eigenvalue equation for  $\hat{H}$  on the space with weight function  $\mu = \lambda$  is  $\hat{H}\Psi = E\Psi$ . Setting  $\Psi = e^{\mathcal{R}}\Phi = \lambda^{1/4}\Phi$  we see that the eigenvalue equation for  $\Phi$  is  $H\Phi = E\Phi$  and the eigenfunctions  $\Phi$  lie in the space with weight function  $\lambda^{3/2}$ .

# Covariant 3D quantum case-4

Note that

$$\sum_{i=1}^3 (\mathcal{R}_{ii} + \mathcal{R}_i^2) / \lambda = -\frac{1}{8}R$$

where  $R$  is the Riemannian scalar curvature. The quantum potential is

$$(20) \quad \tilde{V} = -\frac{1}{8}R + V.$$

# Covariant 3D quantum case-5

If we supplement the classical symmetries with quantum adjustments the corresponding operators are

$$H = \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j) + \frac{1}{8} R,$$

$$S = \frac{1}{\sqrt{g}} \partial_i (a^{ij} \sqrt{g} \partial_j) + \frac{1}{16} a_i^i R - \frac{5}{16} a^{ij} R_{ij} - \frac{1}{16} \nabla_i \nabla_j a^{ij}.$$

Here  $\nabla_j$  is the usual covariant derivative on the Riemannian space.

# Covariant 3D quantum case-6

THEOREM: Let  $\mathcal{H}$ ,  $\hat{H}$  and  $H$  be defined as above where  $\mathcal{H}$  defines a classical superintegrable system with nondegenerate potential  $V$ . Let  $\mathcal{S}^{(1)}$ ,  $\mathcal{S}^{(2)}$  be second order symmetries of  $\mathcal{H}$ , with corresponding symmetry operators  $\hat{S}^{(j)}$ ,  $S^{(j)}$ . Then

$$\{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}\} = 0 \iff [\hat{S}^{(1)}, \hat{S}^{(2)}] = 0 \iff [S^{(1)}, S^{(2)}] = 0.$$

# Covariant 3D quantum case-7

COROLLARY: Every conformally flat 3D classical superintegrable system with nondegenerate potential extends to a unique covariant quantum superintegrable system. The symmetries of the quantum system admit a quadratic algebra structure.

# Fine structure

For fine structure of superintegrable systems we drop the requirement of nondegeneracy and study the various possibilities for systems with potentials depending on fewer parameters. For 2D systems the structure is very simple.

**Theorem 1** *Every 2D system with a one- or two-parameter potential and 3 functionally linearly independent second-order symmetries is the restriction of some nondegenerate (three-parameter) potential.*

# Fine 3D structure 0

For 3D systems the results are much more complicated and have not yet been fully determined. We first consider those systems that just fail to be nondegenerate in the sense that the four functions  $\mathcal{S}_h$  together with  $\mathcal{H}$  are functionally linearly independent in the six-dimensional phase space but that the associated potential functions  $V$  span only a 3 dimensional subspace of the 4 dimensional space of solutions of the potential equations, ignoring the trivial added constant.

# Fine 3D structure 1

This circumstance can occur in only two ways: either the potential is a 3-parameter restriction of a nondegenerate potential, or the integrability conditions for the potential equations are not satisfied identically and an additional condition is imposed.



# Fine 3D structure 2

In either case the canonical potential equations are replaced by the 6 equations

$$(21) \quad V_{ij} = \tilde{A}^{ij}V_1 + \tilde{B}^{ij}V_2 + \tilde{C}^{ij}V_3, \quad i \leq j,$$

whose integrability conditions are satisfied identically. The canonical equations still hold, but with the identifications

$$D^{ij} = \tilde{D}^{ij}, \quad 1 \leq i < j \leq 3, \quad D^{kk} = \tilde{D}^{kk} - \tilde{D}^{11}, \quad k = 2, 3,$$

where  $D = A, B, C$ . For short, we will call the solutions of the potential equations **3-parameter potentials**.

# Fine 3D structure 3

In analogy to the nondegenerate potential case we can compute the full set of integrability conditions satisfied by the potential, and we can use the 10 second order Killing tensor equations and the  $3 \times 3 = 9$  potential conditions for the derivatives  $a_h^{\ell m}$ . There are 19 conditions for the 18 derivatives  $a_h^{\ell m}$ . We get exactly the standard symmetry equations and the remaining condition

$$\begin{aligned} & a^{11}(\tilde{C}^{12} - \tilde{B}^{13}) + a^{22}(\tilde{A}^{23} - \tilde{C}^{12}) + a^{33}(\tilde{B}^{13} - \tilde{A}^{23}) \\ & + a^{12}(\tilde{A}^{13} + \tilde{C}^{22} - \tilde{C}^{11} - \tilde{B}^{23}) + a^{13}(\tilde{C}^{23} + \tilde{B}^{11} - \tilde{B}^{33} - \tilde{A}^{12}) \\ & + a^{23}(\tilde{B}^{12} + \tilde{A}^{33} - \tilde{A}^{22} - \tilde{C}^{13}) = 0, \end{aligned}$$

which we can regard as an obstruction.

# Fine 3D structure 4

The analogous obstruction equation appears for the nondegenerate potential case, but there the linear integrability conditions for the nondegenerate potential cause the obstruction to vanish identically. We have obtained the following results:

**Theorem 2** *A 3D 3-parameter potential is a restriction of a nondegenerate potential if and only if the obstruction vanishes identically. If the obstruction doesn't vanish then the space of second order symmetries is 5 dimensional and the system is uniquely determined by the values of  $\tilde{D}^{ij}, i \leq j, D = A, B, C$  at a single regular point.*

# Fine 3D structure 5

The extended Kepler-Coulomb system

$$V = \frac{\alpha}{\sqrt{x^2 + y^2 + z^2}} + \frac{\beta}{x^2} + \frac{\gamma}{y^2}.$$

is an example of a 3-parameter potential with obstruction.  
Another example is defined by the potential

$$V = \frac{\alpha}{\sqrt{x^2 + y^2 + z^2}} + \frac{\beta}{(x + iy)^2} + \frac{\gamma(x - iy)}{(x + iy)^3}.$$

These are true 3-parameter potentials in the sense that they cannot be extended to nondegenerate potentials.

# Fine 3D structure 6

**Theorem 3** *Let  $V$  be a superintegrable true 3-parameter potential on a conformally flat space. Then the space of third order constants of the motion is 3-dimensional and is spanned by Poisson brackets of the second order constants of the motion. The Poisson bracket of two second order constants of the motion is uniquely determined by the matrix commutator of the second order constants at a regular point.*

**Theorem 4** *Let  $V$  be a superintegrable true 3-parameter potential in a 3D conformally flat space. Then  $V$  defines a multiseparable system.*

# Fine 3D structure 7

**Theorem 5** *Every superintegrable system with true 3-parameter potential on a 3D conformally flat space is Stäckel equivalent to a superintegrable system on either 3D flat space or the 3-sphere.*

Although the spaces of higher order symmetries for true 3-parameter systems have an interesting structure, the quadratic algebra doesn't close.

**Theorem 6** *For a superintegrable system with true 3-parameter potential on a 3D conformally flat space there exist two second order constants of the motion  $\mathcal{S}_1, \mathcal{S}_2$  such that  $\{\mathcal{S}_1, \mathcal{S}_2\}^2$  is not expressible as a cubic polynomial in the second order constants of the motion.*

# Polynomial ideals 1

We introduce a very different way of studying and classifying superintegrable systems, through polynomial ideals. Here we confine our analysis to 3D Euclidean superintegrable systems with nondegenerate potentials, though the approach is also effective in 2D and for spheres. The canonical potential equations are just

$$V_{jj} - V_{11} = A^{jj}V_1 + B^{jj}V_2 + C^{jj}V_3, \quad j = 2, 3$$

$$V_{k\ell} = A^{k\ell}V_1 + B^{k\ell}V_2 + C^{k\ell}V_3, \quad 1 \leq k < \ell \leq 3.$$

# Polynomial ideals 2

All of the functions  $A^{ij}, B^{ij}, C^{ij}$  can be expressed in terms of the 10 basic terms

$$(22) \quad (A^{12}, A^{13}, A^{22}, A^{23}, A^{33}, B^{12}, B^{22}, B^{23}, B^{33}, C^{33}).$$

Since the symmetry equations admit 6 linearly independent solutions  $a^{hk}$  the integrability conditions  $\partial_i a_\ell^{hk} = \partial_\ell a_i^{hk}$  for these equations must be satisfied identically. These conditions plus the integrability conditions for the potential allow us to compute the 30 derivatives  $\partial_\ell D^{ij}$  of the 10 basic terms. Each is a quadratic polynomial in the 10 terms.



# Polynomial ideals 3

In addition there are 5 quadratic conditions remaining. The first 2 are as follows and the others are similar.

$$\begin{aligned} a) \quad & -A^{23} B^{33} - A^{12} A^{23} + A^{13} B^{12} \\ & + B^{22} A^{23} + B^{23} A^{33} - A^{22} B^{23} = 0, \\ (23) \quad b) \quad & (A^{33})^2 + B^{12} A^{33} - A^{33} A^{22} - B^{33} A^{12} \\ & - C^{33} A^{13} + B^{22} A^{12} - B^{12} A^{22} + A^{13} B^{23} \\ & - (A^{12})^2 + = 0 \end{aligned}$$

# Polynomial ideals 4

These 5 polynomials determine an ideal  $\Sigma'$ . Already we see that the values of the 10 terms at a fixed regular point must uniquely determine a superintegral system. However, choosing those values such that the 5 quadratic conditions are satisfied will not guarantee the existence of a solution, because the conditions may be violated for values of  $(x, y, z)$  away from the chosen regular point.

# Polynomial ideals 5

To test this we compute the derivatives  $\partial_i \Sigma'$  and obtain a single new condition, the square of the quadratic expression

$$\begin{aligned} f) \quad & A^{13}C^{33} + 2A^{13}B^{23} + B^{22}B^{33} - (B^{33})^2 + A^{33}A^{22} \\ & - (A^{33})^2 + 2A^{12}B^{22} + (A^{12})^2 - 2B^{12}A^{22} + (B^{12})^2 \\ & + B^{23}C^{33} - (B^{23})^2 - 3(A^{23})^2 = 0. \end{aligned}$$

(24)

This polynomial extends the ideal.

# Polynomial ideals 6

Let  $\Sigma$  be the ideal generated by the 6 quadratic polynomials. It can be verified that  $\partial_i \Sigma \subseteq \Sigma$ , so that the system is closed under differentiation!

This leads us to a fundamental result.

**Theorem 7** *Choose the 10-tuple at a regular point, such that the 6 polynomial identities are satisfied. Then there exists one and only one Euclidean superintegrable system with nondegenerate potential that takes on these values at a point.*

# Polynomial ideals 7

We see that all possible nondegenerate 3D Euclidean superintegrable systems are encoded into the 6 quadratic polynomial identities. These identities define an algebraic variety that generically has dimension 6, though there are singular points, such as the origin  $(0, \dots, 0)$ , where the dimension of the tangent space is greater. This result gives us the means to classify all superintegrable systems.

# Polynomial ideals 8

An issue is that many different 10-tuples correspond to the same superintegrable system. How do we sort this out? The key is that the Euclidean group  $E(3, \mathbb{C})$  acts as a transformation group on the variety and gives rise to a foliation.

The action of the translation subgroup is determined by the derivatives  $\partial_k D^{ij}$  that we have already determined.

The action of the rotation subgroup on the  $D^{ij}$  can be determined from the behavior of the canonical equations under rotations.

The local action on a 10-tuple is then given by 6 Lie derivatives that are a basis for the Euclidean Lie algebra  $e(3, \mathbb{C})$ .

# Polynomial ideals 9

For “most” 10-tuples  $D_0$  on the 6 dimensional variety the action of the Euclidean group is locally transitive with isotropy subgroup only the identity element. Thus the group action on such points sweeps out a solution surface homeomorphic to the 6 parameter  $E(3, C)$  itself. The generic Jacobi elliptic system

$$V = \alpha(x^2 + y^2 + z^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}$$

corresponds to this case. At the other extreme the isotropy subgroup of the origin  $(0, \dots, 0)$  is  $E(3, C)$  itself, i.e., the point is fixed under the group action. This corresponds to the isotropic oscillator

$$V = \alpha(x^2 + y^2 + z^2) + \beta x + \gamma y + \delta z.$$

# Polynomial ideals 10

More generally, the isotropy subgroup at  $D_0$  will be  $H$  and the Euclidean group action will sweep out a solution surface homeomorphic to the homogeneous space  $E(3, C)/H$  and define a unique superintegrable system. For example, the isotropy subalgebra formed by the translation and rotation generators  $\{P_1, P_2, P_3, J_1 + iJ_2\}$  determines the system with potential

$$V = \alpha \left( (x - iy)^3 + 6(x^2 + y^2 + z^2) \right) + \\ \beta \left( (x - iy)^2 + 2(x + iy) \right) + \gamma(x - iy) + \delta z.$$



# Polynomial ideals 11

Each class of Stäckel equivalent Euclidean superintegrable systems is associated with a unique isotropy subalgebra of  $e(3, C)$ , although not all subalgebras occur and one subalgebra could lead to more than one system. (Indeed, there is no isotropy subalgebra conjugate to  $\{P_1, P_2, P_3\}$ .) Thus to find all superintegrable systems we need to determine a list of all subalgebras of  $e(3, C)$ , defined up to conjugacy, and then for each subalgebra to determine if it occurs as an isotropy subalgebra, and the multiplicity of its occurrence.

# Outlook 1

We have given an overview of some of the tools used and results obtained in the study of second order superintegrable systems.

The basic problems for 2D systems have been solved, and the extension of these methods to complete the fine structure analysis for 3D systems appears relatively straightforward.

# Outlook 2

The 3D fine structure analysis can be extended to analyze 2 parameter and 1 parameter potentials with 5 functionally linearly independent second order symmetries. Here first order PDEs for the potential appear, as well as second order, and Killing vectors may occur.

The other class of 3D superintegrable systems is that for which the 5 functionally independent symmetries are functionally linearly dependent. This class contains the Calogero potential and necessarily leads to first order PDEs for the potential, as well as second order.

# Outlook 3

The integrability condition methods discussed here should be able to handle this class with no special difficulties. On a deeper level, we think that algebraic geometry methods can be extended to classify the possible superintegrable systems in all these cases.

# Outlook 4

Whereas 2D superintegrable systems are very special, the 3D systems seem to be good guides to the structure of general  $n$ D systems, and we intend to proceed with this analysis. Finally, the ultimate aim is to understand the structure of superintegrable systems in general.

# Outlook 5

Finally, the algebraic geometry related results that we have sketched suggest strongly that there is an underlying geometric structure to superintegrable systems that is not apparent from the usual presentations of these systems.