SUMS-OF-SQUARES FORMULAS

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Abstract. The following is the extended version of my notes from my ATC talk given on June 4, 2014 at UCLA. I begin with a basic introduction to sums-of-squares formulas, and move on to giving motivation for studying these formulas and discussing some results about them over the reals. More recent techniques have made it possible to obtain similar results over arbitrary fields, and some of these are discussed later in the paper. I finish with some open questions about sums-of-squares formulas. All of the background comes from Daniel Shapiro’s book [1] and online notes [2, 3, 4], and the more recent results in section 7 come from the papers [5, 6, 7] by Daniel Dugger and Daniel Isaksen.

1. Basics

Fix a field $F$. An identity of the form

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2$$

(where the $z_i$ are bilinear expressions in the $x_j$ and $y_k$) is called a sums-of-squares formula of type $[r, s, n]$.

Example 1.1. We have a formula of type $[2, 2, 2]$ over $\mathbb{R}$ from the norm on the complex numbers:

$$z_1 = x_1 y_1 - x_2 y_2$$
$$z_2 = x_1 y_2 + x_2 y_1$$

Similarly, we have examples of types $[4, 4, 4]$ and $[8, 8, 8]$ corresponding to the quaternions and octonions, respectively.

Example 1.2. To show that formulas of more general types exist, we give a formula of type $[3, 5, 7]$:

$$z_1 = x_1 y_1 + x_2 y_2 - x_3 y_3$$
$$z_2 = x_2 y_1 - x_1 y_2 + x_3 y_4$$
$$z_3 = x_1 y_3 + x_3 y_1 - x_2 y_4$$
$$z_4 = x_1 y_4 + x_2 y_3 + x_3 y_2$$
$$z_5 = x_1 y_5$$
$$z_6 = x_2 y_5$$
$$z_7 = x_3 y_5$$

(Note that this formula is found using a consistently signed intercalculate matrix, discussed later.)

The primary question we would like to answer about sums-of-squares formulas is:
Question 1.3. For what \([r, s, n]\) do such formulas exist?

Sums-of-squares formulas are related to several important problems in topology, formulated through the following observation:

Fact 1.4. There is a composition formula of size \([r, s, n]\) over \(\mathbb{R}\) if and only if there is a normed bilinear map \(f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n\).

Here, a map \(f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n\) is normed if \(|f(x, y)| = |x||y|\). The proof of the fact is trivial, since the norm comes from a sum of squares.

If a map \(f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n\) is normed and bilinear, then it is nonsingular, bi-skew, and skew-linear. (These weaker conditions enable us to give some of the topological results as equivalent statements.)

2. Motivation

Now, we give some motivation for studying sums-of-squares formulas:

- Sums-of-squares formulas are related to the immersion problem: there is an immersion \(\mathbb{P}^{r-1} \to \mathbb{R}^{n-1}\) if and only if there is a nonsingular bi-skew map of size \([r, r, n]\). (It is unknown if this equivalent to the existence of a nonsingular bilinear map of size \([r, r, n]\).)
- Existence of sums-of-squares formulas of type \([n, n, n]\) is equivalent to the existence of an \(n\)-dimensional real normed division algebra (as in the example of a \([2, 2, 2]\) formula for \(\mathbb{C}\)). An early use of sums-of-squares formulas was to show that the only real normed division algebras are \(\mathbb{R}, \mathbb{C}, \mathbb{H}, \text{and } \mathbb{O}\).
- There is a nonsingular skew-linear map of size \([r, s, n]\) over \(\mathbb{R}\) if and only if \(n \cdot \xi_{r-1}\) over \(\mathbb{P}\) admits \(s\) linearly independent sections. Here, \(n \cdot \xi_{r-1}\) is the direct sum of \(n\) copies of the canonical line bundle on \(\mathbb{P}^{r-1}\).
- A normed pairing of size \([r, s, n]\) yields a map
  \[ H : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R} \times \mathbb{R}^n \]
  given by \(H(x, y) = (|x|^2 - |y|^2, 2f(x, y))\). This restricts to a Hopf map
  \[ H : S^{r+s-1} \to S^n \]
  which is a nontrivial quadratic map between spheres. Note that the Hopf map that we get from the formula of type \([2, 2, 2]\) is just the usual Hopf map \(S^3 \to S^2\).
- Over arbitrary fields, sums-of-squares formulas are an example of a composition of quadratic forms. Furthermore, it may be the case that existence of a formula of a given type is independent of the base field (for fields of characteristic not 2), in which case we could restrict our study to finite fields, over which formulas could be found with a computer.
3. A Special Case

Over \( \mathbb{R} \), the special case \([r, n, n]\) is done: a formula of size \([r, n, n]\) exists if and only if \( r \leq \rho(n) \).

\( \rho \) is the Hurwitz-Radon function, defined, for \( n = 2^m n_0 \) where \( n_0 \) is odd, by:

\[
\rho(n) = \begin{cases} 
2m + 1 & m \equiv 0 \pmod{4} \\
2m & m \equiv 1 \pmod{4} \\
2m & m \equiv 2 \pmod{4} \\
2m + 2 & m \equiv 3 \pmod{4}
\end{cases}
\]

This result was proved independently by Hurwitz and Radon in the 1920’s. It was reformulated as the existence of a certain system of matrix equations, and then proved using elementary linear algebra.

Earlier results of conditions for existence of this type of formula led to Hurwitz’s proof of the 1, 2, 4, 8 Theorem in 1898, showing that the only real normed division algebras are \( \mathbb{R} \), \( \mathbb{C} \), \( \mathbb{H} \), and \( \mathbb{O} \). In particular, Hurwitz showed that if a formula of type \([r, n, n]\) exists, then \( 2^{r-2} \leq n^2 \) (when \( n = r \), this implies \( n \leq 8 \)).

4. The General Case over the Reals

Most of the results on existence of formulas of general type come from topology. In particular, one of the main methods for studying the formula is to use the fact that a formula of type \([r, s, n]\) induces a map

\[
\mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \to \mathbb{P}^{n-1}
\]

This follows because the induced map \( \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n \) is nonsingular, so restricts to a map

\[
(\mathbb{R}^r - \{0\}) \times (\mathbb{R}^s - \{0\}) \to \mathbb{R}^n - \{0\}
\]

Since the original map was bilinear, we get a well-defined map on projective spaces, as desired.

Note that for this map we only use that the induced map \( \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n \) is nonsingular and bilinear, not that it is normed. At the end of this section, we will discuss one way to distinguish between the normed and nonsingular cases.

We begin with Hopf’s theorem, giving a necessary condition for existence:

**Theorem 4.1.** (Hopf’s Theorem) If there is a formula of type \([r, s, n]\) over \( \mathbb{R} \), then \( \binom{n}{k} \) is even for \( n - s < k < r \).

**Proof.** A formula of type \([r, s, n]\) induces a map

\[
f : \mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \to \mathbb{P}^{n-1}
\]

which in turn induces a map on singular mod 2 cohomology:

\[
f^* : (\mathbb{Z}/2\mathbb{Z})[x]/(x^n) \to (\mathbb{Z}/2\mathbb{Z})[a]/(a^r) \otimes (\mathbb{Z}/2\mathbb{Z})[b]/(b^s)
\]

We claim that under this homomorphism, we have \( x \mapsto a + b \). Since \( f^* \) is an induced map on cohomology, it preserves degree. In particular, we must have \( f^*(x) = Ca + Db \) for some \( C, D \). To see that we have, in fact, \( f^*(x) = a + b \), choose basepoints of \( \mathbb{P}^{r-1} \) and \( \mathbb{P}^{s-1} \), then the restriction of \( f \) to

\[
\mathbb{P}^{r-1} \vee \mathbb{P}^{s-1} \to \mathbb{P}^{n-1}
\]
is homotopic to the canonical inclusion on each factor of the wedge (this uses the
fact that $f$ is bilinear).

Then, since $x^n = 0$, we must also have $(a + b)^n = 0$ on the right side of the
morphism. Expanding this, we must then have $(\binom{n}{k}) \equiv 0 \pmod{2}$ for $n - s < k < r$,
completing the proof. \hfill \Box

Another result uses $K$-theory to eliminate some possible sums-of-squares formu-
las:

**Theorem 4.2.** If there is a formula of type $[r, s, n]$ over $\mathbb{R}$, then
\[
\binom{n}{i} \equiv 0 \pmod{2^{\phi(r-1)-i+1}}
\]
for $n - s < i \leq \phi(r-1)$.

Here, $\phi(m)$ is the number of integers $j$ with $0 < j \leq m$ and $j \equiv 0, 1, 2, 4 \pmod{8}$.

**Proof. (Very Rough Sketch)** We can show that
\[n \cdot \xi = s \cdot \epsilon \oplus \eta\]
for some $(n - s)$-plane bundle $\eta$ over $\mathbb{P}^{r-1}$. Here, $\xi$ is the canonical bundle and $\eta$
the trivial bundle on $\mathbb{P}^{r-1}$.

We apply Grothendieck operators to this identity, and get the desired condition. \hfill \Box

We will prove this (as well as the Hopf Theorem) for general fields later, and
give some more details of the proof.

So far, we have not distinguished between normed and nonsingular maps. The
following construction enables us to show, in some cases, that a normed map cannot
exist even though a nonsingular one does. In particular, this construction can
be used to show that formulas of type $[16, 16, 23]$ do not exist, while there is a
nonsingular map of that type.

**Construction 4.3.** If $f$ is a normed pairing of size $[r, s, n]$, then we have a map
\[H : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R} \times \mathbb{R}^n\]
given by $H(x, y) = (|x|^2 - |y|^2, 2f(x, y))$. This restricts to a Hopf map
\[H : S^{r+s-1} \to S^n\]
For any point $q$ in the image of $H$, $H^{-1}(q)$ is a great circle cut out by a linear
subspace $W_q \subset \mathbb{R}^{r+s}$. The differential $dH$ induces a nonsingular bilinear pairing
of type $[k, r+s-k, n]$, where $k$ is the dimension of $W_q$. Then we can use nonexistence
of some nonsingular maps to show nonexistence of the original normed map.

5. Sums-of-Squares Formulas over the Integers

It can be shown that the existence of sums-of-squares formulas over the integers
is equivalent to existence with coefficients in $\{0, \pm 1\}$. These formulas have been
studied using consistently signed intercalculate matrices.

**Definition 5.1.** Suppose $M$ is an $r \times s$ matrix with entries taken from a set of
“colors”. Let $M(i, j)$ denote the $(i, j)$th entry of $M$.

$M$ is **intercalculate** if:

1. The colors along each row are distinct.
2. If $M(i, j) = M(i', j')$, then $M(i, j') = M(i', j)$.
and has type $[r, s, n]$ if it is $r \times s$ with at most $n$ colors.

An intercalculate matrix is \textit{consistently signed} if there are $\epsilon_{ij} \in \{\pm 1\}$ such that $
abla_{ij} = \nabla_{i'j'} = -1$ whenever $M(i, j) = M(i', j')$ (with $i \neq i'$ and $j \neq j'$).

A formula of type $[r, s, n]$ exists over $\mathbb{Z}$ if and only if there is a consistently signed intercalculate matrix of type $[r, s, n]$.

We can read off the formula directly from the matrix: the color $k$ occurs in $M(i, j)$ if and only if the term $x_iy_j$ occurs in $z_k$ (and the sign given by $\epsilon_{ij}$ gives the sign of $x_iy_j$).

\textbf{Example 5.2.} We have a consistently signed intercalculate matrix of type $[3, 5, 7]$:

\[
\begin{bmatrix}
1 & -2 & 3 & 4 & 5 \\
2 & 1 & 4 & -3 & 6 \\
3 & 4 & -1 & 2 & 7
\end{bmatrix}
\]

This gives us the sums-of-squares formula from Example 1.2:

\[
\begin{align*}
z_1 &= x_1y_1 + x_2y_2 - x_3y_3 \\
z_2 &= x_2y_1 - x_1y_2 + x_3y_4 \\
z_3 &= x_1y_3 + x_3y_1 - x_2y_4 \\
z_4 &= x_1y_4 + x_2y_3 + x_3y_2 \\
z_5 &= x_1y_5 \\
z_6 &= x_2y_5 \\
z_7 &= x_3y_5
\end{align*}
\]

\section{Formulas over Fields of Characteristic 0}

Although we do not even know if the existence or nonexistence of formulas is independent of the base field in characteristic 0, the following results provide some support for that conjecture.

\textbf{Theorem 6.1.} (Lam-Lam Lemma) \textit{If there is a formula of type $[r, s, n]$ over $\mathbb{C}$, then there is a nonsingular bilinear pairing of size $[r, s, n]$ over $\mathbb{R}$.}

Note that this does NOT say that existence of sums-of-squares formulas over $\mathbb{R}$ and over $\mathbb{C}$ are equivalent.

\textbf{Proof.} Suppose we have \[(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2\]
with the $z_k$ bilinear in $x_i$ and $y_j$ with coefficients in $\mathbb{C}$.

Write $z_k = u_k + iv_k$, with $u_k$ and $v_k$ bilinear in $x_i$ and $y_j$ with coefficients in $\mathbb{R}$.

Looking at the real parts of the formula, we have:

\[(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = u_1^2 - v_1^2 + \cdots + u_n^2 - v_n^2\]

Define $f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ by

\[f(a, b) = (u_1(a, b), ..., u_n(a, b))\]

Then $f$ is bilinear, and the above multiplication formula gives us that $f$ is nonsingular. $\square$

We can then immediately generalize this result to any field of characteristic 0.
Theorem 6.2. If there is a formula of type $[r, s, n]$ over a field $F$ of characteristic 0, then there is a nonsingular pairing of size $[r, s, n]$ over $\mathbb{R}$.

Proof. A formula of type $[r, s, n]$ over $F$ involves only finitely many coefficients $\alpha_j \in F$, so the formula will be valid over $\mathbb{Q}(\{\alpha_j\})$. We can embed this field in $\mathbb{C}$, so the formula can be viewed over $\mathbb{C}$, and the result follows from the Lam-Lam Lemma. □

7. Results over General Fields

The following result is proved by Dugger and Isaksen in [5]. It is Hopf’s theorem 4.1 stated for general fields.

Theorem 7.1. If $F$ is a field of characteristic not equal to 2, and a sums-of-squares formula of type $[r, s, n]$ exists over $F$, then $n_i$ must be even for $n_i < r < i < s$.

Note that over fields of characteristic 2, sums-of-squares formulas always exist and are completely trivial, so we always consider only the case where $F$ has characteristic not 2.

Proof. (Sketch) The proof follows the same idea as the original proof of Hopf’s theorem over $\mathbb{R}$.

Let $Q_k$ denote the projective quadric $\{w_1^2 + \cdots + w_{k+2}^2 = 0\} \subset \mathbb{P}^{k+1}$. Then the bilinear map $\phi : F^r \times F^s \to F^n$ from the formula induces

$$ p : (\mathbb{P}^{r-1} - Q_{r-2}) \times (\mathbb{P}^{s-1} - Q_{s-2}) \to \mathbb{P}^{n-1} - Q_{n-2} $$

The idea is that once we get rid of the “singular part,” we are left with a well defined map on projective space (but since we now might have a sum of squares equal to 0, we have to remove more than we did over $\mathbb{R}$).

Let $DQ_k$ denote the deleted quadric $\mathbb{P}^k - Q_{k-1}$.

$p$ then induces a map on motivic cohomology,

$$ p^* : H^{*,*}(DQ_{n-1}; \mathbb{Z}/2\mathbb{Z}) \to H^{*,*}(DQ_{r-1}; \mathbb{Z}/2\mathbb{Z}) \otimes_{M_2} H^{*,*}(DQ_{s-1}; \mathbb{Z}/2\mathbb{Z}) $$

The following result tells us enough about the motivic cohomology of $DQ_n$ to be able to complete the proof. (The motivic cohomology of $DQ_n$ is unknown in general; we do not know what $\tau$ is.)

Theorem 7.2. If every element of $F$ is a square and the $F$ has characteristic not 2,

$$ H^{*,*}(DQ_n; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} M_2[a, b]/(a^2 - \tau b, b^{k+1}) & n = 2k + 1 \\ M_2[a, b]/(a^2 - \tau b, b^{k+1}, ab^k) & n = 2k \end{cases} $$

Here, $M_2 = H^{*,*}(\text{Spec } F; \mathbb{Z}/2\mathbb{Z})$.

Also, $M_2^{0,0} \cong \mathbb{Z}/2\mathbb{Z}$ and $M_2^{1,1} = 0$.

We can assume that every element of $F$ is a square: if a sums-of-squares formula holds over $F$, it certainly holds over any extension of $F$. So we can assume $F$ contains all square roots without loss of generality.

One important consequence of the computation is:

Corollary 7.3. In $H^{*,*}(DQ_n; \mathbb{Z}/2\mathbb{Z})$, $a^{n+1} = 0$ and $a^i \neq 0$ for $i \leq n$. 

Let $a$ and $b$ be the generators of $H^{*,*}(DQ_{n-1}; \mathbb{Z}/2\mathbb{Z})$, $a_1$ and $b_1$ the generators of $H^{*,*}(DQ_{r-1}; \mathbb{Z}/2\mathbb{Z})$, and $a_2$ and $b_2$ the generators of $H^{*,*}(DQ_{s-1}; \mathbb{Z}/2\mathbb{Z})$. As in the real case, one can show that $p^*(a) = a_1 + a_2$. The proof follows the same idea as in the real case, showing that restricting to factors is $\mathbb{A}^1$-homotopic to a standard inclusion. Then, since $a^n = 0$, we have $(a_1 + a_2)^n = 0$. By the corollary, we then get $\binom{n}{i} \equiv (\text{mod } 2)$ for $n - r < i < s$, as desired. □

In carrying out the details of the argument, it is actually convenient to make a change of coordinates and consider a quadric isomorphic to what we call $Q_k$. Details can be found in [5].

Another result over the reals was generalized by Dugger and Isaksen in [6]:

**Theorem 7.4.** Assume $F$ is not of characteristic 2. If a sums-of-squares formula of type $[r,s,n]$ exists over $F$, then $2\left(\frac{s+1}{2}\right)^{i+1}$ divides $\binom{n}{i}$ for $n - i < i \leq \left\lfloor \frac{s+1}{2} \right\rfloor$

**Proof. (Sketch)** One can show that if a sums-of-squares formula exists, then there is an algebraic vector bundle $\zeta$ on $DQ_{s-1}$ of rank $n - r$ such that

$$r[\xi] + [\zeta] = n$$

in the Grothendieck group $K^0(DQ_{s-1})$ of locally free coherent sheaves on $DQ_{s-1}$. Here, $\xi$ is the restriction of the tautological line bundle on $\mathbb{P}^{s-1}$ to $DQ_{s-1}$. If $F$ contains a square root of $-1$ (which we can assume without loss of generality), $	ext{char } F \neq 2$, and $c = \left\lfloor \frac{s+1}{2} \right\rfloor$, then we have:

$$K^0(DQ_{s-1}) \cong \mathbb{Z}[\nu]/(2^s \nu, \nu^2 - 2 \nu)$$

where $\nu = [\xi] - 1$ generates the reduced Grothendieck group. From $r[\xi] + [\zeta] = n$ in $K^0(DQ_{s-1})$, we get

$$r([\xi] - 1) + ([\zeta] - (n - r)) = 0$$

in $K^0(DQ_{s-1})$. Applying $\gamma_t$ (the generating function for the Grothendieck operators $\gamma^t$ on $K^0(X)$) to this, we get

$$\gamma_t(\nu)^r \cdot \gamma_t([\zeta] - (n - r)) = 1$$

From this, we have:

$$\gamma_t([\zeta] - (n - r)) = \gamma_t(\nu)^{-r} = (1 + t \nu)^{-r}$$

On the right side of this equality, the coefficient of $t^i$ is:

$$(-1)^i \binom{r + i - 1}{i} \nu^i = -2^{r-1} \binom{r + i - 1}{i} \nu$$

(since $\nu^2 = -2 \nu$).

$\zeta$ has rank $n - r$, so on the left side the coefficient of $t^i$ is 0 for $i > n - r$. This gives us $2^s$ divides $2^{r-1} \binom{r + i - 1}{i}$ for $i > n - r$ and $c = \left\lfloor \frac{s+1}{2} \right\rfloor$, and so we get that $2^{c-i+1} \binom{r + i - 1}{i}$ for $n - r < i \leq c$.

A straightforward manipulation shows that this is equivalent to the theorem as stated. □

This result was improved by Heng Xie in [8] using Hermitian $K$-theory, matching the exponent in the result over the reals in Theorem 4.2. Explicitly, we have:
Theorem 7.5. If a formula of type \([r, s, n]\) exists over a field \(F\) of characteristic not 2, then \(2^{(r-1) - i + 1}\) divides \(\binom{n}{i}\) for \(n - s < i \leq \phi(r - 1)\).

The following theorem was proved by Dugger and Isaksen in [7].

Theorem 7.6. Suppose a sums-of-squares formula of type \([2a+1, 2b+1, 2m]\) exists over a field \(F\), where \(\max\{a, b\} < m \leq a + b\) and \(\text{char} F \neq 2\). Then the vector

\[
(-1)^a \binom{m}{a}, (-1)^{a-1} \binom{m}{a-1}, \ldots, (-1)^{m-b} \binom{m}{m-b}
\]

is in the \(\mathbb{Z}\)-linear span of a certain set of relations (which can be found in [7]).

Proof. (Very Rough Sketch) If a sums-of-squares formula of type \([r, s, n]\) over \(\mathbb{R}\) exists, we have a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} & \xrightarrow{j \times j} & \mathbb{R}P^{\infty} \times \mathbb{R}P^{\infty} \\
\downarrow & & \downarrow \mu \\
\mathbb{R}P^{m-1} & \xrightarrow{j} & \mathbb{R}P^{\infty}
\end{array}
\]

in the homotopy category of topological spaces. Here, \(j\) is the usual inclusion, and \(\mu\) is the usual multiplication (which classifies tensor products of line bundles).

Using this diagram, one can show that an expression vanishes in \(E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})\), where \(E^*\) is a complex-oriented cohomology theory which is \(k\)-connected for some \(k\), and such that each coefficient group \(E^q\) is a finite 2-group. In particular, taking \(E^*(X) = BP^2(X; \mathbb{Z}/2N)\) for large \(N\) works.

If a formula exists over a field of characteristic not 2, a related diagram exists in the homotopy category of pro-simplicial sets and \(\mathbb{Z}/2\)-cohomological equivalences.

One can use this diagram to show that the same expression as above vanishes in \(E^*(\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1})\), and the result follows.

\(\square\)

8. Open Problems

We finish by listing some open questions on sums-of-squares formulas:

- Is existence/non-existence of formulas of type \([r, s, n]\) independent of the field? Or at least independent of the characteristic if the field is algebraically closed? This is true as far as we know (for small \(r, s, n\) and some special cases) but we really don’t know many formulas, so there is not much evidence for this.
- Since we don’t know many formulas, can we find some more? For finite fields, this can be done computationally.
- Can we prove that formulas of certain types cannot exist? (For example, using topological methods as discussed earlier.)
- In the last result (from [7]), we saw that results from topology using complex oriented cohomology theories can be brought into the general setting. Can this be done for other cohomology theories?
- Can we formulate the relationship between sums-of-squares formulas and immersions in the algebraic context?
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References