Assignment 1

1. Since $\|f_n - 0\|_2 = \sqrt{\int_0^1 |f_n(t)|^2} = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$. Thus, f_n converges in $L^2[0,1]$ norm to 0. Assume that given any $\epsilon > 0$ $\exists N(\epsilon)$ such that $|f_n(t) - 0| < \epsilon$ $\forall t \in [0,1]$ and $\forall n > N(\epsilon)$. But, since $|f_{N+1}(t) - 0| = 1 > \epsilon$ for $0 \le t \le 1/(N+1)$ and $\epsilon < 1$, we can always establish a contradiction.
   Note that the given sequence $\{f_n(t)\}$ converges pointwise to the following $f(t)$
   $$f(t) = \begin{cases} 0 & t \in (0,1] \\ 1 & t = 0 \end{cases} \tag{1}$$
   Since $\lim M_n = 1$ where $M_n = \sup_{t\in[0,1]} |f_n(t) - f(t)| = 1$, $\{f_n\}$ does not converge uniformly to $f(t)$.

2. Since $\|f_n - 0\|_2 = \sqrt{\int_0^1 |f_n(t)|^2} = \sqrt{\int_0^{\frac{1}{n}} n dt} = 1/\sqrt{n} \to 0$ as $n \to \infty$, we conclude that $f_n \to 0$ in $L^2[0,1]$ norm as $n \to \infty$.
   $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \sqrt{n} = \infty$.

3. Given $\epsilon > 0$, choose $N(\epsilon) = 1/\epsilon$ so that $|f_n(t) - 0| < \epsilon$ $\forall n > N$ and $\forall t \in [0,1]$. Thus $\{f_n(t)\}$ converges pointwise uniformly to 0.
   Since $\|f_n(t) - 0\|_1 = \int_0^\infty f_n(t) = \int_0^n 1/n dt = 1$, $\|f_n - 0\|_1 \nrightarrow 0$ as $n \to \infty$. Thus, $\{f_n\}$ does not converge in $L^1[0,\infty]$ norm to 0.

4. Change of variable problem. Easy!

5. An inner product calculation. Note that both $\phi(t)$ and $\psi(t)$ have norm 1. Problem 4 shows that we need to multiply both $\psi(2t)$ and $\psi(2t-1)$ by $2^{1/2}$ to normalize them. After normalization, these four functions form an ON basisof a subspace of $L^2[0,1]$. $\hat{f}(t) = \frac{1}{2}\phi(t) - \frac{1}{4}\psi(t) - \frac{1}{8}\psi(2t) - \frac{1}{8}\psi(2t-1)$; where $\hat{f}(t)$ is the projection of t on the subspace spanned by these four functions.

6. Apply Gram-Schmidt orthogonalization process. The first four set of polynomials: 1, x, $(3x^2 - 1)/2$, $(5x^3 - 3x)/2$. Of course, you can normalize these polynomials to get an ON basis.

7. Since $P_0, P_1, \cdots, P_{n+1}$ form an ON basis for the set of polynomials of order $n+1$: we can write
   $$tP_n(t) = \Sigma_{k=0}^{n+1} c_{nk} P_k(t) \tag{2}$$
   where
   $$c_{nk} = \langle tP_n, P_k \rangle \tag{3}$$
   It follows from the fact that P_n is orthogonal to every polynomial of order less than n, that
   $$c_{nk} = \langle tP_n, P_k \rangle = \langle P_n, tP_k \rangle = 0 \tag{4}$$

for $k < n-1$. Therefore, equation 2 simplifies to

$$tP_n(t) = c_{n,n-1}P_{n-1}(t) + c_{n,n}P_n(t) + c_{n,n+1}P_{n+1}(t) \tag{5}$$

Using the notation of the question:

$$tP_n(t) = a_nP_n(t) + b_nP_n(t) + c_nP_{n-1}(t) \tag{6}$$

where $a_n = \langle tP_n, P_{n+1}\rangle = \langle P_n, tP_{n+1}\rangle = \langle tP_{n+1}, P_n\rangle$; and $c_n = \langle tP_n, P_{n-1}\rangle$. So, $c_n = a_{n-1}$.

8. Note that for t^3, there are discontinuities at $\pm k\pi$ for $k = 1, 2, \cdots$, so more terms are needed to get better approximation.

9. See Sec 1.7 of the notes on Linear Operators.

   We have to consider the following two cases:

   (a) Assume that (1) holds. We need to prove that (2) can NOT hold. if (1) holds, then for any $v \in V\ \exists u \in U$ such that $\mathbf{T}u = v$. Assume that there is a nonzero $v_0 \in V$ such that $\mathbf{T}^*v_0 = 0$. Then $\langle u, \mathbf{T}*v_0 = 0\rangle_U\ \forall u \in U$. Use the adjoint of $\mathbf{T}^*$, $\langle \mathbf{T}u, v_0\rangle_V = 0$, and $\forall u \in U$. Based on our assumption, $R(\mathbf{T}) = V$. So, $v_0 \perp v\ \forall v \in V$. Thus, $v_0 = 0$, contradicting our assumption that $v_0 \neq 0$. Q.E.D.

   (b) If $v \notin R(\mathbf{T})$, that means $\nexists u \in U$ such that $\mathbf{T}u = v$. So let us find v_0 such that $\mathbf{T}^*v_0 = 0$ (i.e. the least squares problem). Find the projection of v on $R(T)$, and let $v_0 = v - proj(v) \perp R(\mathbf{T})$. That means $\langle \mathbf{T}u, v_0\rangle_V = 0\ \forall u \in U$. Now using the adjoint operator, we have $\langle \mathbf{T}u, v_0\rangle_V = \langle u, \mathbf{T}^*v_0\rangle_U = 0$, $\forall u \in U$. Note that $v_0 \neq 0$ because $v \notin R(\mathbf{T})$ and $pro(v) \in R(\mathbf{T})$. Thus, $\mathbf{T}^*v_0 = 0$.

10. $\log y = \log a + bx$. You can construct a system of 7 linear equations in two unknowns $\log a$ and b. You can use matlab to find the solution. In general, if you want to estimate the solution of $Ax = b$, then solve $A^TAx = A^Tb$. If A^TA happens to be invertible (as is the case here), then $x = (A^TA)^{-1}A^Tb$. The final solution is y(2.0) = 2.22.