VECTORS IN THE PLANE

Chester Miracle

School of Mathematics
University of Minnesota

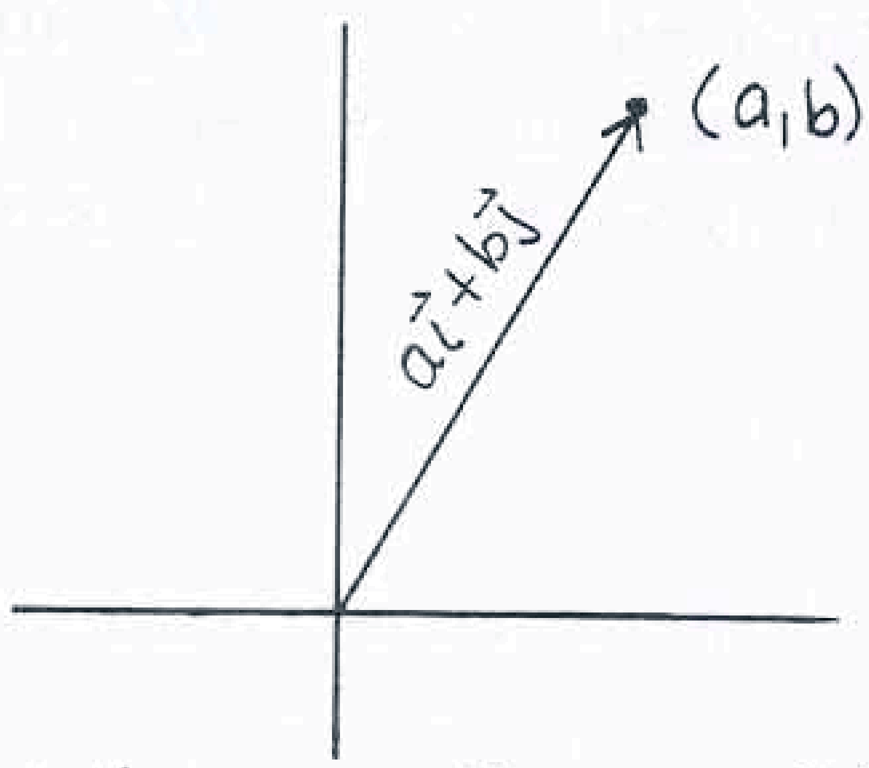
6751 Introduction to Vectors

Quanities such as force, displacement, and velocity can not be represented by a single number. In order to completely describe these quanities we must give not only their magnitude but also their direction. This causes us to introduce the idea of vectors.

The geometric way to define vectors is to define a vector as a directed line segment. Directed means that the line segment has an initial point at one end and a terminal point at the other end. The direciton is the direction from the initial point toward the terminal point. The magnitude of the vector is the length of the line segment. For example, we can represent the velocity of an object as a vector. The length of the vector is the same as the speed of the object and the direction of the vector indicates the direction of the movement.

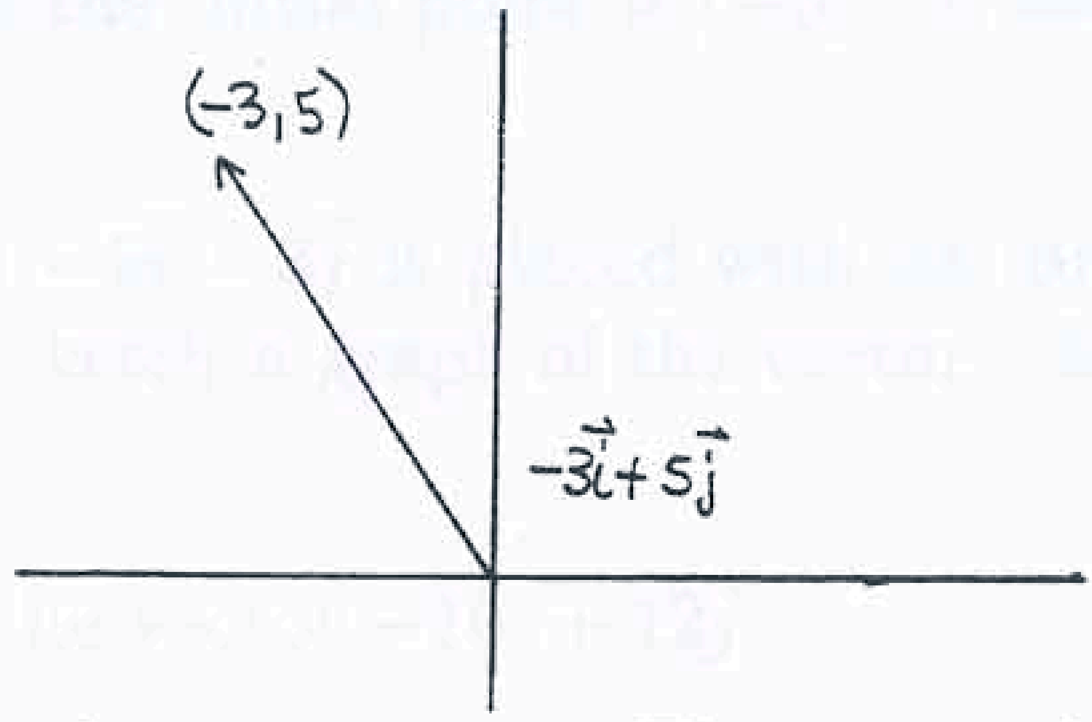
It is very difficult to discuss vectors using just old fashion geometry. It is even difficult to determine the length of a given vector in such a discussion. We need a method for locating points. We are going to discuss vectors in a plane. We will always assume that there is a given coordinate system for this plane. This enables us to specify any point in the plane by giving its coordinates. This enables us to discuss vectors algebrically.

Definition. The vector $\vec{v} = a\vec{i} + b\vec{j}$ in standard position is the directed line segment in the plane with initial point (0,0) and terminal point (a, b).



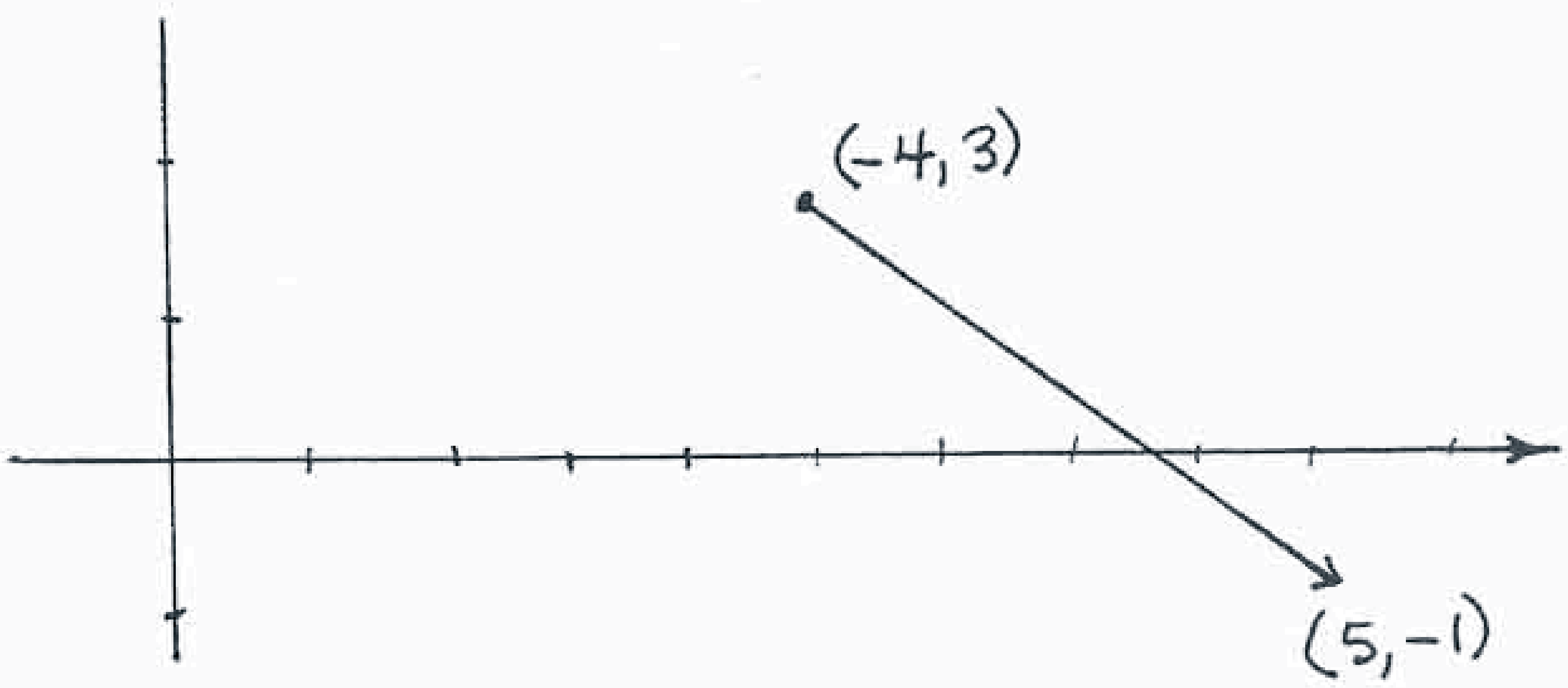
Other notations sometimes used for the vector $\vec{v}$ are $[a, b]$, $< a, b >$, and (a, b). The vector $\vec{v}$ need not always be in standard position. If two line segments have the same direction and the same length, then they represent the same vector. The vector $\vec{i}$ is of length one from (0,0) to (1,0). The vector $\vec{j}$ is of length one from (0,0) to (0,1). In order to save time we sometimes write just i and j instead of $\vec{i}$ and $\vec{j}$.

Example. Suppose the vector $4\vec{\imath} - 3\vec{\jmath}$ is placed with its initial point at (5,2). What is the terminal point?



Solution. $5 + 4 = 9$ and $-3 + 2 = -1$. The terminal point is $(9, -1)$.

Example. Find the vector $\vec{v}$ if the initial point is $(-4, 3)$ and the terminal point is $(5, -1)$.



Solution. $[5 - (-4)]\vec{\imath} + [-1 - 3]\vec{\jmath} = 9\vec{\imath} - 4\vec{\jmath}$.

Theorem. We denote the length of the vector $\vec{v} = a\vec{\imath} + b\vec{\jmath}$ by $|\vec{v}|$ or $||\vec{v}||$. When $\vec{v}$ is placed in standard position it is an easy application of the Pythagorean theorem to see that

$$||\vec{v}|| = \sqrt{a^2 + b^2}$$

Example. Find the length of the vector $12\vec{\imath} - 5\jmath$.

Solution. $\sqrt{12^2 + (-5)^2} = 13$. The length is 13.

Given the vector $\vec{v} = a\vec{\imath} + b\vec{\jmath}$ the number a is called the component of $\vec{v}$ in the direction of $\vec{\imath}$ or in the x direction and b is called the component of $\vec{v}$ in the $\vec{\jmath}$ direction or in the y direction.

Exercises

1. Find the vector $\vec{v}$ if the initial point is $(-5, -3)$ and the terminal point is $(2, 7)$.

2. Suppose the vector $-5\vec{\imath} + 3\vec{\jmath}$ is placed with its initial point (2,4) what is its terminal point? Sketch a graph of the vector $-5\imath + 3\jmath$ with its initial point at $(2, 4)$.

3. Find the length of the vector $-16\vec{\imath} + 12\vec{\jmath}$.

6755 The Dot Product

Geometric definition of dot product. The dot product of two vectors is a scalar which may be either positive or negative. When discussing the dot product we define the angle θ between two vectors with the same initial point to be the angle satisfying $0 \leq \theta \leq \pi$. If $\vec{u}$ and $\vec{v}$ are two vectors, then the dot product $\vec{u} \cdot \vec{v}$ is given by

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta,$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

Algebraic definition of dot product. Given the vectors $\vec{u} = a\vec{i} + b\vec{j}$ and $\vec{v} = c\vec{i} + d\vec{j}$, the dot product is

$$\vec{u} \cdot \vec{v} = ac + bd.$$

Now we should, of course, prove that these two definitions are equivalent, that is,

$$||\vec{u}||||\vec{v}|| \cos\theta = ac + bd.$$

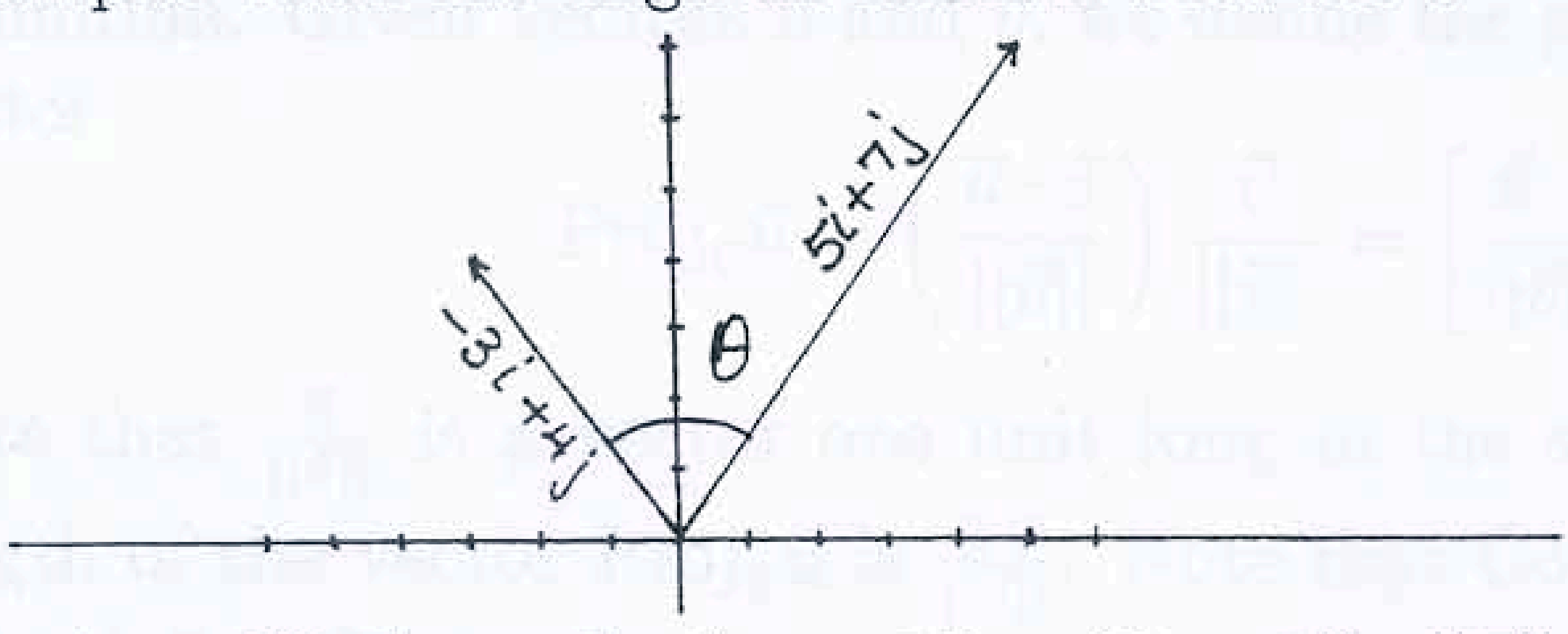
We will not give the proof here. It is an exercise in trigonometry involving the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

When asked to find the dot product of two vectors $a\vec{i} + b\vec{j}$ and $c\vec{i} + d\vec{j}$ we would always use the algebraic form $(a\vec{i} + b\vec{j}) \cdot (c\vec{i} + d\vec{j}) = ac + bd$. On the other hand the geometric definition can be very useful. It says

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}.$$

This formula can be used to find the angle between two vectors.

Example. Find the angle in radians between $\vec{v} = 5\vec{i} + 7\vec{j}$ and $\vec{v} = -3\vec{i} + 4\vec{j}$.



Solution. We have $\vec{u} \cdot \vec{v} = -15 + 28 = 13$. $||\vec{u}|| = \sqrt{5^2 + 7^2} = \sqrt{74}$, and

$||\vec{v}|| = \sqrt{(-3)^2 + 4^2} = 5$. Therefore,

$$\cos\theta = \frac{13}{5\sqrt{74}} = 0.3022$$
$$\theta = 1.2638 \text{ or } 72.41^o.$$

In math we usually measure this angle in radians, but in some practical problems we might want to measure it in degrees.

Theorem. If $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, but $\vec{u} \cdot \vec{v} = 0$, then the vectors $\vec{u}$ and $\vec{v}$ are prependicular.

Given the vector $\vec{v} = 3i - 4j$, as stated earlier the coefficient 3 of $\vec{i}$ is called the component of $\vec{v}$ in the direction of $\vec{i}$ and the coefficient -4 of $\vec{j}$ is called the component of $\vec{v}$ in the direction of $\vec{j}$. We also refer to 3 and -4 as the components of the vector $\vec{v}$. We also use the word component in a more general way.

Definition. Given vectors $\vec{u}$ and $\vec{v}$ we say that the component of $\vec{u}$ in the direction of $\vec{v}$ is given by

$$\text{Comp}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||}.$$

Note that $\text{Comp}_{\vec{v}}\vec{u}$ is a scalar also given by $||\vec{u}|| \cos\theta$.

Example. Let $\vec{u} = 7\vec{i} - 8\vec{j}$ and $\vec{v} = -3\vec{i} + 4\vec{j}$ find $\text{Comp}_{\vec{v}}\vec{u}$.

Solution. We have $\vec{u} \cdot \vec{v} = -21 - 32 = -53$ and $||\vec{v}|| = \sqrt{(-3)^2 + 4^2} = 5$. Therefore,

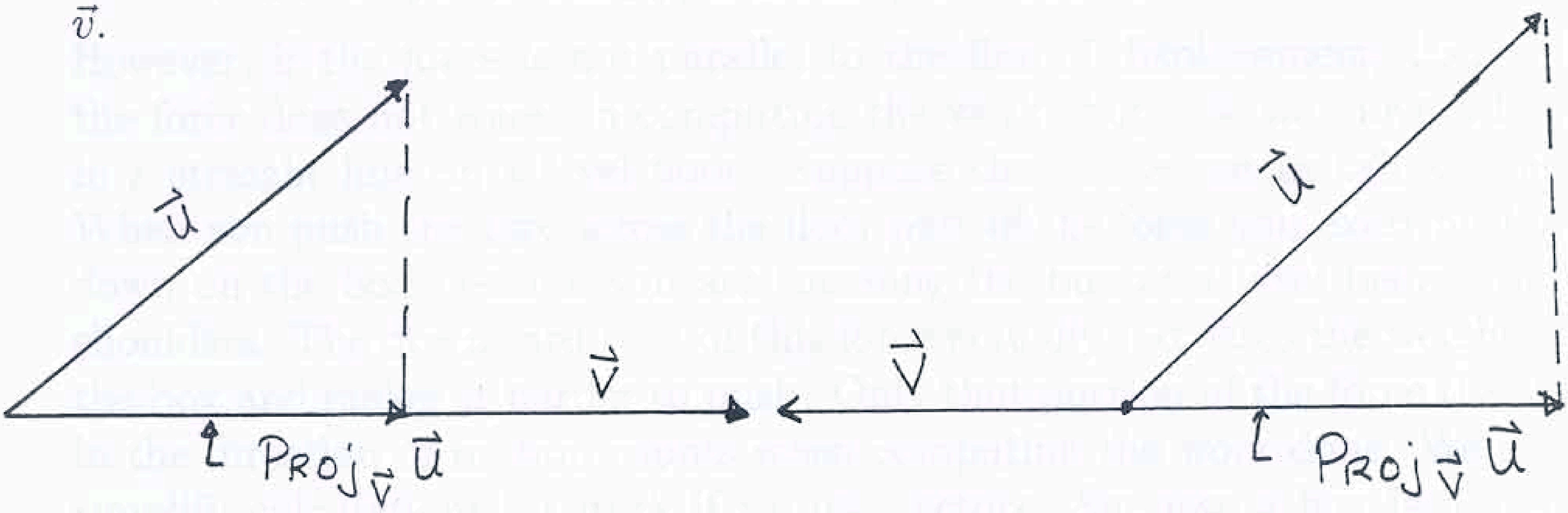
$$\text{Comp}_{\vec{v}}\vec{u} = \frac{-53}{5} = -10.6.$$

Definition. Given vectors $\vec{u}$ and $\vec{v}$, we define the projection of $\vec{u}$ on $\vec{v}$ as the vector

$$\text{Proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||}\right) \frac{\vec{v}}{||\vec{v}||} = \left[\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\right] \vec{v}.$$

Note that $\frac{\vec{v}}{||\vec{v}||}$ is a vector one unit long in the same direction as $\vec{v}$. The length of the vector $\text{Proj}_{\vec{v}}\vec{u}$ is $\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||}$. Note that $\text{Comp}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{||v||}$. The length of $\text{Proj}_{\vec{v}}\vec{u}$ is $\text{Comp}_{\vec{v}}\vec{u}$.

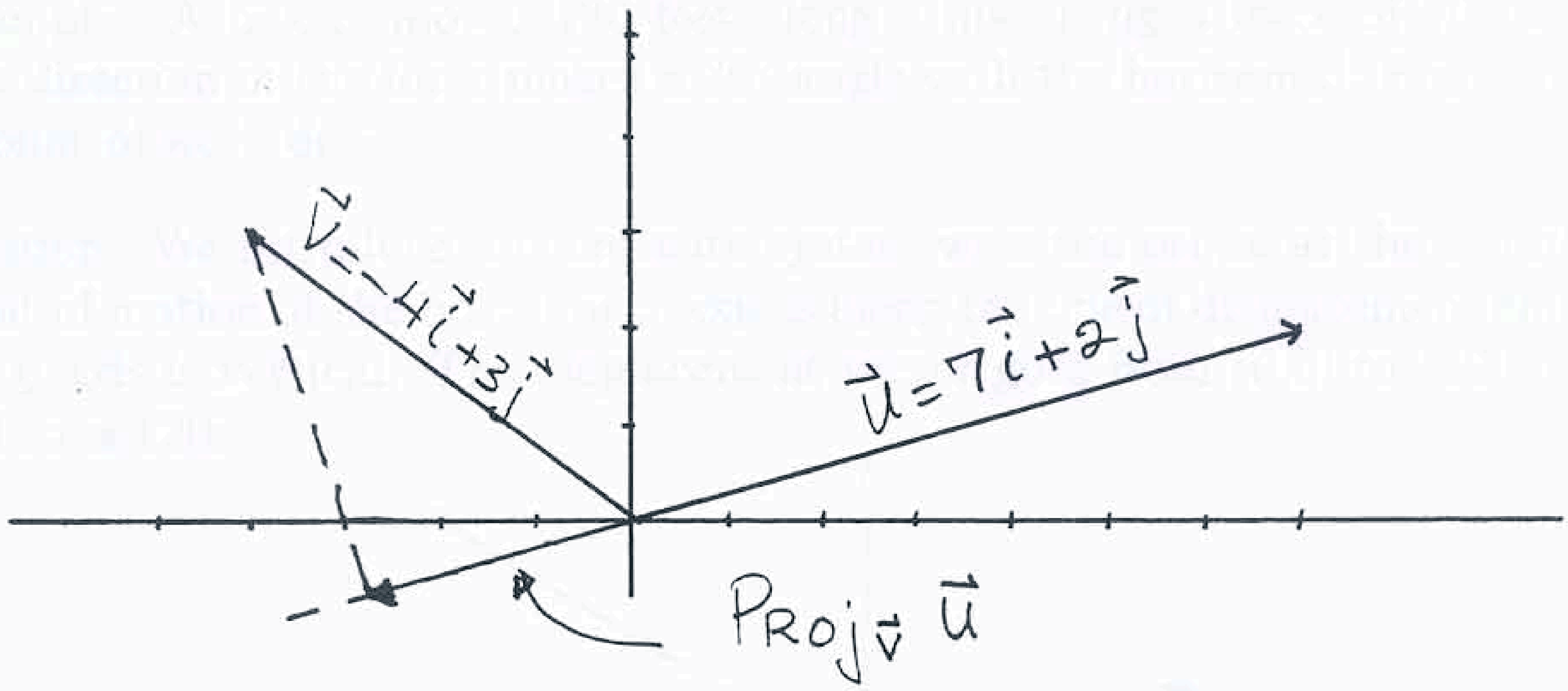
It is easy to picture geometrically the relationship between $\vec{u}, \vec{v}$, and $\text{Proj}_{\vec{v}}\vec{u}$ when all vectors are in standard position. Draw a line perpendicular to $\vec{v}$ through the tip of $\vec{u}$. The tip of vector $\text{Proj}_{\vec{v}}\vec{u}$ is where this line intersects $\vec{v}$.



Example. Given the vectors $\vec{u} = 7i + 2j$ and $\vec{v} = -4i + 3j$, find $\text{Proj}_{\vec{v}}\vec{u}$. Draw a sketch of these three vectors in standard position.

Solution. We have $\vec{u} \cdot \vec{v} = -28 + 6 = -22$, $||\vec{u}|| = \sqrt{53}$ and $||\vec{v}|| = 5$.

$$\text{Proj}_{\vec{v}}\vec{u} = \frac{-22}{25}(7i + 2j) = -6.16\vec{i} - 1.76\vec{j}.$$



The most famous application of the dot product and projection vector is work. Suppose we are moving a box along a line and applying a force of 80 Newtons in the direction parallel to the line of motion, then the work done is equal to the force times the displacement. If we move this box 30 meters applying this force, then the work done is

$$\text{work} = (80 \text{ Newtons})(30 \text{ meters}) = 2400 \text{ Newton Meters.}$$

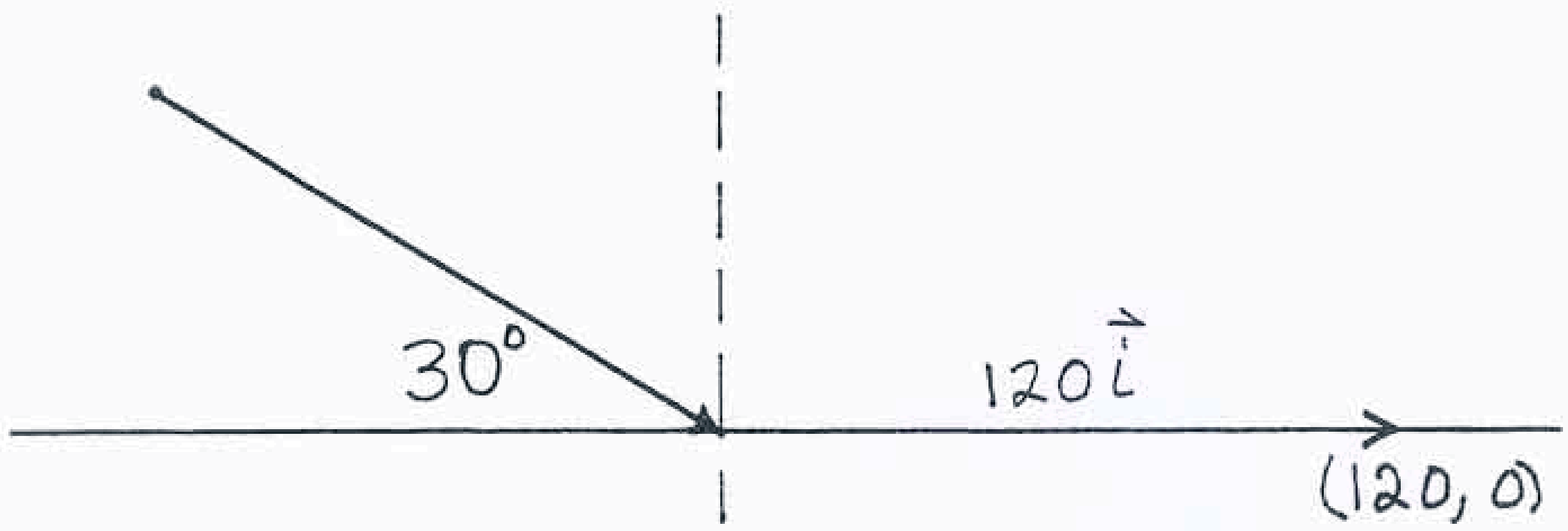
However, if the force is not parallel to the line of displacement, then all the force does not count in computing the work. Suppose we move a box in a straight line on a level floor. Suppose the box is not as tall as you. When you push the box across the floor part of the force you exert pushes down on the box because you are touching the box at a level below your shoulders. The downward part of this force actually increases the weight of the box and makes it harder to push. Only that portion of the force that is in the direction of motion counts when computing the work done. We can simplify calculations for work if we use vectors. Suppose a box is moved along the line segment given by the vector $\vec{D}$. Suppose while it is moving a constant force $\vec{F}$ is exerted on the box, then the work done by the force in moving the box through the displacement $\vec{D}$ is

$$\vec{F} \cdot \vec{D}.$$

All the motion which we will consider takes place in plane with a given coordinate system. If no coordinate system is given, then we must make one up.

Example. A box is moved 120 feet along a line using a force of 70 lbs. The direction of the force makes a 30^o angle with the horizontal. Find the amount of work done.

Solution. We introduce a coordinate system with the origin at the initial point of motion of the box. The x axis is along the line of displacement and the y axis is vertical. The displacement vector goes from (0,0) to (120,0) and so is $120\vec{\imath}$.



The force vector is

$$70(\cos 30^o)\vec{\imath} + 70(-\sin 30^o)\vec{\jmath} = (35\sqrt{3})\vec{\imath} - 35\vec{\jmath}.$$

$$\text{work} = [(35\sqrt{3})\vec{\imath} - 35\vec{\jmath}] \cdot [12\vec{\imath}] = 4200\sqrt{3} \text{ ft lbs.}$$

Exercises

1. Find the angle in radians between the vectors $\vec{u} = -5\vec{\imath} + 6\vec{\jmath}$ and $\vec{v} = 2\vec{\imath} + 7\vec{\jmath}$.

2. Given $\vec{u} = 4\vec{\imath} - 3\vec{\jmath}$, $\vec{v} = 2\vec{\imath} - 5\vec{\jmath}$, find $\text{Comp}_{\vec{v}}\vec{u}$.

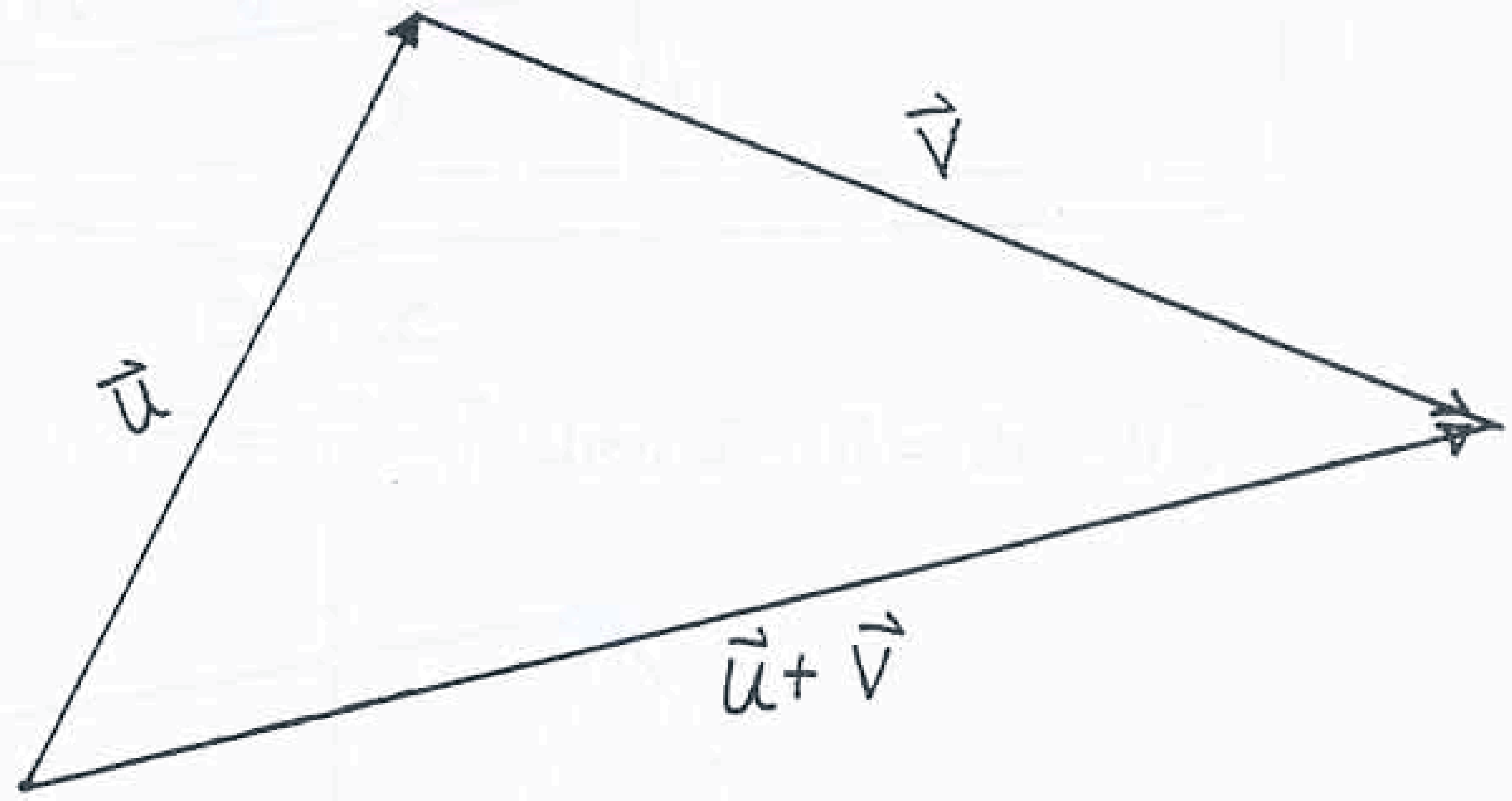
3. Given $\vec{u} = -3\vec{\imath} + 5\vec{\jmath}$, $\vec{v} = 6\vec{\imath} + 2\vec{\jmath}$, find $\text{Proj}_{\vec{v}}\vec{u}$.

4. Given $\vec{u} = 7\vec{\imath} - 2\vec{\jmath}$ and $\vec{u} = -3\vec{\imath} + 8\vec{\jmath}$, find $\text{Proj}_{\vec{v}}\vec{u}$. Sketch a graph of the three vectors.

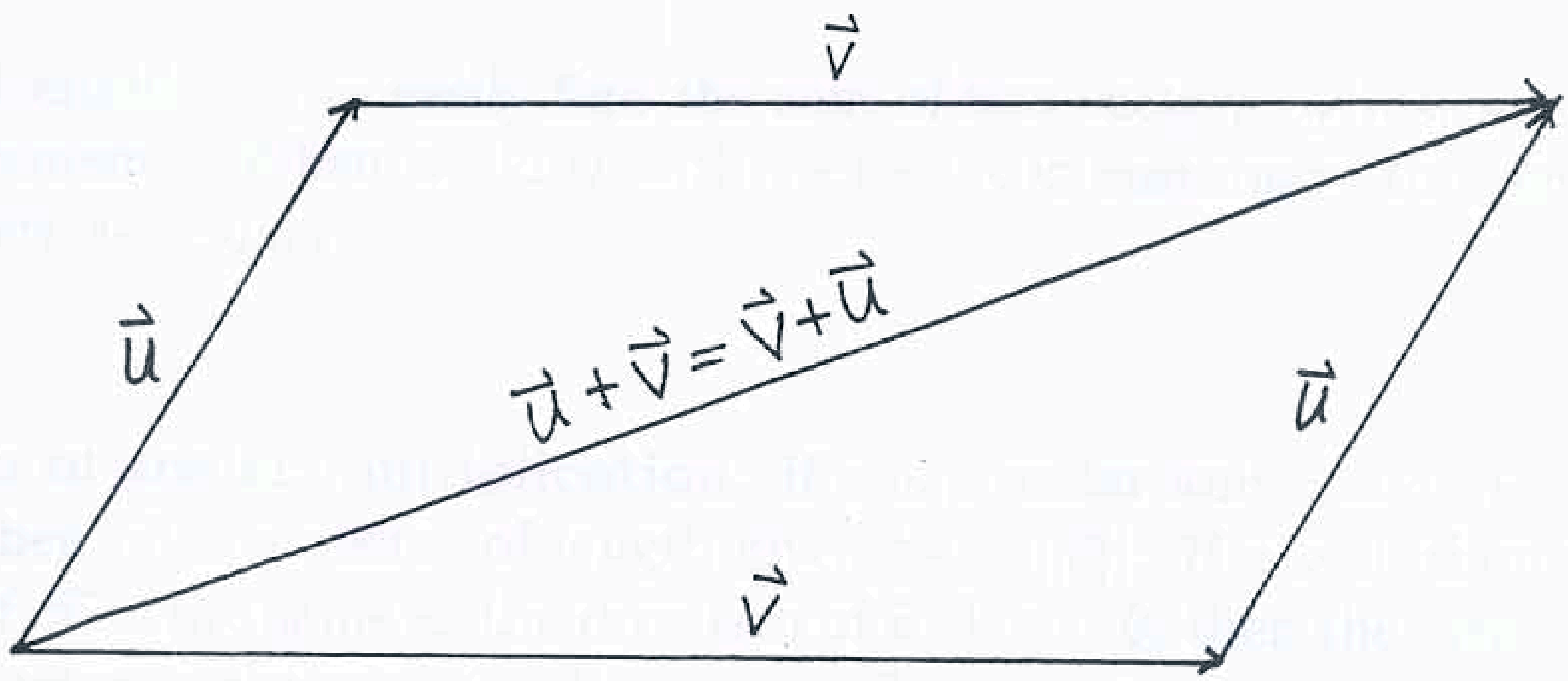
5. A box is moved 150 feet along a line using a force of 90 lbs. The direction of the force makes an angle of 40^o with the horizontal. Find the amount of work done.

6753 Addition of Vectors

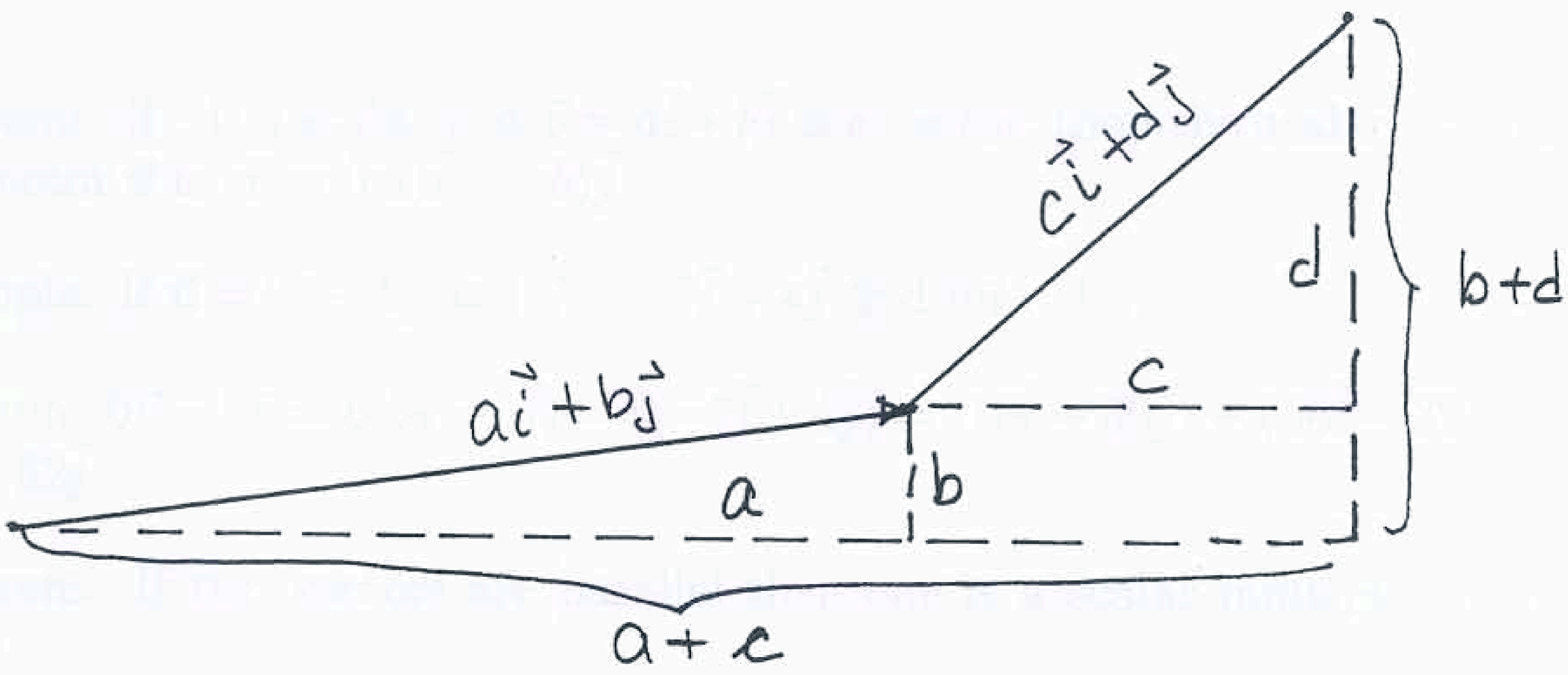
Definition of Addition of Vectors. If $\vec{u} = a\vec{i} + b\vec{j}$ and $\vec{v} = c\vec{i} + d\vec{j}$ are vectors positioned so that the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then the sum $\vec{u} + \vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.



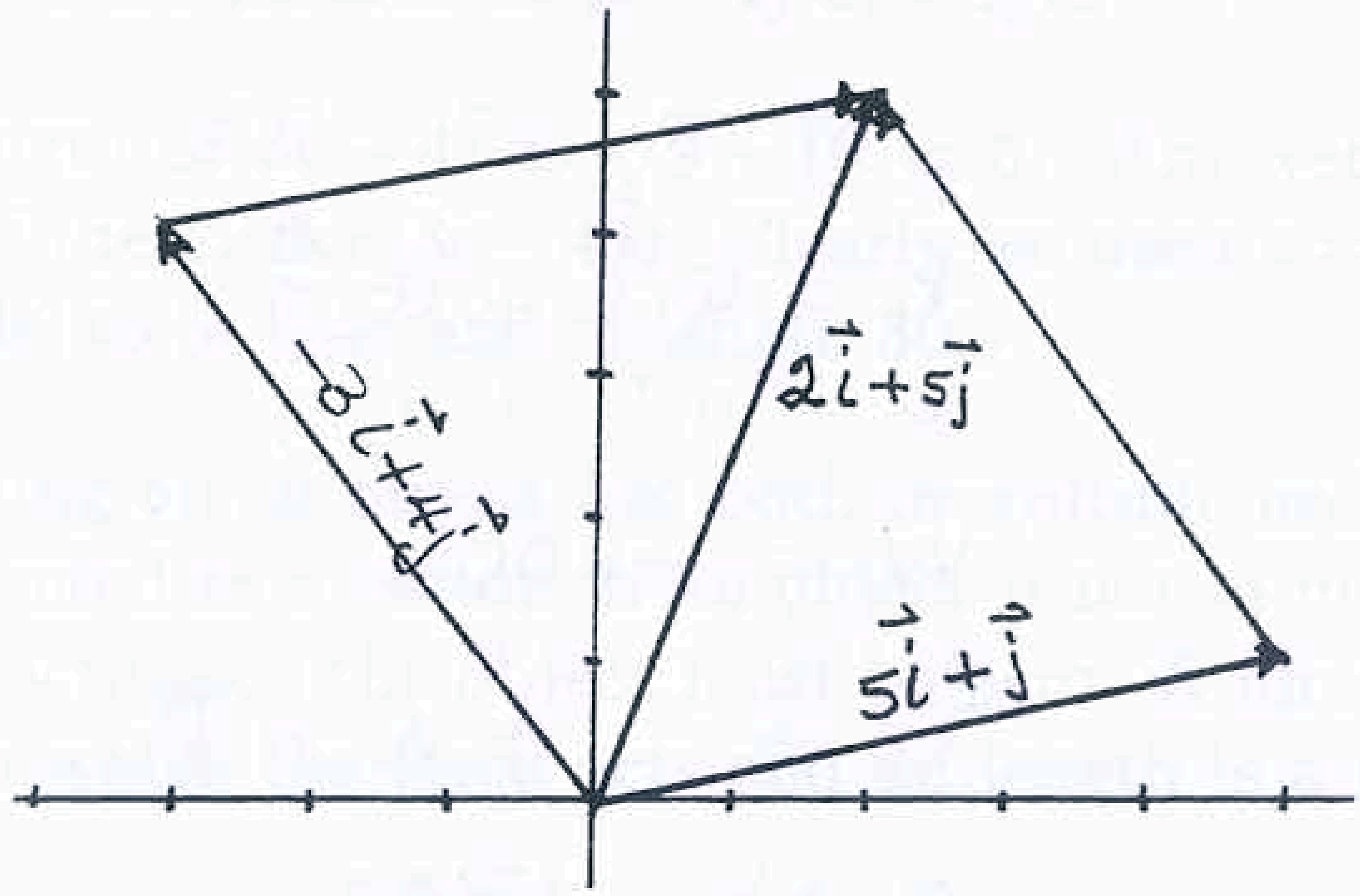
Parallelogram Law. Suppose the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$. Let $\vec{z}$ denote the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$. Second, suppose the initial point of $\vec{u}$ is at the terminal point of $\vec{v}$. Let $\vec{w}$ denote the vector from the initial point of $\vec{v}$ to the terminal point of $\vec{u}$, then $\vec{z} = \vec{w}$. We might also express this as $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.



Theorem 1. If $\vec{u} = a\vec{i} + b\vec{j}$ and $\vec{v} = c\vec{i} + d\vec{j}$ then $\vec{u} + \vec{v} = (a+c)\vec{i} + (b+d)\vec{j}$.

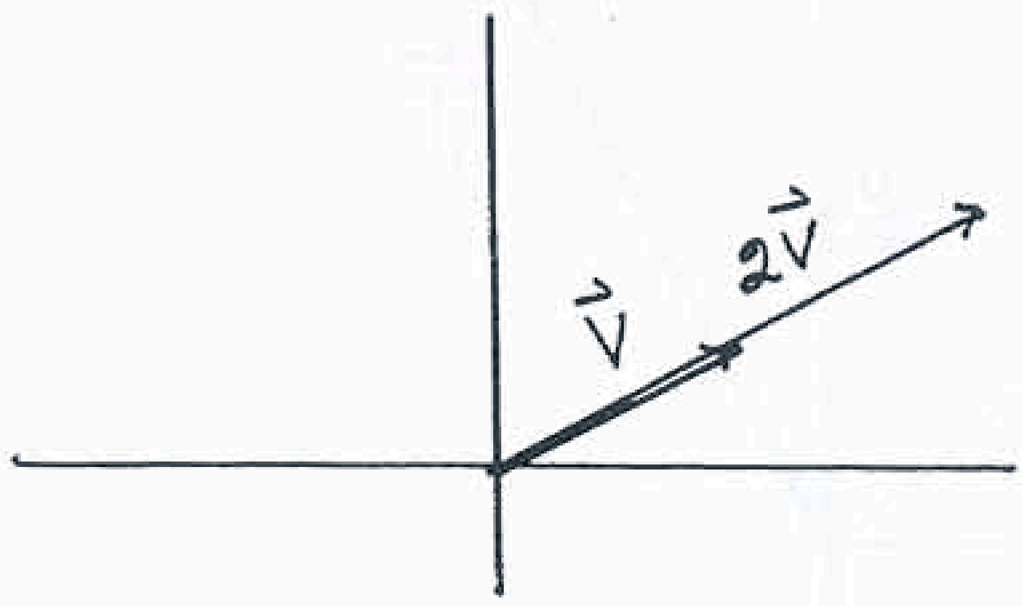


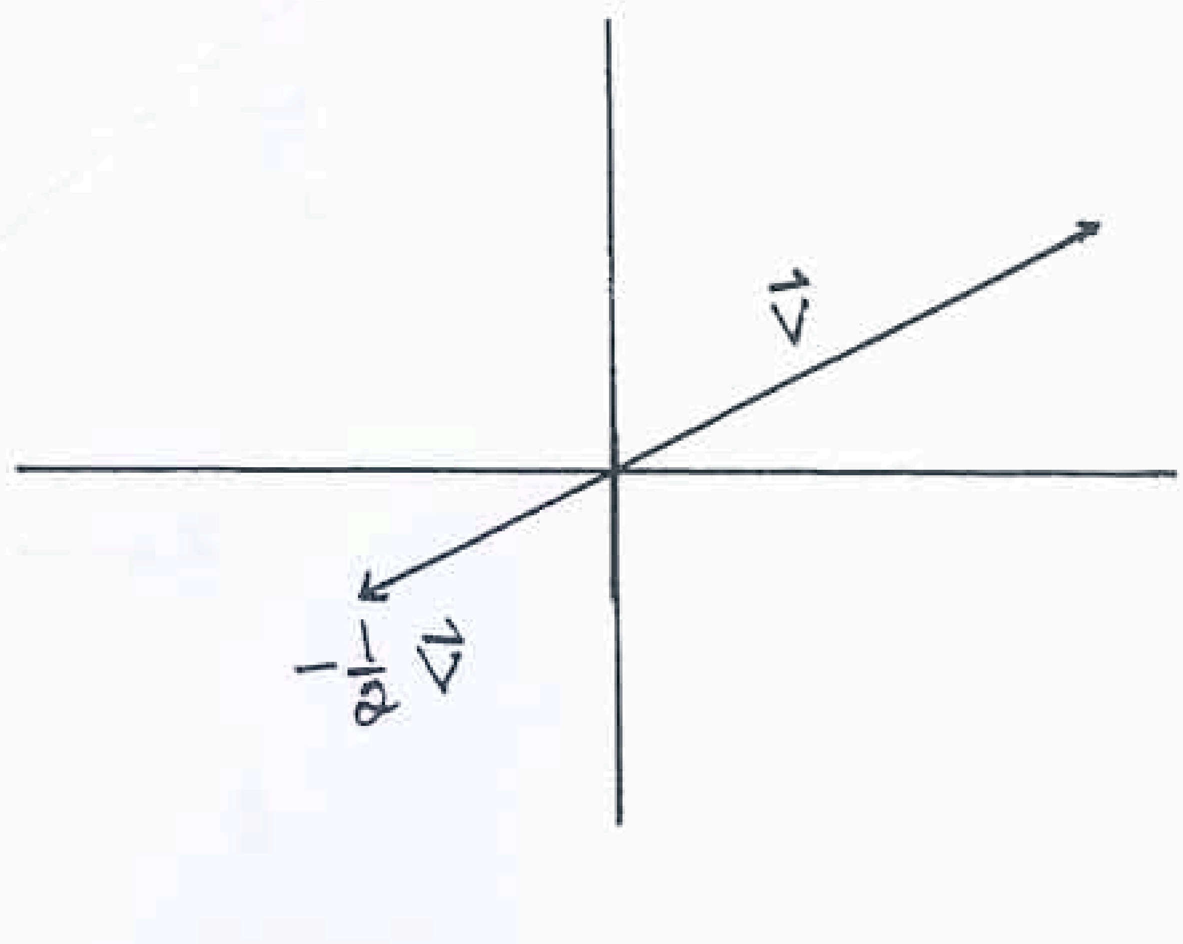
Example. If $\vec{u} = -3\vec{i} + 4\vec{j}$ and $\vec{v} = 5\vec{i} + \vec{j}$, then $\vec{u} + \vec{v} = 2\vec{i} + 5\vec{j}$.



Theorem 1 enables us to easily find the sum of two vectors by just adding their components. When working with vectors it is customary to refer to real numbers as scalars.

Definition of scalar multiplication. If c is a scalar and $\vec{v} = a\vec{i} + b\vec{j}$ is a vector, then $c\vec{v}$ is a vector of length given by $|c|||\vec{v}||$. If $c > 0$, then the direction of $c\vec{v}$ is the same as the direction of $\vec{v}$. If $c < 0$, then the direction of $c\vec{v}$ is a vector in the opposite direction of $\vec{v}$.





Theorem. If c is a scalar and $\vec{v} = a\vec{i} + b\vec{j}$ is a vector, then the scalar c times the vector $\vec{v}$ is $c\vec{v} = (ca)\vec{i} + (cb)\vec{j}$.

Example. If $\vec{u} = 3\vec{i} - 5\vec{j}$ and $\vec{v} = -7\vec{i} + 4\vec{j}$ find $6\vec{u} - 8\vec{v}$.

Solution. $6\vec{u} - 8\vec{v} = 6(3\vec{i} - 5\vec{j}) - 8(-7\vec{i} + 4\vec{j}) = (18\vec{i} - 30\vec{j}) + (56\vec{i} - 32\vec{j}) = 74\vec{i} - 62\vec{j}$.

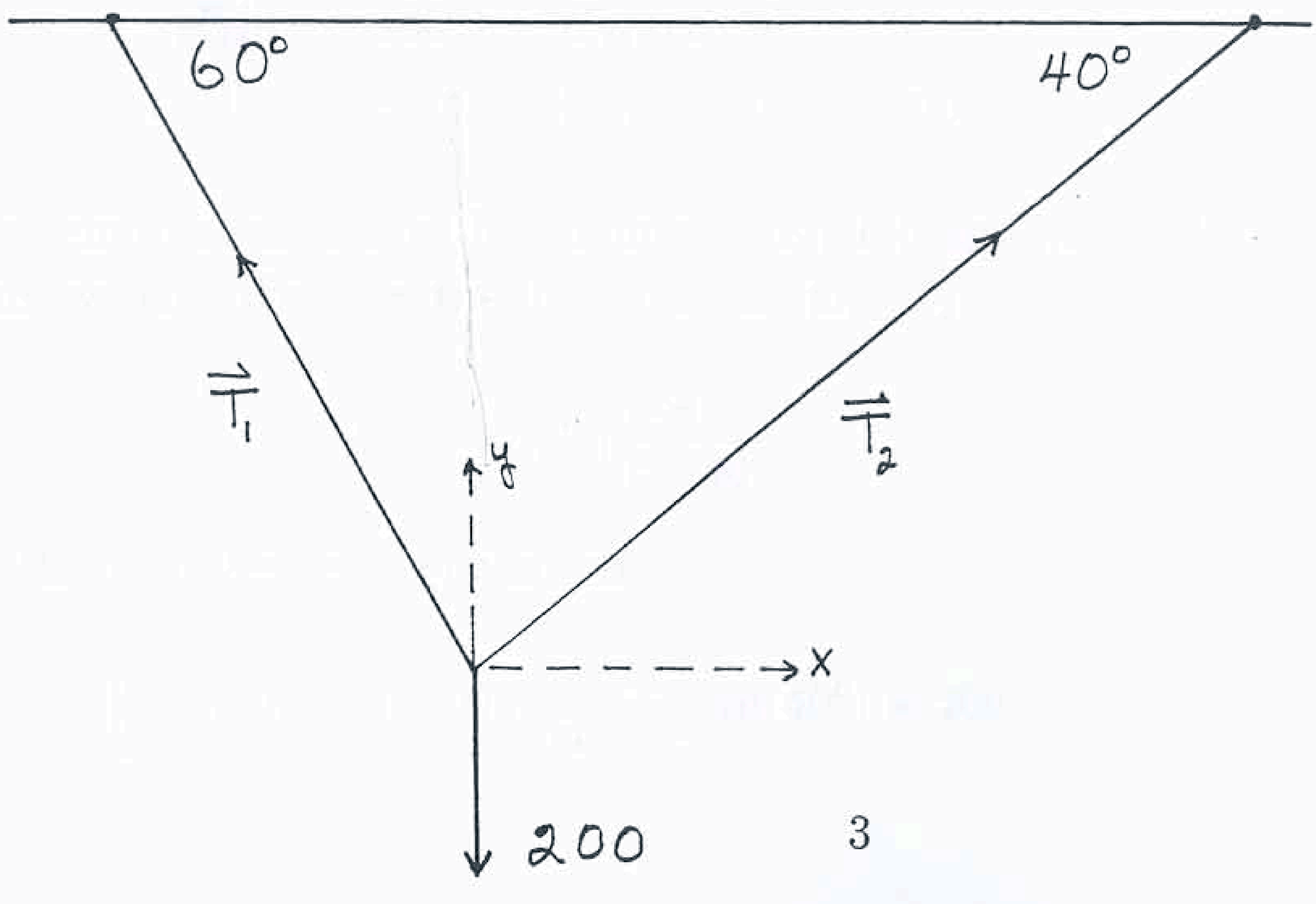
Theorem. If two vectors are parallel then one is a scalar multiple of the other.

Example. Find a vector parallel to $3\vec{i} - 4\vec{j}$ of length 30.

Solution. The length of $3\vec{i} - 4\vec{j}$ is $\sqrt{9+16} = 5$. Any vector parallel to $3\vec{i} - 4\vec{j}$ must be of the form $c(3\vec{i} - 4\vec{j})$. Clearly we need $c = 6$. The vector $18\vec{i} - 24\vec{j}$ is parallel to $3\vec{i} - 4\vec{j}$ and of length 30.

The force acting on an object has both magnitude and direction and so must be represented as a vector. If an object is not in motion, then the sum of the forces acting on the object must be zero. A force vector points in the direction in which the force acts and its length is a measure of the force's strength.

Example. A 200 pound weight hangs from two wires as shown below. Find the forces (tensions) $\vec{T}_1$ and $\vec{T}_2$ in the wires.



Solution. The first thing we must do is to decide on a coordinate system. As indicated in the above diagram we place the origin of the coordinate system on the object. We place the x axis parallel to the ground and the y axis vertical. In this coordinate system we can easily specify the direction of all forces. The two things that we do not know are the magnitudes of the forces $\vec{T}_1$ and $\vec{T}_2$. From the given geometry we easily see that

$$\begin{aligned}\vec{T}_1 &= ||\vec{T}_1||[(-\cos 60^o)\vec{\imath} + (\sin 60^o)\vec{\jmath}] \\ &= ||\vec{T}_1||[(-1/2)\vec{\imath} + (\sqrt{3}/2)\vec{\jmath}], \\ \vec{T}_2 &= ||\vec{T}_2||[(\cos 40^o)\vec{\imath} + (\sin 40^o)\vec{\jmath}]\end{aligned}$$

Gravity exerts a force of $-200\jmath$ on the object. The sum of the forces acting on the object is zero.

$$\begin{aligned}\vec{T}_1 + \vec{T}_2 - 200\vec{\jmath} &= \vec{0} \\ \vec{T}_1 + \vec{T}_2 &= 200\vec{\jmath}.\end{aligned}$$

Therefore,

$$\begin{aligned}||\vec{T}_1||[(-\cos 60^o)\vec{\imath} + (\sin 60^o)\vec{\jmath}] + ||\vec{T}_2||[(\cos 40^o)\vec{\imath} + (\sin 40^o)\vec{\jmath}] \\ = 0\vec{\imath} + 200\vec{\jmath}\end{aligned}$$

If two vectors are equal, then each of their components must be equal.

$$\begin{aligned}||\vec{T}_1||(-\cos 60^o) + ||\vec{T}_2||(\cos 40^o) &= 0 \\ ||\vec{T}_1||(\sin 60^o) + ||\vec{T}_2||(\sin 40^o) &= 200.\end{aligned}$$

This is a system of equations with two unknowns $||\vec{T}_1||$ and $||\vec{T}_2||$. We need to solve this system. We solve the first equation for $||\vec{T}_2||$,

$$||\vec{T}_2|| = ||\vec{T}_1|| \frac{\cos 60^o}{\cos 40^o} \tag{$*$}$$

Substitute this into the second equation.

$$||\vec{T}_1||(\sin 60^o) + ||T_1|| \frac{\cos 60^o}{\cos 40^o}(\sin 40^o) = 200.$$

Multiply both sides by $(\cos 40^o)$.

$$||\vec{T}_1||(\sin 60^o)(\cos 40^o) + ||\vec{T}_1||(\cos 60^o)(\sin 40^o) = 200(\cos 40^o)$$
$$||\vec{T}_1||(\sin 100^o) = 200(\cos 40^o)$$

$$||\vec{T}_1|| = 200\frac{\cos 40^o}{\sin 100^o} = 155.57.$$

Substituting this value in $(*)$, we get

$$||\vec{T}_2|| = 200\frac{\cos 40^o}{\sin 100^o}\frac{\cos 60^o}{\cos 40^o} = 200\frac{\cos 60^o}{\sin 100^o} = 101.54.$$

It follows that

$$\begin{aligned}\vec{T}_1 &= (155.57)[(-\cos 60^o)\vec{i} + (\sin 60^o)\vec{j}] \\ &= -77.79\vec{i} + 134.73\vec{j} \\ \vec{T}_2 &= (101.54)[(\cos 40^o)\vec{i} + (\sin 40^o)\vec{j}] \\ &= 77.79\vec{i} + 65.27\vec{j}.\end{aligned}$$

Note that $134.73 + 65.27 = 200$ and $-77.79 + 77.79 = 0$.

Exercises

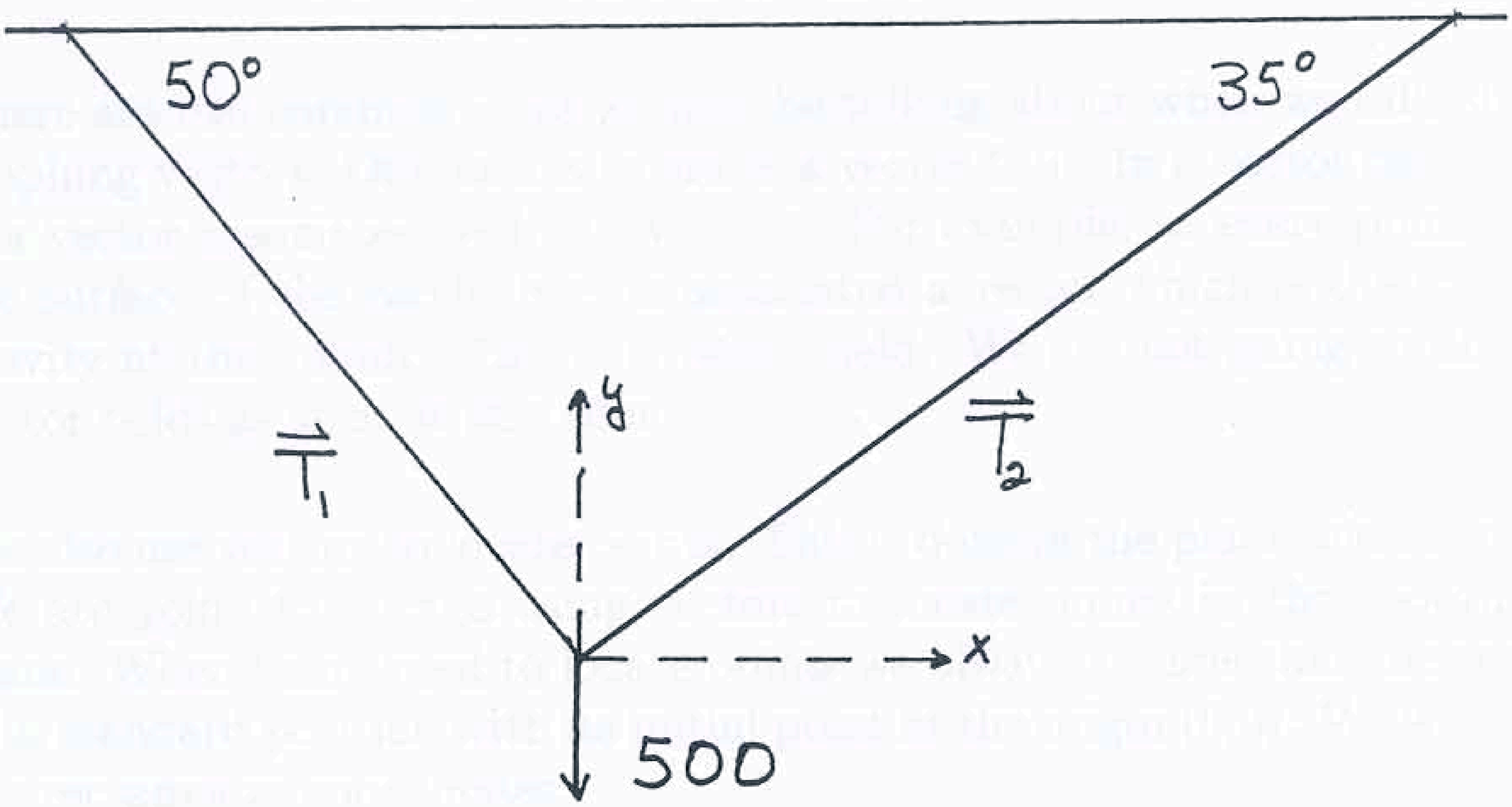
1. Let $\vec{u} = -4\vec{i} + 3\vec{j}$ and $\vec{v} = 5\vec{i} + 2\vec{j}$. On the same set of axis draw $\vec{u}, \vec{v}$, and $\vec{u} + \vec{v}$.

2. Let $\vec{u} = -2\vec{i} + 6\vec{j}$. On the same set of axis draw $\vec{u}$, $2\vec{u}$, and $(-1/2)\vec{u}$.

3. Find a vector parallel to $-4\vec{i} + 3\vec{j}$ that is 20 units long.

4. If $\vec{u} = 5\vec{i} - 2\vec{j}$ and $\vec{v} = 3\vec{i} + 7\vec{j}$ find $6\vec{u} - 4\vec{v}$ and $-3\vec{u} + 10\vec{v}$. No need to give a sketch.

5. A 500 pound weight hangs from two wires as shown below. Find the forces (tensions) $\vec{T}_1$ and $\vec{T}_2$ in the wires.


50°
35°
y
$\vec{T}_1$
$\vec{T}_2$
x
500

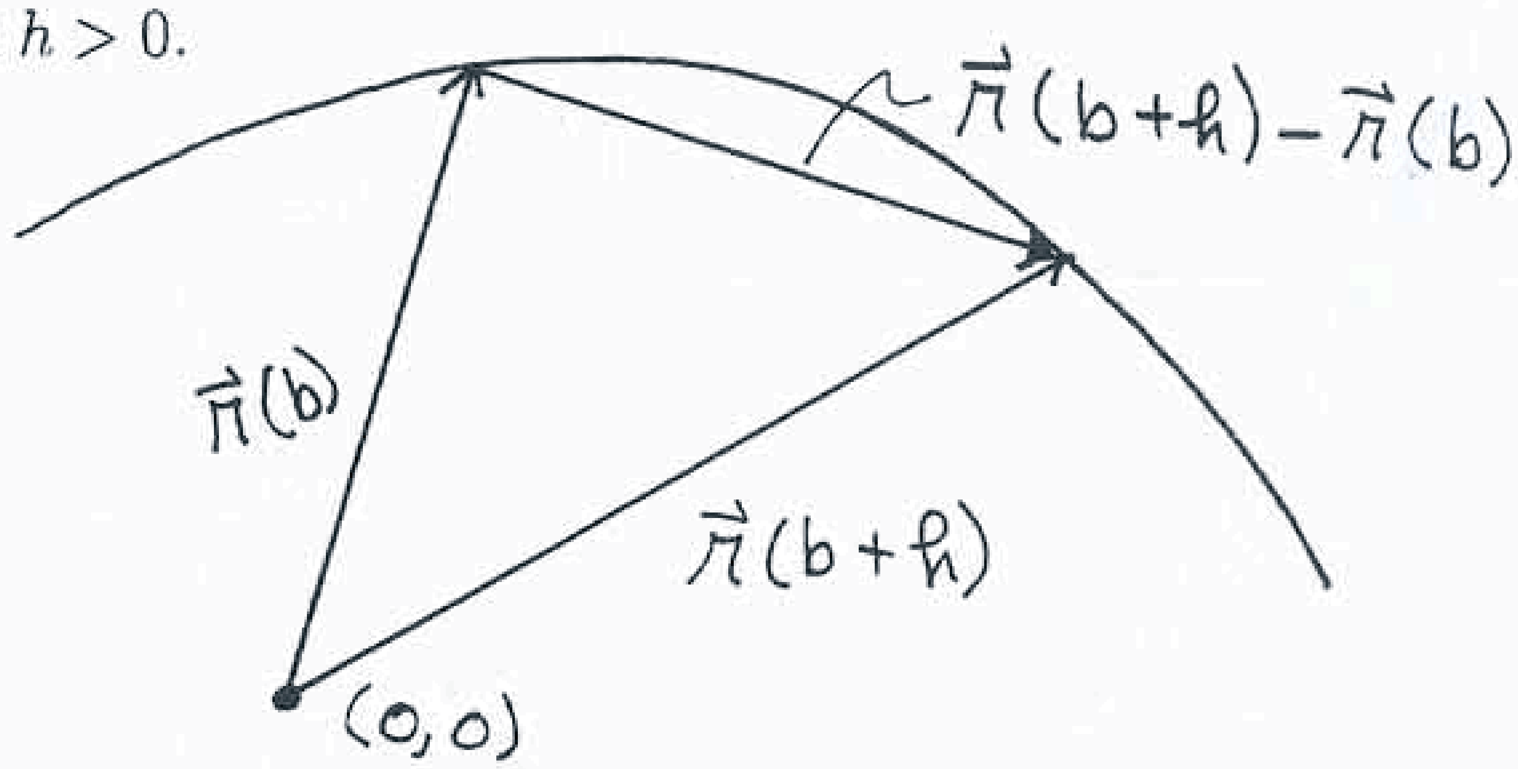
6759 Calculus of Vector Functions

Theorem. If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$, then the derivative $\vec{r}\,'(t)$ is given by $\vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j}$.

Example. Let $\vec{r}(t) = (t^3 + 8t)\vec{i} + (\sin 3t)\vec{j}$, then $\vec{r}\,'(t) = (3t^2 + 8)\vec{i} + (3\cos 3t)\vec{j}$.

Theorem. Consider the curve which is the graph of $\vec{r}(t)$. Assume that $\vec{r}\,'(b)$ exists, then $\vec{r}\,'(b)$ is a vector tangent to this curve at the point located by the vector $\vec{r}(b)$.

Proof. The vector $\vec{r}(b+h) - \vec{r}(b)$ is the vector whose initial point is the point located by the vector $\vec{r}(b)$ and whose terminal point is the point located by the vector $\vec{r}(b+h)$. For ease in graphing assume that $h > 0$.



The vector $\vec{r}(b+h) - \vec{r}(b)$ is a secant line or chord for the curve $\vec{r}(t)$. The number h is a scalar and so $1/h$ is a scalar. The vector $\frac{1}{h}[\vec{r}(b+h) - \vec{r}(b)]$ is a vector with initial point the point located by $\vec{r}(b)$. It is parallel to the vector (chord) $\vec{r}(b+h) - \vec{r}(b)$. Suppose $\vec{r}\,'(b)$ exists. It follows that

$$\lim_{h \to 0+} \frac{\vec{r}(b+h) - \vec{r}(b)}{h}$$

exists and is a vector with initial point the point located by the vector $\vec{r}(b)$. We see from the graph that this vector is tangent to the curve $\vec{r}(t)$ at the point $\vec{r}(b)$. If $\vec{r}\,'(b)$ exists, then it follows that

$$\lim_{h \to 0-} \frac{\vec{r}(b+h) - \vec{r}(b)}{h} = \lim_{h \to 0+} \frac{\vec{r}(b+h) - \vec{r}(b)}{h}$$

Hence, $\vec{r}\,'(b) = \lim_{h \to 0} \frac{\vec{r}(b+h) - \vec{r}(b)}{h}$ is a tangent vector to curve at point located by $\vec{r}(b)$.

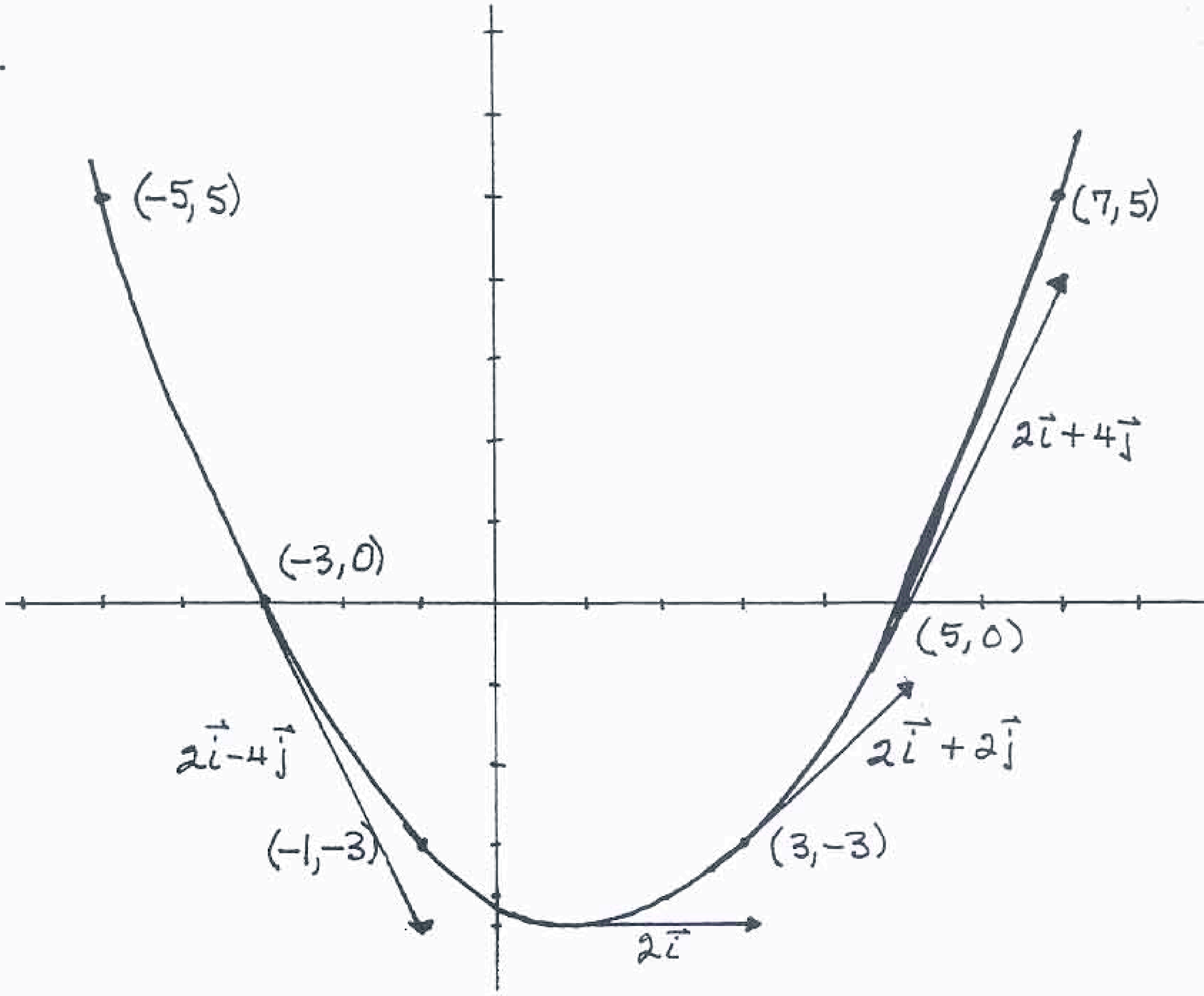
Example. Sketch a graph of the curve

$$\vec{r}(t) = (2t+1)\vec{i} + (t^2 - 4)\vec{j} \text{ for } -3 \le t \le 3,$$

and then sketch the tangent vectors $\vec{r}\,'(2)$, $\vec{r}\,'(0)$, $\vec{r}\,'(1)$, and $\vec{r}\,'(2)$ on the same graph.

Solution. First, we make a chart of values and use these values to plot some points on the curve. We then connect these points using a smooth curve.

t	x	y
-3	-5	5
-2	-3	0
-1	-1	-3
0	1	-4
1	3	-3
2	5	0
3	7	5



Next we find the tangent vectors. Differentiating

$$\vec{r}\,'(t) = 2\vec{i} + (2t)\vec{j}.$$

Substituting we find the tangent vectors $\vec{r}\,'(-2) = 2\vec{i} - 4\vec{j}$, $\vec{r}\,'(0) = 2\vec{i}$, $\vec{r}\,'(1) = 2\vec{i} + 2\vec{j}$ and $\vec{r}\,'(2) = 2\vec{i} + 4\vec{j}$. We then plot these vectors on the graph. Note that $\vec{r}(-2) = -3\vec{i}$, $\vec{r}(0) = \vec{i} - 4\vec{j}$, $\vec{r}(1) = 3\vec{i} - 3\vec{j}$, and $\vec{r}(2) = 5\vec{i}$.

Example. Find the vector equation of the tangent line to the graph of

$$\vec{r}(t) = t(t^2 - 9)i + t(t + 2)\vec{j}$$

at the point where $t = 2$.

Solution. Since $\vec{r}(2) = 2(4 - 9)\vec{i} + 2(2 + 2)\vec{j} = -10\vec{i} + 8\vec{j}$, the point on the curve has the coordinates $(-10, 8)$. Next,

$$\begin{aligned} \vec{r}\,'(t) &= (3t^2 - 9)\vec{i} + (2t + 2)\vec{j} \\ \vec{r}\,'(2) &= 3\vec{i} + 6\vec{j}. \end{aligned}$$

The vector $3\vec{i} + 6\vec{j}$ is a tangent vector to the curve $\vec{r}(t)$ at the point $(-10, 8)$. The tangent line to the curve is a line through the point $(-10, 8)$ with direction vector the same as the tangent vector. The direction vector of the tangent line is the tangent vector $-3\vec{i}+6\vec{j}$. The equation of the tangent line is

$$\begin{aligned}\vec{R}(t) &= (-10\vec{i}+8\vec{j})+t(3\vec{i}+6\vec{j})\\ &= (3t-10)\vec{i}+(6t+8)\vec{j}.\end{aligned}$$

Definition. If $\vec{r}(t)=f(t)\vec{i}+g(t)j$, then

$$\int \vec{r}(t)dt = \left[\int f(t)dt\right]\vec{i}+\left[\int g(t)dt\right]\vec{j}$$

and

$$\int_a^b \vec{r}(t)dt = \left[\int_a^b f(t)dt\right]\vec{i}+\left[\int_a^b g(t)dt\right]\vec{j}$$

Example. Let $\vec{r}(t)=[4t^3-10t]\vec{i}+[3t^2+4t]\vec{j}$ then

$$\int \vec{r}(t)dt = [t^4-5t^2]\vec{i}+[t^3+2t^2]\vec{j}+a\vec{i}+b\vec{j}$$

where $a\vec{i}+b\vec{j}$ is a constant vector.

$$\begin{aligned}\int_0^5 \vec{r}(t)dt &= [5^4-5(5)^2]\vec{i}+[5^3+2(5)^2]\vec{j}\\ &= 500\vec{i}+175\vec{j}.\end{aligned}$$

Let us consider the problem of finding the length of the curve $\vec{r}(t)$, $a\le t\le b$.

Divide the interval $a\le t\le b$ into n subintervals of length $\Delta t=h=\frac{b-a}{n}$. Using the points $a=t_0,t_1,t_2,\cdots t_n=b$. Plot the points $\vec{r}(t_0),\vec{r}(t_1),\vec{r}(t_2),\cdots,\vec{r}(t_n)$ on the curve. Let $\vec{r}(c)$ and $\vec{r}(c+h)$ denote two successive points on the curve. How long is the chord (vector) from $\vec{r}(c)$ to $\vec{r}(c+h)$. This is the vector $\vec{r}(c+h)-\vec{r}(c)$. The length of this vector is

$$||\vec{r}(c+h)-\vec{r}(c)|| = ||[f(c+h)-f(c)]\vec{i}+[g(c+h)-g(c)]\vec{j}||$$

$$=\sqrt{[f(c+h)-f(c)]^2+[g(c+h)-g(c)]^2}$$

$$= h\sqrt{\left[\frac{f(c+h)-f(c)}{h}\right]^2 + \left[\frac{g(c+h)-g(c)}{h}\right]^2}$$

The sum of the lengths of all the chords is

$$\sum_{k=1}^{n} h\sqrt{\left[\frac{f(c+h)-f(c)}{h}\right]^2 + \left[\frac{g(c+h)-g(c)}{h}\right]^2}$$

The sum of the lengths of the chords is approximately equal to the length of the curve and is actually equal to the length of the curve in the limit as $h \to 0$. Note that $\lim_{h\to 0}$ is the same as $\lim_{n\to\infty}$. Note that

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = f'(c).$$

Also recall that the limit as $n \to \infty$ of a sum like the above sum is a definite integral. Taking the limit of the above sum as $n \to \infty$ $(h \to 0)$ it follows that the length of the curve $\vec{r}(t)$ for $a \le t \le b$ is given by

$$\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt = \int_a^b ||\vec{r}\,'(t)||dt.$$

Example. Find the arc length of the curve $\vec{r}(t) = (12t - t^3)\vec{i} + (6t^2)\vec{j}$ for $0 \le t \le 3$.

Solution. We have $\vec{r}\,'(t) = (12 - 3t^2)\vec{i} + (12t)\vec{j}$.

$$\begin{aligned} ||\vec{r}\,'(t)|| &= \sqrt{(12-3t^2)^2 + (12t)^2} \\ &= \sqrt{144 + 72t^2 + 9t^4} = 12 + 3t^2 \end{aligned}$$

$$\int_0^3 [12 - 3t^2]dt = 12t + t^3 \mid_0^3 = 63$$

If the curve $\vec{r}(t)$ for $a \le t \le b$ is on one side of the x axis then a surface is generated when this curve is revolved about the x axis. An interesting problem is to find the surface area of such a surface. However, we are not going to discuss this idea here.

When discussing the definite integral $\int_a^b f(x)dx$ we spent a lot of time finding the area of the region bounded by the curve $y = f(x)$, the line $x = a$, the line $x = b$, and the x axis. The situation is more complicated when discussing area enclosed by the graph of $\vec{r}(t)$. The curve $y = f(x)$ intersects the line $x = a$ exactly one time. The curve $\vec{r}(t)$ may intersect the line $x = a$ any number of times. This makes it more difficult to exactly describe the region we are discussing. There is one situation which is pretty easy to deal with. Just so we have looked at this idea we will discuss this situation very briefly. It is possible that the curve $\vec{r}(t)$ for $a \le t \le b$ exactly encloses a region in the xy plane. The formula for the area of such a region is easy to state but it is a little bit different. Suppose a region R in the xy plane is exactly enclosed by the curve $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$ for $a \le t \le b$, then the area of is given by

$$\text{Area } = \int_a^b f(t)g'(t)dt = -\int_a^b g(t)f'(t)dt.$$

Example. The curve $\vec{r}(t) = (4\cos t)\vec{i} + (9\sin t)\vec{j}$ for $0 \le t \le 2\pi$ exactly encloses an ellipse. Find the area of this ellipse.

Solution. We have $f(t) = 4\cos t$, $g(t) = 9\sin t$, $f'(t) = -4\sin t$, and $g'(t) = 9\cos t$. The area of the ellipse is given by either of the two integrals

$$-\int_0^{2\pi} (9\sin t)(-4\sin t)dt = 36\pi$$

$$\int_0^{2\pi} (4\cos t)(9\cos t)dt = 36\pi.$$

Exercises

1. If $\vec{r}(t) = [4t^3 + 8t]\vec{i} + [3t(t+4)]\vec{j}$, find $\vec{r}\,'(t)$, $\int \vec{r}(t)dt$, and $\int_0^2 \vec{r}(t)dt$.

2. Sketch the graph of $\vec{r}(t) = [t(t^2-9)]\vec{i} + [9-t^2]\vec{j}$ for $-3 \le t \le 3$. On the same graph sketch the tangent vectors $\vec{r}\,'(-2)$, $\vec{r}\,'(0)$, $\vec{r}\,'(1)$, and $\vec{r}\,'(2)$.

3. Consider the curve $\vec{r}(t) = (2t^3 + t)\vec{i} + (4t^2 + t)\vec{j}$. Find the point on this curve corresponding to $t = 2$. Find the vector equation of the tangent line to this curve at the point where $t = 2$.

4. Find the arc length of the curve $\vec{r}(t) = (t^2)\vec{i} + (t^3)\vec{j}$ for $1 \le t \le 4$.

5. Sketch the graph of the curve $\vec{r}(t) = [t(t^2-9)]\vec{i} + [4-t^2]\vec{j}$ for $-3 \le t \le 3$. Note that this curve exactly encloses a region R. Find the area of the region R.

6. Find the arc length of the curve

$$\vec{r}(t) = [(1/2)t^4 - t^2]\vec{i} + [(4/3)t^3]\vec{j} \text{ for } 1 \le t \le 3.$$

Optional

Many problems about polar coordinates can be resolved by reference to the same problem in vector equations. Suppose we wish to graph the polar equation $r = g(\theta)$. The formulas for polar coordinates are $x = r\cos\theta$ and $y = r\sin\theta$. When $r = g(\theta)$, this is $x = g(\theta)\cos\theta$ and $y = g(\theta)\sin\theta$. Suppose a certain curve is obtained by graphing $r = g(\theta)$, $\alpha \le \theta \le \beta$, using polar coordinates. The same curve is obtained by graphing

$$\vec{r}(\theta) = [g(\theta)\cos\theta]\vec{i} + [g(\theta)\sin\theta]\vec{j} \text{ for } \alpha \le \theta \le \beta.$$

Graphing the polar equation $r = 2\cos\theta$, $0 \leq \theta \leq \pi$, is the same as graphing the vector equation $[(2\cos\theta)\cos\theta]\vec{\imath} + [(2\cos\theta)\sin\theta]\vec{\jmath}$, $0 \leq \theta \leq \pi$. In either case we get the circle with center (1,0) and radius 1.

Suppose we want to find the arc length of $\vec{r}(\theta) = [g(\theta)\cos\theta]\vec{\imath} + [g(\theta)\sin\theta]\vec{\jmath}$, $\alpha \leq \theta \leq \beta$.

$$\vec{r}\,'(\theta) = [-g(\theta)\sin\theta + g'(\theta)\cos\theta]\vec{\imath} + [g(\theta)\cos\theta + g'(\theta)\sin\theta]\vec{\jmath}.$$

After a little work, we get

$$||\vec{r}\,'(\theta)||^2 = [g(\theta)]^2 + [g'(\theta)]^2.$$

Therefore, the arc length for $r = g(\theta)$, $\alpha \leq \theta \leq \beta$, is

$$\int_\alpha^\beta \sqrt{[g(\theta)]^2 + [g'(\theta)]^2}\,d\theta.$$

6761 Velocity and Acceleration

Suppose that the position of an object moving in the coordinate plane is given at time t by $\vec{r}(t)$. For example, if $\vec{r}(t) = (t^3+2)\vec{i} + (2t^2+5t)\vec{j}$ then when $t = 1$ the object is at $\vec{r}(1) = 3\vec{i}+7\vec{j}$ or the object is located at (3,7). When $t = 2$ the object is at $\vec{r}(2) = 10\vec{i}+18\vec{j}$ or the object is located at (10,18). If $h > 0$, then $\vec{r}(b+h) - \vec{r}(b)$ is the vector from the position of the object at time $t = b$ to the position of the object at time $t = b + h$. The vector $\vec{r}(b+h) - \vec{r}(b)$ gives the change in position. Change in position divided by change in time is average velocity.

$$\text{Average velocity } = \frac{\vec{r}(b+h) - \vec{r}(b)}{h}.$$

The instantaneous velocity is obtained by taking the limit as $h \to 0$.

Theorem. If the position of an object at time t is given by $\vec{r}(t)$, then the instantaneous velocity of the object is given by the derivative $\vec{r}\,'(t)$.

Example. A particle is moving in the coordinate plane such that its position at time t is given by $\vec{r}(t) = (2t^2+5)\vec{i} + (t^2-3t)\vec{j}$. Find the velocity of this particle at $t = 1$, $t = 2$, and $t = 4$.

Solution. We have $\vec{r}\,'(t) = (4t)i + (2t-3)\vec{j}$.

$$\vec{r}\,'(1) = 4\vec{i} - \vec{j} \quad \vec{r}\,'(2) = 8\vec{i} + \vec{j} \quad \vec{r}\,'(4) = 16\vec{i} + 5\vec{j}$$

If a particle is moving in the plane such that its position at time t is given by $\vec{r}(t)$, then the acceleration of the particle is given by the second derivative.

Example. Suppose a particle is moving around a circle of radius 1 such that its position at time t is given by $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$. Find the velocity and acceleration at $t = \pi/4$ and $t = \pi/2$.

Solution. The derivative is $\vec{r}\,'(t) = (-\sin t)i + (\cos t)j$ so that the velocities are $\vec{r}\,'(\pi/4) = (-\sqrt{2}/2)\vec{i} + (\sqrt{2}/2)\vec{j}$ and $\vec{r}\,'(\pi/2) = -\vec{i}$. Note that speed $= ||\vec{r}\,'(t)||$ is always 1. We have $\vec{r}\,''(t) = (-\cos t)\vec{i} + (-\sin t)\vec{j} = -\vec{r}(t)$. The accelerations are $\vec{r}\,''(\pi/4) = (-\sqrt{2}/2)\vec{i} + (-\sqrt{2}/2)\vec{j}$ and $\vec{r}\,''(\pi/2) = -\vec{j}$. Note that the acceleration vector always points toward the origin.

Example. A particle is moving in the plane such that its velocity vector is given by $\vec{r}\,'(t) = (6t^2+5)\vec{i} + (4t+3)\vec{j}$. It's initial position is $\vec{r}(0) = 7i + 11j$. Find the position of the particle $\vec{r}(t)$.

Solution. Since $\vec{r}\,'(t)$ is the velocity the position is given by

$$\begin{aligned}\vec{r}(t) = \int \vec{r}\,'(t)dt &= \int [(6t^2+5)\vec{i} + (4t+3)\vec{j}]dt \\ &= (2t^3+5t)\vec{i} + (2t^2+3t)\vec{j} + a\vec{i} + b\vec{j}.\end{aligned}$$

We must choose the constant vector $a\vec{i} + b\vec{j}$ such that $\vec{r}(0) = 7i + 11j$. This means $\vec{r}(0) = a\vec{i} + b\vec{j} = 7\vec{i} + 11\vec{j}$. Therefore,

$$\begin{aligned}\vec{r}(t) &= (2t^3 + 5t)\vec{i} + (2t^2 + 3t)\vec{j} + 7i + 11\vec{j} \\ &= (2t^3 + 5t + 7)\vec{i} + (2t^2 + 3t + 11)\vec{j}\end{aligned}$$

Exercises

1. If the position of a particle in the coordinate plane is given by

$$\vec{r}(t) = (t^4 + 8t^2)i + (3t^3 + 7t)j$$

Find the velocity and acceleration of this particle at times $t = 1$ and $t = 2$.

2. A particle is moving in the plane such that its velocity vector is given by

$$\vec{r}\,'(t) = (8t^3 + 7)\vec{i} + (9t^2 + 4t)\vec{j}.$$

It's initial position is $\vec{r}(0) = 5i - 3j$. Find the expression $\vec{r}(t)$ for its position at any time t.

6983 Special Problems

1. Find the general solution of the following differential equation:

$$x\frac{dy}{dx} - 2y = 12x^3 \cos 3x.$$

2. Solve the initial value problem:

$$x\frac{dy}{dx} + 3y = 8x \text{ and } y(1) = 6.$$

3. Solve the initial value problem:

$$\frac{dy}{dt} = 12 - \frac{6y}{(100 + 2t)} \text{ and } y(0) = 100.$$

4. A tank contains 25 kg of salt dissolved in 4,000 liters of water. Brine that contains (1/10) kg of salt per liter of water enters the tank at the rate of 20 liters/min. Brine is drained from the tank at the slower rate of 15 liters/min. Find an expression for the amount of salt in the tank at time t.

5. A tank contains 80 lbs of salt dissolved in 600 gallons of water. Brine that contains (1/3) lbs of salt per gallon of water enters the tank at the rate of 15 gal/min. Brine is drained from the tank at the slower rate of 10 gal/min. Find an expression for the amount of salt in the tank at time t.

6. The following is the value chart for a certain polar function $r = f(\theta)$. Sketch the graph of this function.

θ	r
0	2
$\pi/4$	3.4
$\pi/2$	4
$3\pi/4$	3.4
π	2
$5\pi/4$	0.6
$3\pi/2$	0
$7\pi/4$	0.6
2π	2

7. The following is the value chart for a certain polar function $r = f(\theta)$. Sketch the graph of this function.

θ	r
0	3
$\pi/6$	2.6
$\pi/3$	1.5
$\pi/2$	0
$2\pi/3$	-1.5
$5\pi/6$	-2.6
π	-3

θ	r
$7\pi/6$	-2.6
$4\pi/3$	-1.5
$3\pi/2$	0
$5\pi/3$	1.5
$11\pi/6$	2.6
2π	3

8. Show that the following vector equations are each a vector equation for the same line.

$$\vec{r}(t) = (-2t + 3)\vec{\imath} + (2t - 5)\vec{\jmath}$$
$$\vec{R}(s) = (2s + 1)\vec{\imath} + (-2s - 3)\vec{\jmath}$$

9. Show that the following vector equations are each a vector equation for the same parabola. Recall that a parabola is determined by three points.

$$\vec{r}(t) = (t^2 - 3t)\vec{\imath} + (4 - t^2)\vec{\jmath}$$
$$\vec{R}(s) = (s^2 - 7s + 10)\vec{\imath} + (4s - s^2)\vec{\jmath}$$

10. Consider the parabola which is the graph of the vector equation

$$\vec{r}(t) = (t^2 + 3t - 4)\vec{\imath} + (6 + 4t - t^2)\vec{\jmath}.$$

Find the vector equation of the tangent line to this curve at the point (6,10).

11. Consider the curve which is the graph of the vector equation

$$\vec{r}(t) = (t^3 - t^2 + 4)\vec{\imath} + (-t^2 + 3t + 3)\vec{\jmath}.$$

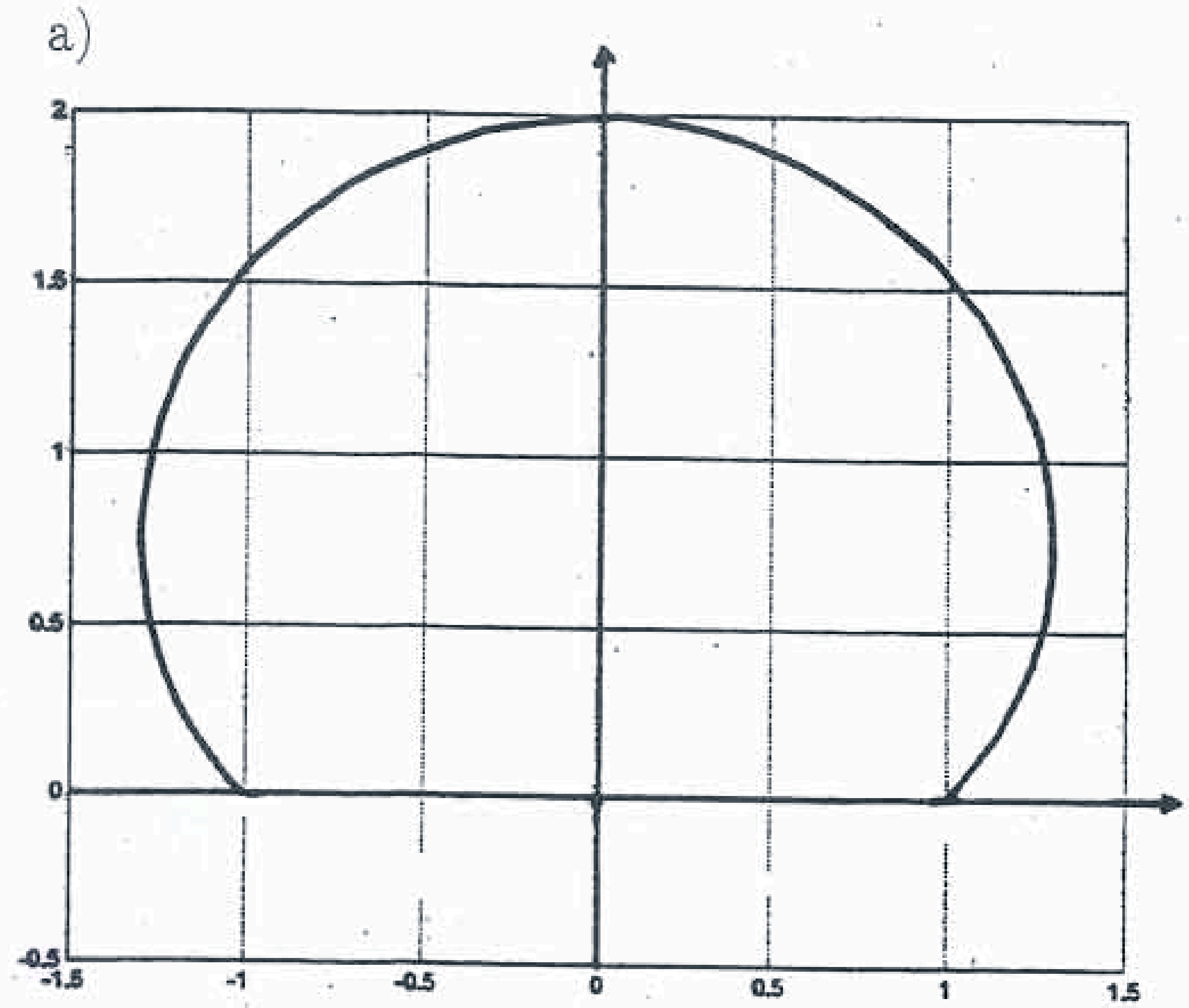
Find the vector equation of the tangent line to this curve at the point corresponding to $t = 2$.

12. Find four different sets of polar coordinates for the following points. All coordinates must be such that $-2\pi < \theta \leq 2\pi$. Give exact values of θ in terms of π.

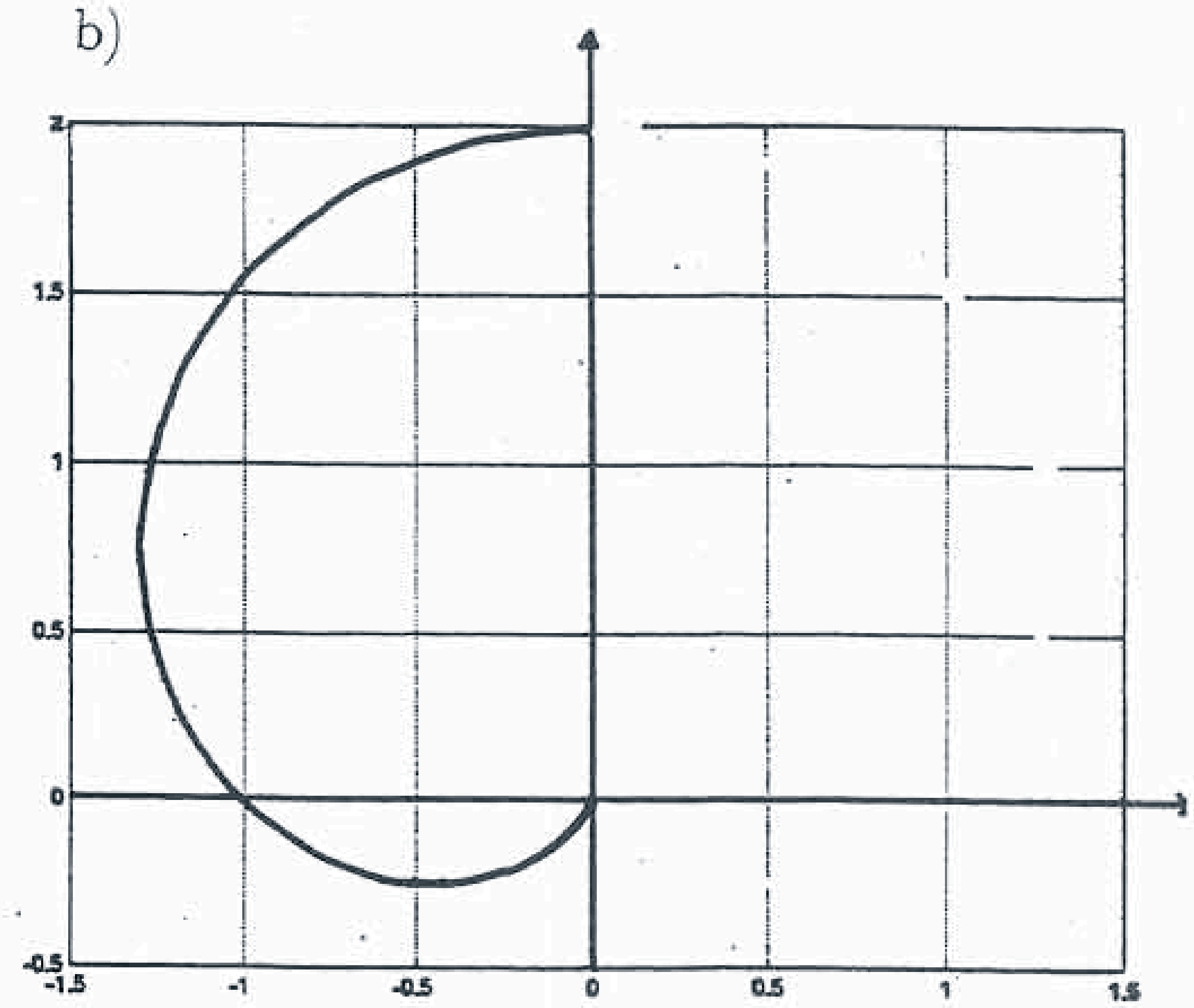
a) $(-\sqrt{2}, \sqrt{2})$ b) $(\sqrt{3}, -1)$

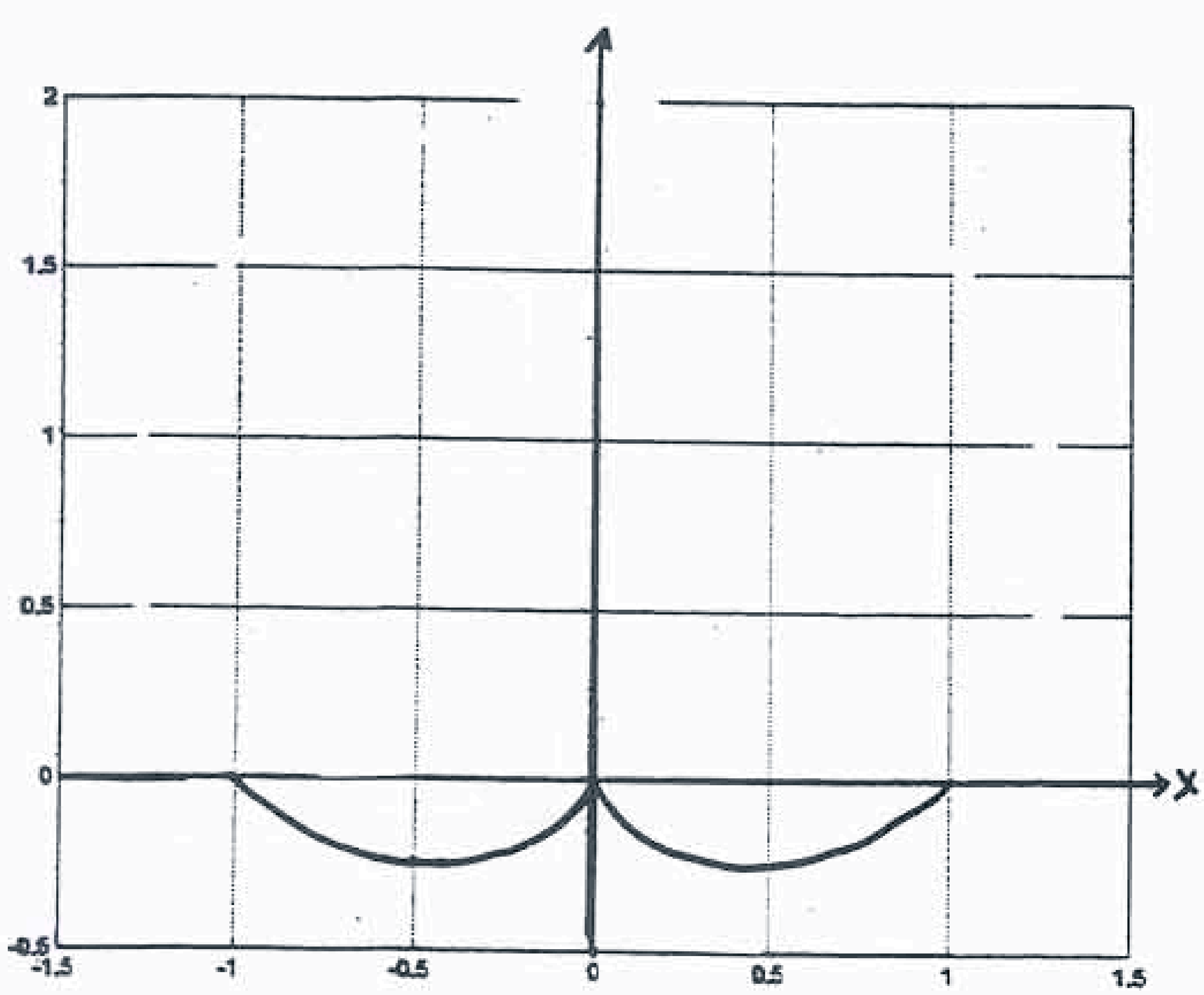
13. 25. Below are some parts of the graph of the polar equation $r = 1 + \sin\theta$. In each case state exactly the values of θ which correspond to the section of the graph shown.

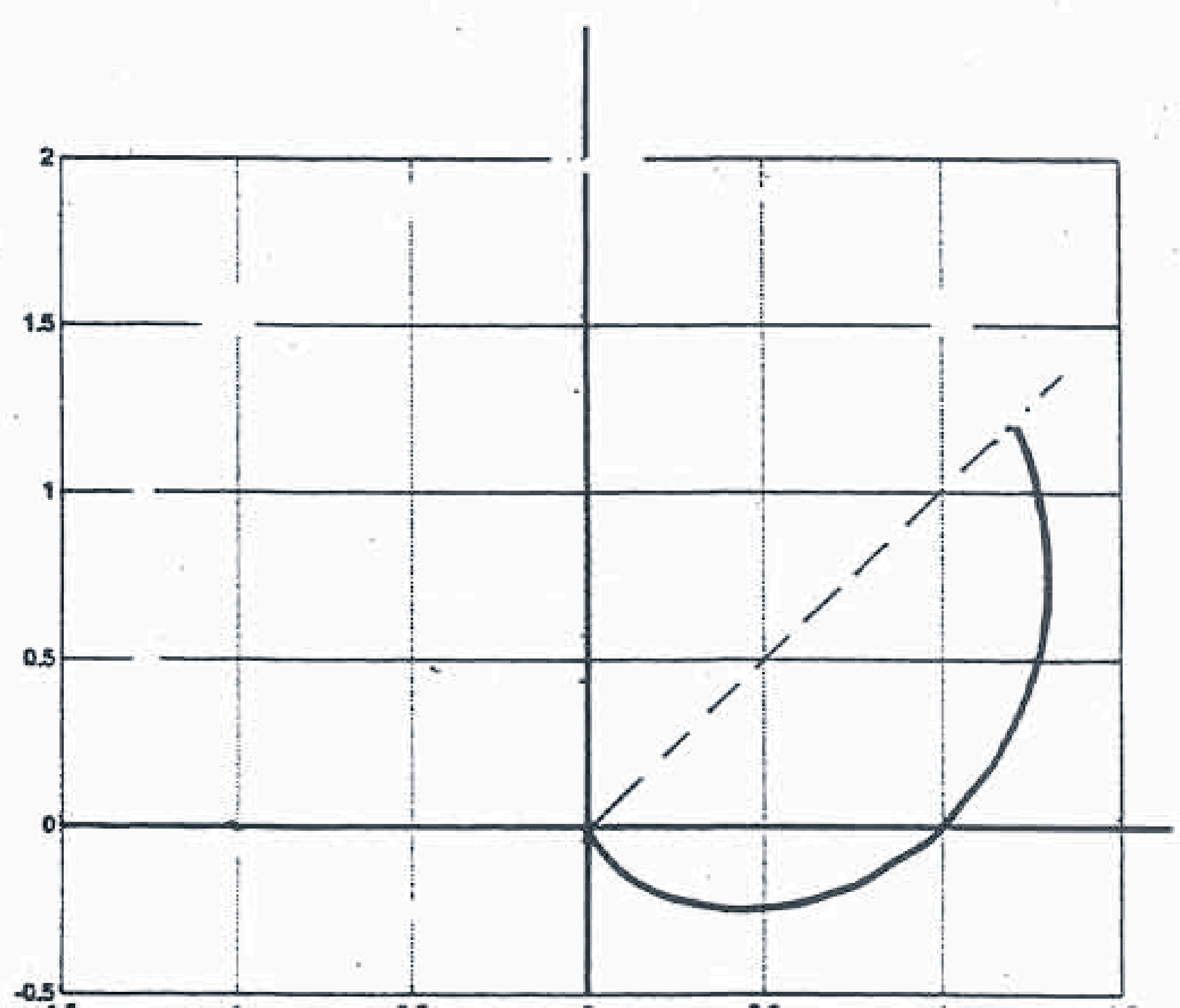
a)



b)

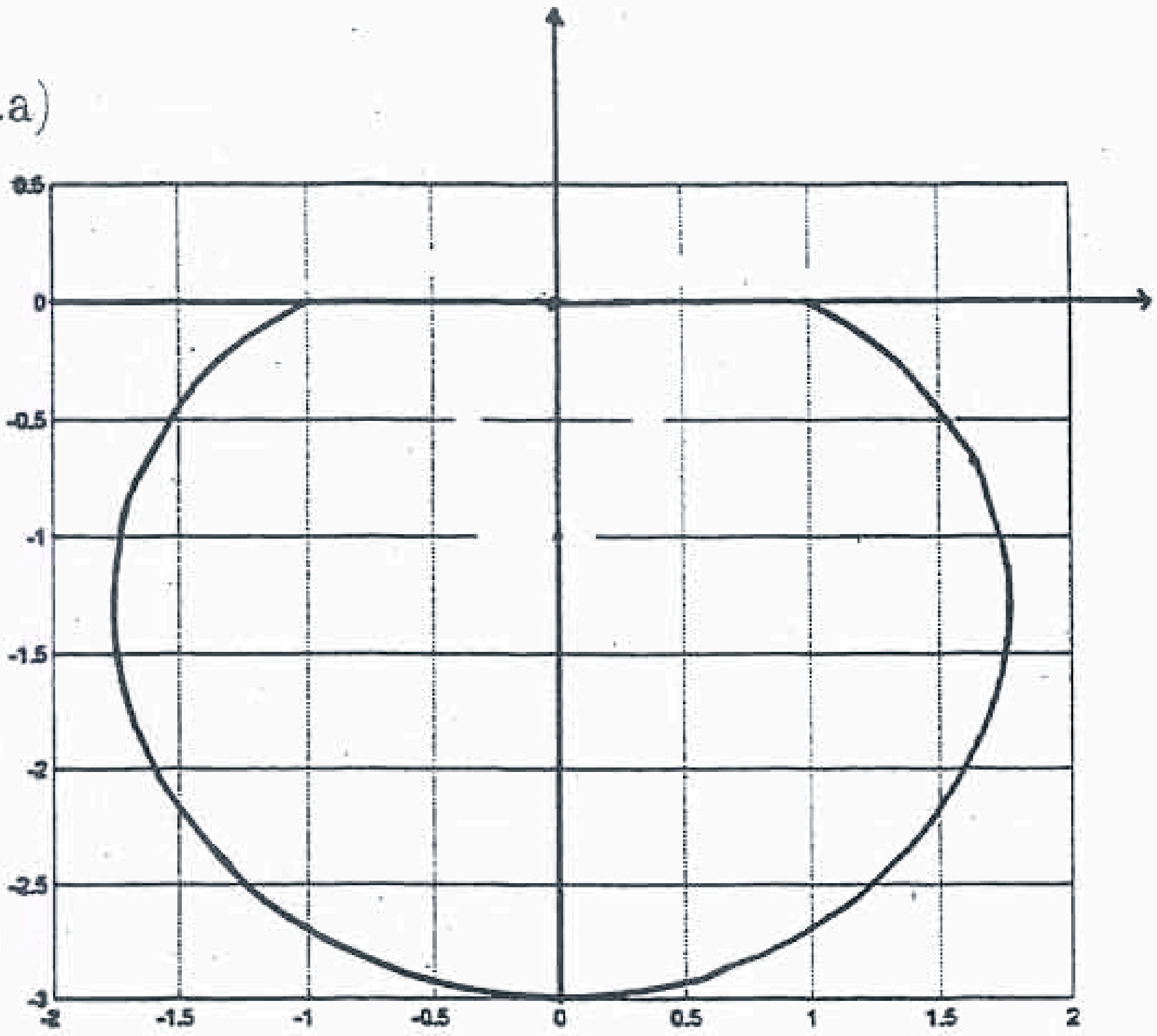




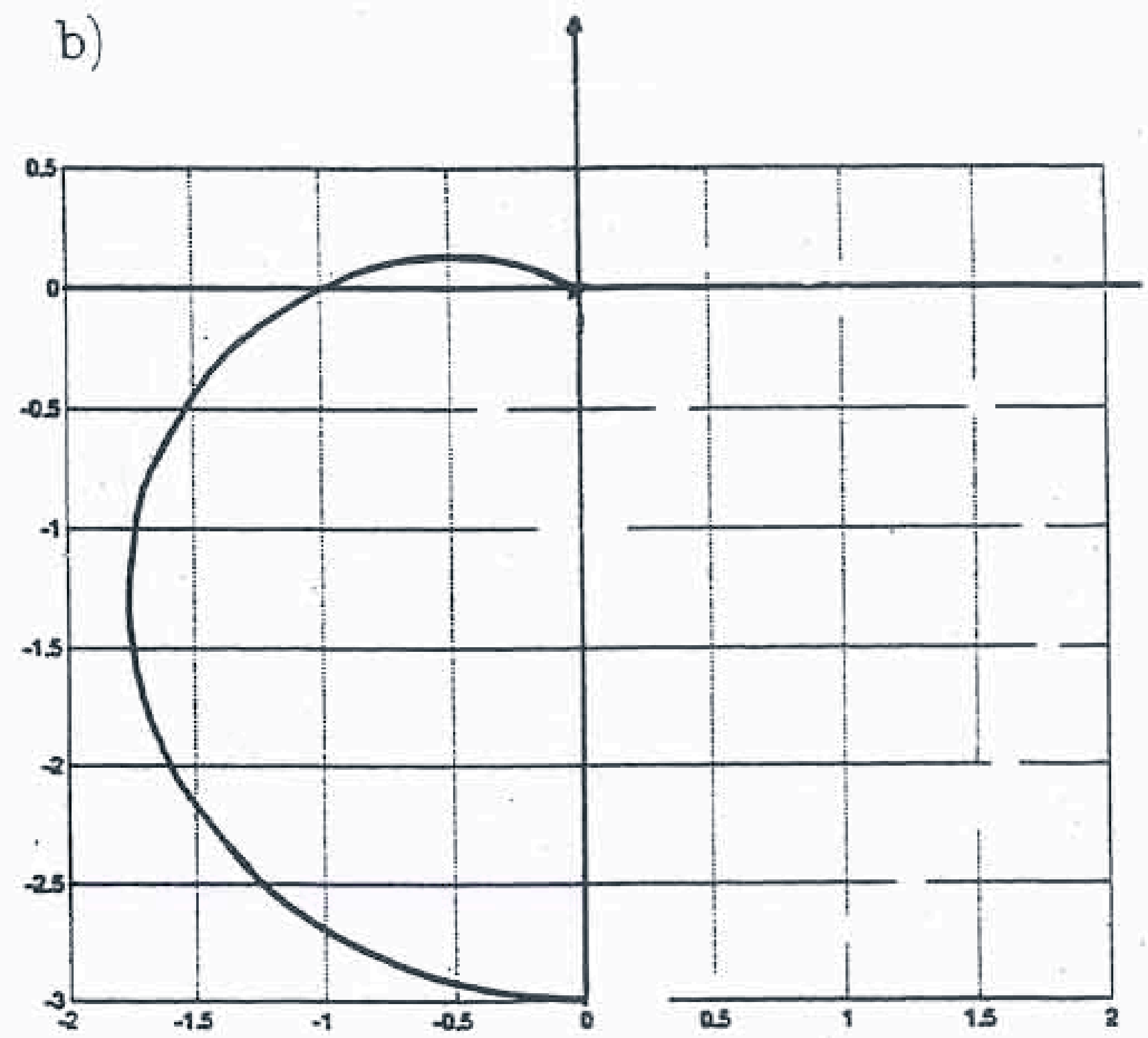


14. Below are some parts of the graph of the polar equation $r = 1 - 2\sin\theta$. State exactly the values of θ which correspond to the sections of the graph shown.

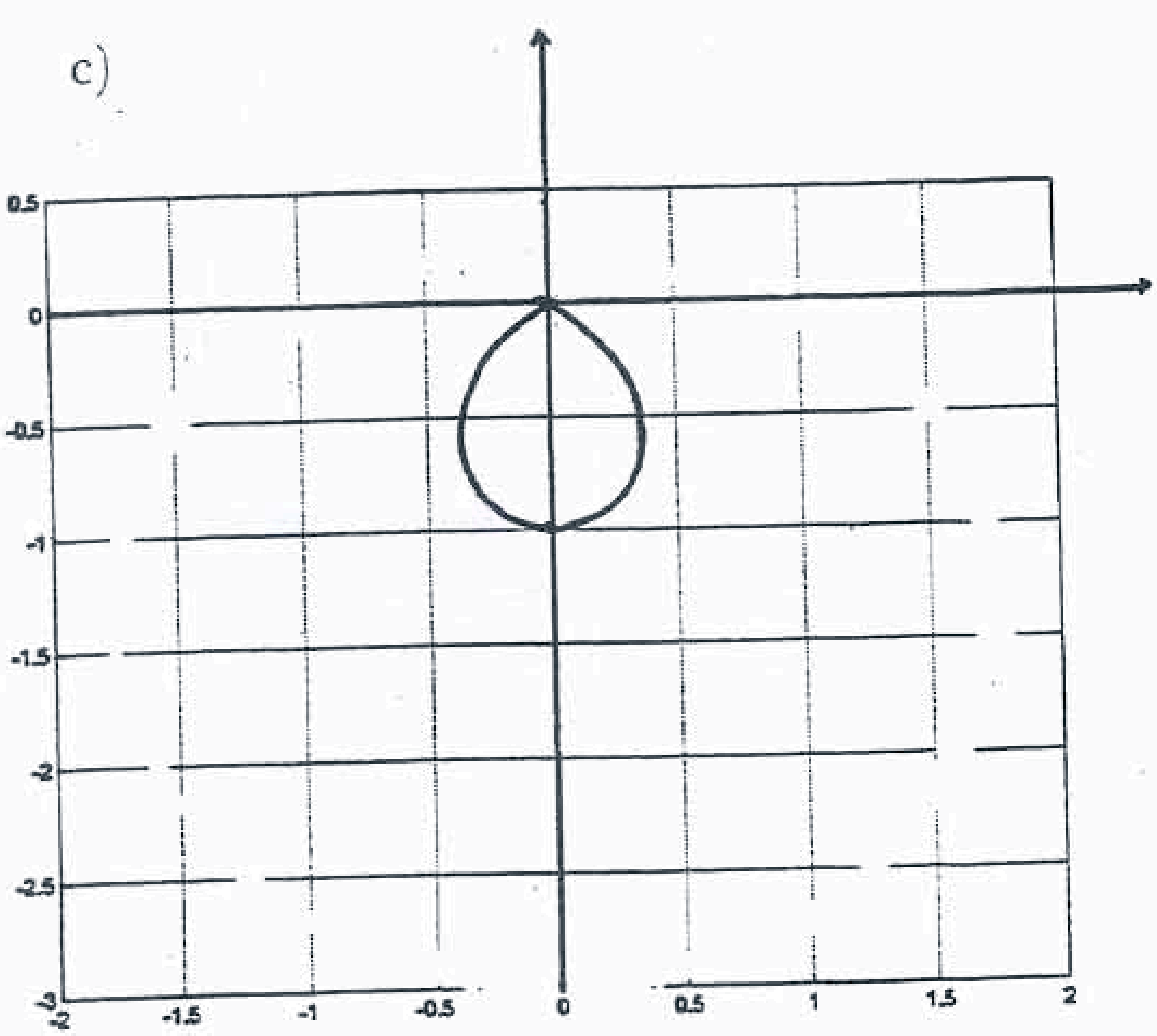
a)



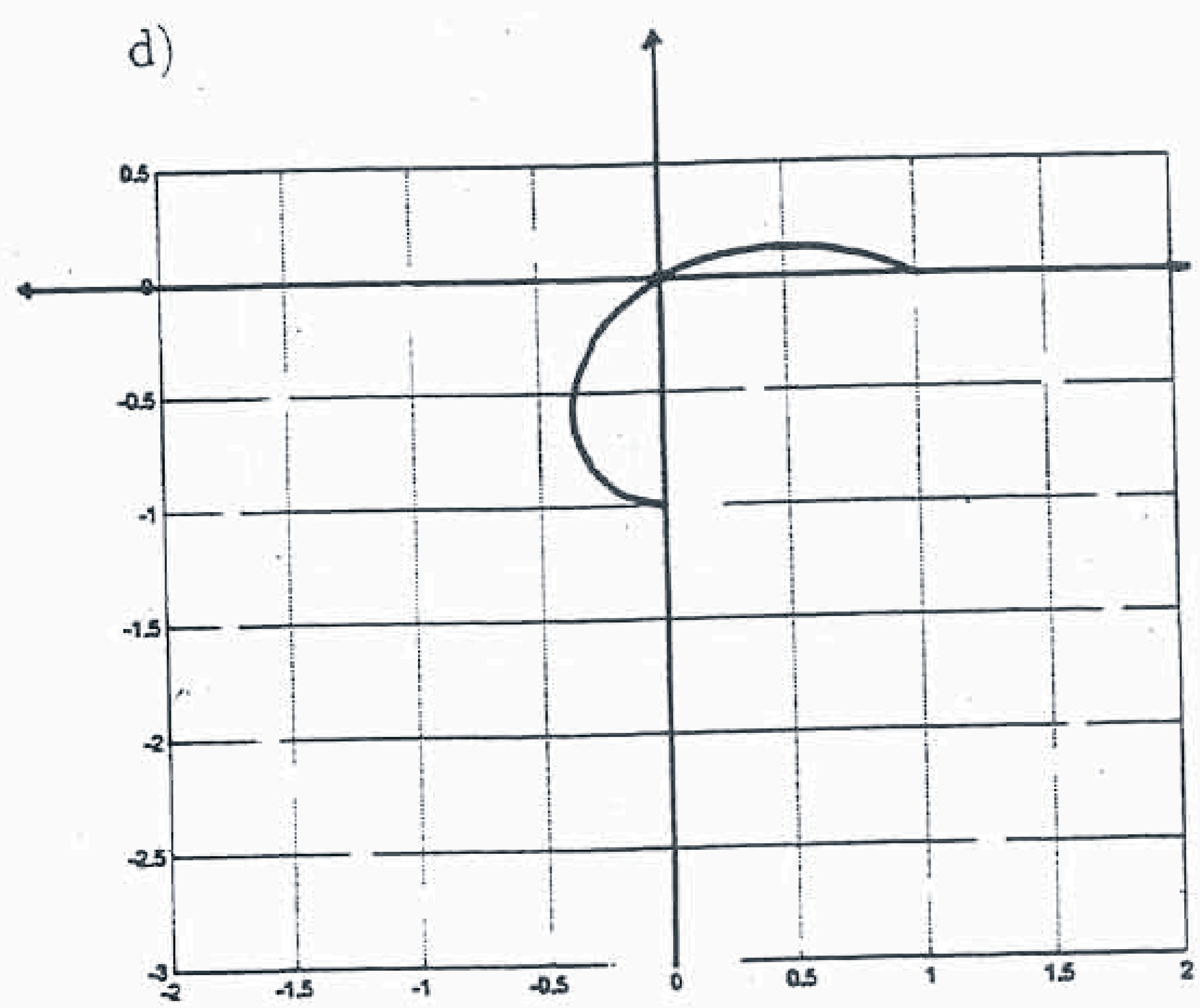
b)



c)

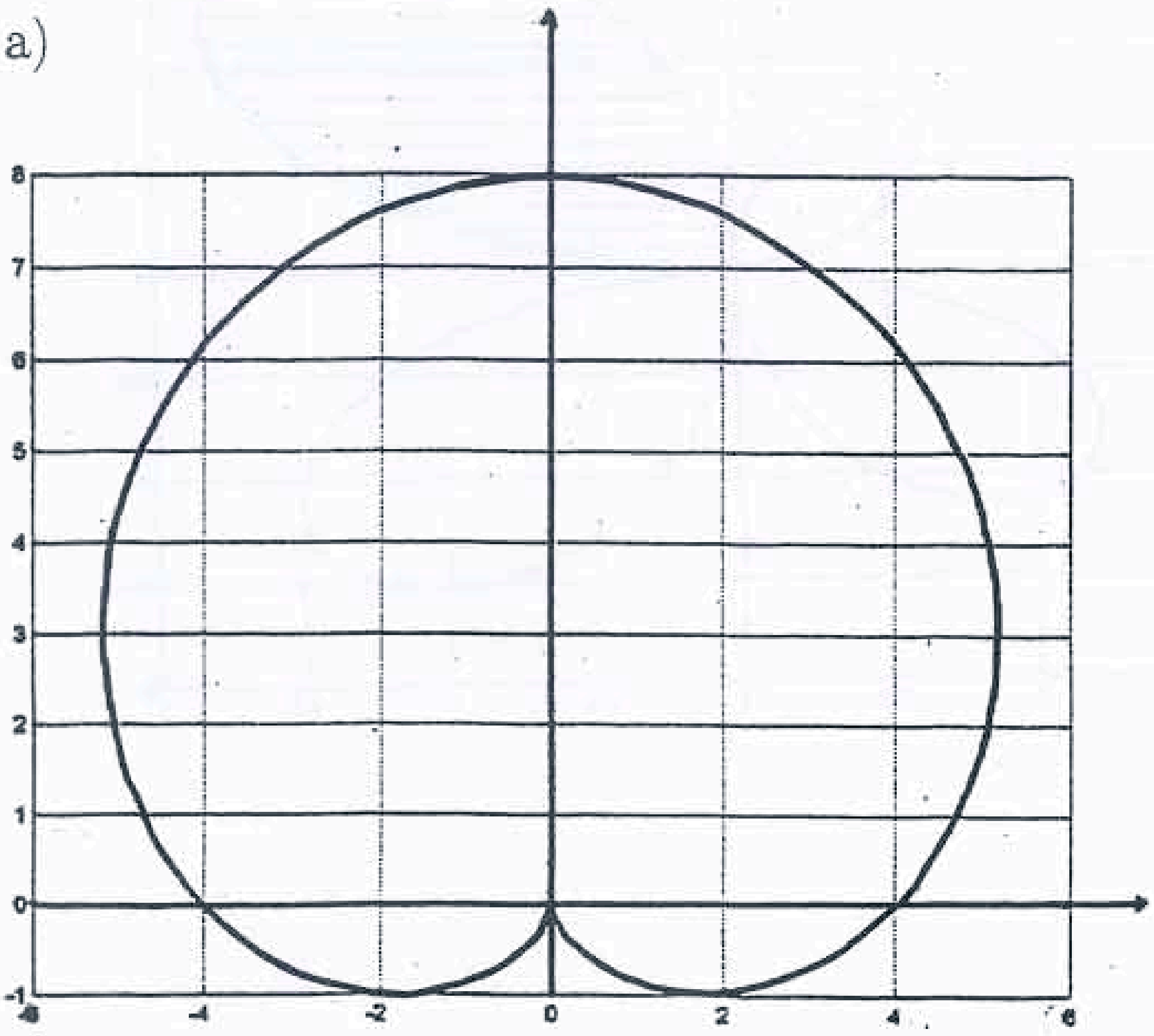


d)

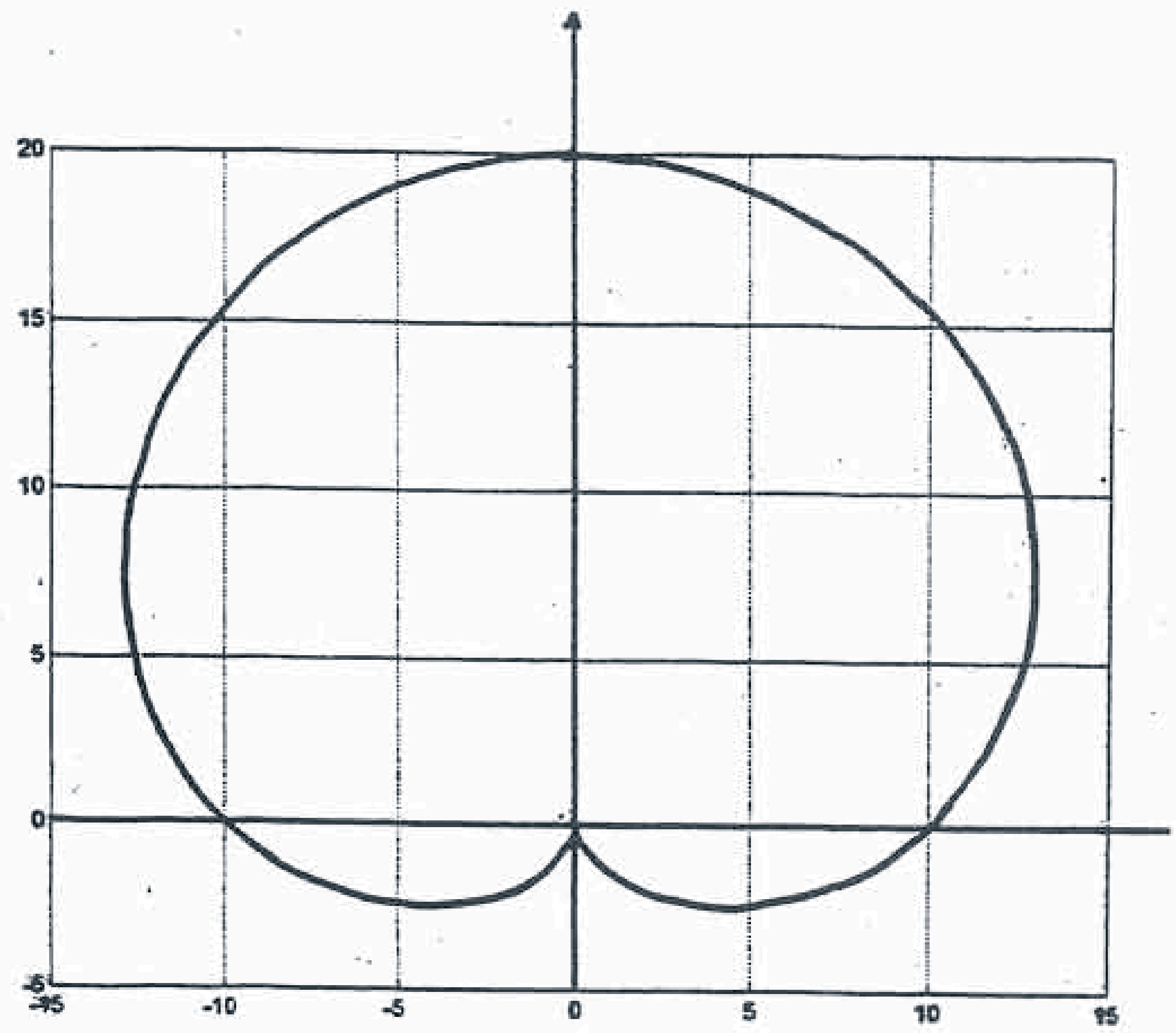


15. Below are the graphs of some polar curves of the form $r = a(1 + \sin\theta)$. What is the value of a in each case?

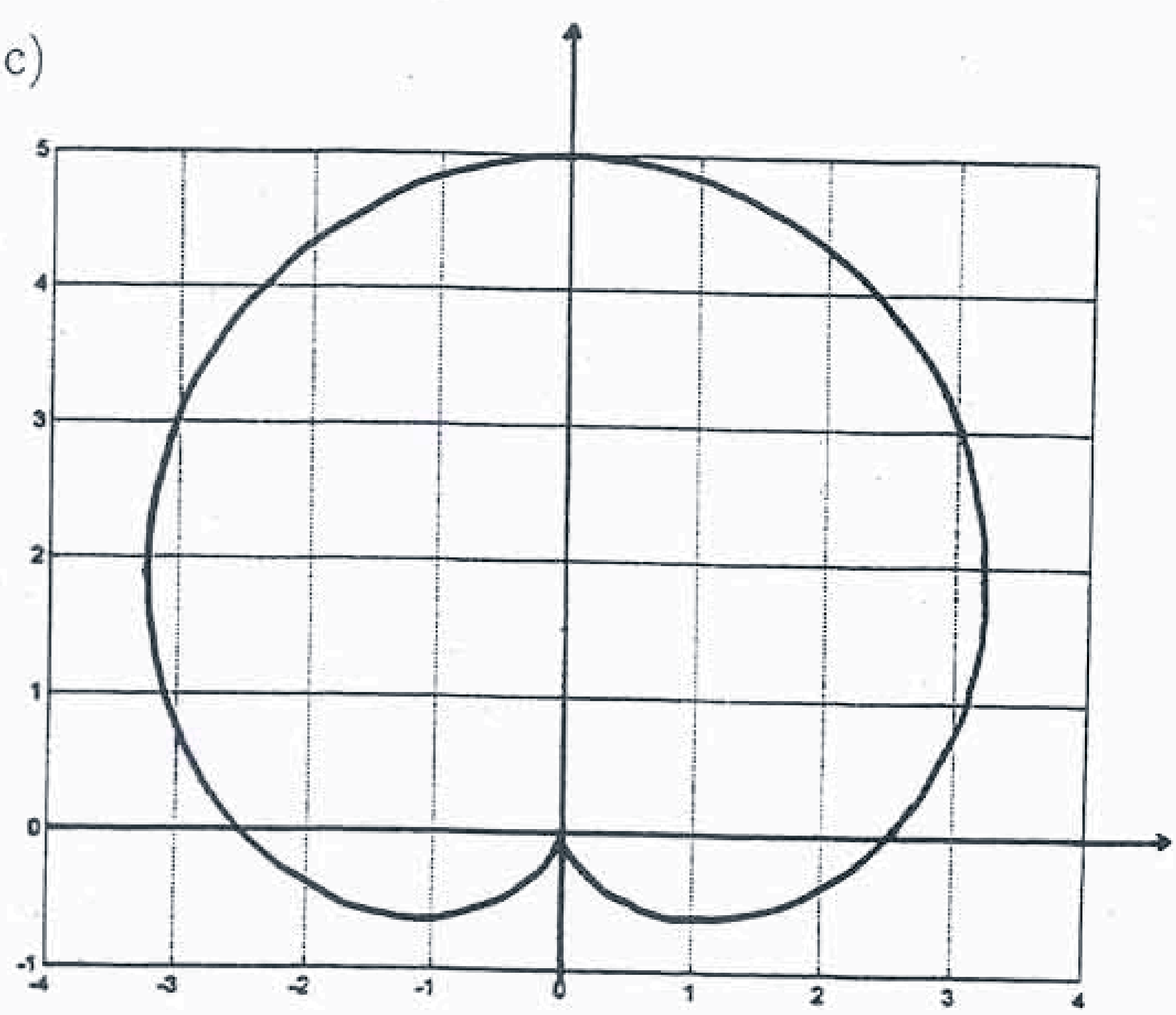
a)



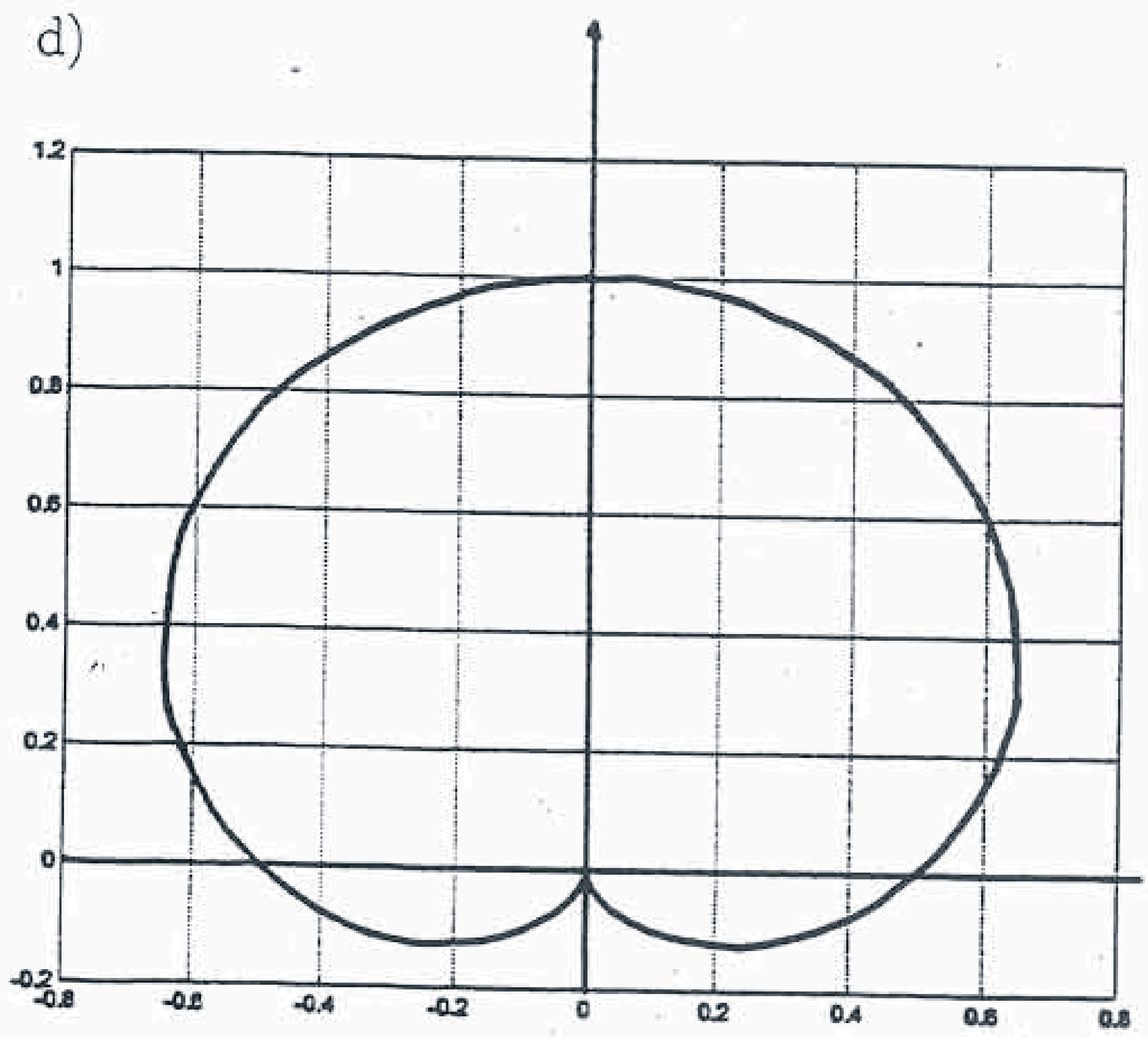
b)



c)



d)



16. Find the area of the shaded leaf of the four leaved rose $r = \sqrt{\theta}\sin 2\theta,\ \theta \geq 0$.