Assignment 4

1. The maxflat Daubechies filter with $p = 2$ is $h(n) = \{0.489, 0.836, 0.2241, -0.129\}$ (see the figure below). The definition of symmetry of four types is given on page 55 of our textbook. Also, see the example on pp. 169. For maxflat Daubechies filter, there is no symmetry. There is a theorem stating that: Suppose ϕ and ψ, the scaling function and wavelet associated with a multiresolution analysis, are both *real* and *compactly* supported. If ψ has either a symmetry or an antisymmetry axis, then ψ is the Haar function. Orthogonality and symmetry are **NOT** compatible for compact support filters, except for Haar wavelets.

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2. For $p = 5$ (use equation 2.31 on page 60 of our textbook):

$$H_5(\omega) = \left(\frac{1+\cos\omega}{2}\right)^5 \sum_{k=0}^{4} \binom{4+k}{k} \left(\frac{1-\cos\omega}{2}\right)^k \tag{1}$$

The coefficients of the corresponding filter can be easily obtained from the z-transform; therefore to obtain a polynomial in z we use: $\frac{1-\cos\omega}{2} = \frac{1}{4}(2 - z - z^{-1})$, and $\frac{1+\cos\omega}{2} = \frac{1}{4}(2 + z + z^{-1})$. So,

$$\begin{aligned} P(z) = \frac{1}{131072}(&35z^9 - 405z^7 + 2268z^5 - 8820z^3 + 39690z + \\ &65536 + 35z^{-9} - 405z^{-7} + 2268z^{-5} - 8820z^{-3} + 39690z^{-1}) \end{aligned} \tag{2}$$

This factors into $P(z) = C(z)C(z^{-1})$, and $H(z) = \frac{1}{\sqrt{2}}C(z)$. Half of $2p$ zeros of $P(z)$ at $z = -1$ go to $C(z)$, and the $p-1$ zeros inside the unit circle also go to $C(z)$. See page 173 for more the coefficients of $h(n)$. The Daubechies filter response is a halfband filter. Unlike the truncated ideal response and the equiripple filter it is maximally flat on the passband and stopband (it does not exhibit any ripple) and its first $p-1$ derivatives are zero at $\omega = 0$ and $\omega = \pi$ (See section 2.3 for comparison between different filters).

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3. Use, $P_0(z) = F_0(z)H_0(z)$, $F_1(z) = -H_0(-z)$, and $H_1(z) = F_0(-z)$.

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4. The ON basis for V_2 is given by $\phi_{2k}(t) = 2\phi(4t - k)$, where $k = 0, \pm1, \pm2, \cdots$. The given $f(t)$ is of special form, and we can conclude that $f(t) = \frac{-1}{2}\phi_{20}(t) + 2\phi_{21}(t) + \frac{1}{2}\phi_{22}(t) + \frac{-3}{2}\phi_{23}(t)$. Note that $w_{00}(t) = w(t) = \phi(2t) - \phi(2t-1)$. $w_{1,0} = \sqrt{2}w(2t)$. $w_{11} = \sqrt{2}w(2t-1)$. $V_2 = V_0 \bigoplus W_0 \bigoplus W_1$. We can show that $a_{0,0} = 1/2$, $b_{0,0} = 1$, $b_{10} = -5\sqrt{2}/4$, and $b_{11} = 5\sqrt{2}/4$.

5.
   - $a_{00} = \int \Pi\phi(t)dt = \int_0^1 \phi(t)$. $a_{0k} = \int_k^{k+1} \phi(t)dt$. So, $\Pi_0 = \sum_k a_{0k}\phi_{0k}(t)$. For $j > 0$, $a_{j0} = \int \Pi(t)2^{j/2}\phi(2^j t)dt = 2^{-j/2}\int_0^{2^j} \phi(t)dt$. Also, we can derive that $a_{jk} = 2^{-j/2}\int_{-k}^{2^j-k} \phi(t)dt$.

- Suppose that the support of ϕ lies in $[-N, N]$. Then the support of $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$ lies in $\left[\frac{-N+k}{2^j}, \frac{N+k}{2^j}\right] = I_{N,jk}$. If this interval is completely inside $[0,1]$ then $\langle \Pi, \phi_{jk}\rangle = 0$. If this interval, $I_{N,jk}$ is completely outside $[0,1]$, then $\langle \Pi, \phi_{jk}\rangle = 0$, also. Nonzero coefficients occur when there is an overlap between $I_{N,jk}$ and $[0,1]$. This happens iff and only if $k = -N+1, \cdots, 0, 1, \cdots, N-1$, or $k = 2^j - N + 1, \cdots, 2^j + N - 1$ (You can derive this easily from constraint on the end points). Let $M = \max \phi$. Thus,

$$|a_{jk}| = |2^{j/2} \int \phi(2^j t - k) dt \leq M(2N)| \tag{3}$$

So, $a_{jk} \leq MN/2^{j/2-1}$ for $k = -N+1, \cdots, N-1$, or $k = 2^j - N + 1, \cdots, 2^j + N - 1$; otherwise $a_{jk} = 0$. Hence,

$$\Pi_j = \sum a_{jk}\phi_{jk} \tag{4}$$

and

$$\|\Pi_j\|^2 = \sum_k a_{jk}^2 \leq \frac{M^2 N^2}{2^{j-2}} \tag{5}$$

Since the projection is orthogonal, $\|\Pi_j\|^2 + \|\Pi_j - \Pi\|^2 = 1$, we can choose j large enough such that $\|\Pi_j\|^2 \leq 0.5$. Q.E.D.

- From the previous result, we can conclude that Π_j does not converge to Π. This contradicts with the fact that $\bigcup_{-\infty}^{\infty} V_j = L^2[-\infty, \infty]$, and $\Pi \in L^2$.
As a special case, recall from a previous homework that if a scaling function satisfies the dilation equation and the cascade algorithm converges, then its mean value can not equal zero; otherwise its frequency response is zero for all frequencies.

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Needs["Wavelets`Wavelets`"]

d2 = DaubechiesFilter[2]

{{0.482963, 0.836516, 0.224144, -0.12941}, 0}

philist = ScalingFunction[d2, 5];

Short[philist]

{{0, 0.}, {1/32, 0.20305}, <<94>>, {3, 0}}

ListPlot[philist, PlotJoined → True,
 AxesLabel → {"t", "phi(t)"}]
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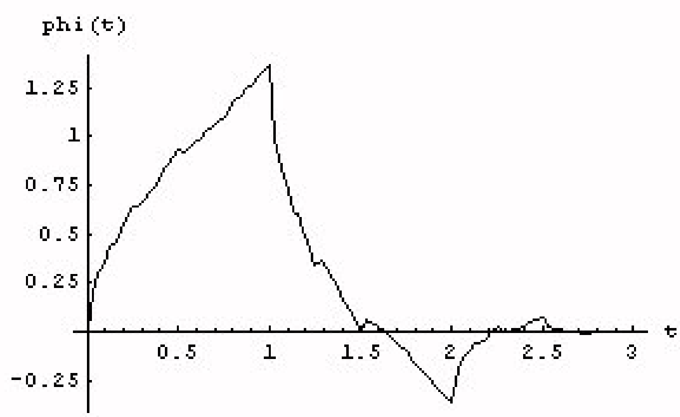

Figure 1: D4 filter.