MATH 2574H: Review Worksheet - Exam 2

(With Answers)
March 30, 2016

1. Let \( u_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \). Let \( B \) be the matrix whose columns are the vectors \( u_1, u_2, u_3 \).

(a) Determine if the vectors are linearly independent.

(b) Just by using the result of part (a) (without doing any row reductions), what can you tell about the dimension of the row space of \( B \)?

(c) Find a vector that is not in the subspace spanned by the columns of \( B \).

(d) Find a basis for \( \mathbb{R}^3 \) containing a basis for the column space of \( B \).

2. Let \( M \) be a \( 4 \times 3 \) matrix such that the first three rows of \( M \) equal the rows of the matrix in problem \#1(b). Then what is the maximum possible rank of \( M \)?

3. Let \( v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \\ -7 \end{bmatrix} \). Let \( A \) be the matrix whose columns are the vectors \( v_1, v_2, v_3 \).

(a) Suppose that it is known that the vectors \( v_1, v_2, v_3 \) are linearly dependent. What can you infer about the dimension of the row space of \( A \)? Compare with problems \#1 and \#2.

(b) Determine whether or not the vectors are linearly independent.

(c) Find a basis for the column space of \( A \).

(d) Is \( v_3 \) in span\( \{v_1, v_2\} \)?

(e) Is \( v_2 \) in span\( \{v_1, v_3\} \)?

(f) Find another basis (different from the one you found in part (c)) for the column space of \( A \). Explain why it is a basis.

(g) In general, if \( v_1, v_2, v_3 \) are arbitrary vectors in \( \mathbb{R}^n \) (with say, \( n \geq 3 \)) such that they are linearly dependent, is it always true that \( v_3 \) is in span\( \{v_1, v_2\} \) and \( v_2 \) is in span\( \{v_1, v_3\} \)? (What is the additional condition that is required in order to assert that \( v_3 \) is in span\( \{v_1, v_2\} \), for example?)

4. State true or false. Justify by proving or by giving a counterexample.

(a) Let \( A \) be a \( 3 \times 4 \) matrix. Then the system of equations \( Ax = 0 \) has a unique solution.
(b) Let $A$ be a $3 \times 4$ matrix. Then any vector $b$ in $\mathbb{R}^3$ can be written as a linear combination of the columns of $A$. In other words, the column space of $A$ is (all of) $\mathbb{R}^3$.

(c) Let $A$ be a nonsingular $3 \times 3$ matrix. Then any vector $b$ in $\mathbb{R}^3$ is can be written as a linear combination of the columns of $A$.

(d) Let $A$ be a $4 \times 3$ matrix such that one of the $3 \times 3$ sub-determinants is nonzero. Then the system of equations $Ax = b$ is consistent for every $b$ in $\mathbb{R}^4$.

5. Find a basis for $\mathbb{R}^4$ that contains a basis for the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$ 

In other words, extend a basis for the null space to a basis for (all of) $\mathbb{R}^4$.

6. Suppose that $A$ is a $7 \times 9$ matrix whose rank is 4. State true or false. Justify by proving or providing a counterexample.

(a) The row space of $A$ is a 5-dimensional subspace of $\mathbb{R}^9$.
(b) The row space of $A$ is a 4-dimensional subspace of $\mathbb{R}^7$.
(c) The null space of $A$ is a 5-dimensional subspace of $\mathbb{R}^9$.
(d) The column space of $A$ is a 4-dimensional subspace of $\mathbb{R}^7$.
(e) The system of equations $Ax = b$ is consistent for every $b$ in $\mathbb{R}^7$.
(f) The last column of $A$ is in the subspace spanned by the remaining columns.
(g) At least 5 columns of $A$ are linearly independent.
(h) At most 3 columns of $A$ are linearly independent.
(i) At least 5 columns of $A$ are required to span the column space of $A$.
(j) At most 3 columns of $A$ are required to span the column space of $A$.
(k) At least 4 columns (respectively, rows) of $A$ are required to span the column space (respectively, row space) of $A$.
(l) At most 4 columns (respectively, rows) of $A$ are linearly independent.

7. Let $P$ be the collection of all polynomials of degree less than or equal to 3. Is the subset of $P$ consisting of all polynomials whose $x^3$-coefficient is thrice the negative of the $x$-coefficient a subspace of $P$? What about the subset consisting of those polynomials whose constant term is always one more than the $x$-coefficient?
Answers:

1. (a) The vectors are linearly dependent because the determinant of the matrix $B$ is zero (One can easily see this by observing that the third row is a scalar multiple of the second row).

(b) Part (a) tells us that the three columns of $B$ do not span (all of) $\mathbb{R}^3$.

(Why? Recall that any spanning set for a vector space contains a basis for that vector space. The three columns are obviously a spanning set for the column space of $B$ (why? what is the meaning of the column space of a matrix?).

If the column space was (all of) $\mathbb{R}^3$, which is a three dimensional vector space, then the spanning set which is the set of all columns of the matrix contains a basis for $\mathbb{R}^3$. But then since there are only three columns, this can happen only when the basis coincides with the spanning set in which case the columns have to be linearly independent contradicting the result of part (a)).

Thus, we deduce that the dimension of the column space is less than or equal to 2. But since the row rank is the same as the column rank, this means that the dimension of the row space is also less than or equal to 2. In short, the rank of $B$ is less than or equal to 2.

Alternatively, one could argue that since the columns are linearly dependent, the determinant of $B$ is zero. This means that $\det B^T = \det B = 0$. This implies that the rows of $B$ are also linearly dependent.

(Why? The columns of the transpose matrix, $B^T$ are essentially the rows of $B$. Hence the column space of $B^T$ is the same as the row space of $B$).

Then we may apply the same argument as before to conclude directly that the dimension of the row space of $B$ is less than or equal to 2.

Remark 0.1. With the additional information that the first two columns of $B$ are linearly independent (why? neither is a scalar multiple of the other), one can also infer that the column rank of $B$ is at least 2.

(How? Recall that a basis is a maximal linearly independent set. We know that the set $S = \{u_1, u_2, u_3\}$ is a spanning set for the column space of $B$, and therefore it contains a basis for the column space. Now, we also know that the set $T = \{u_1, u_2\}$ is a linearly independent subset of the column space. Hence, any basis for the column space must contain at least two elements, that is, the column rank is at least 2).

And with both these observations (column rank $\leq 2$ and column rank $\geq 2$), we have nailed that the column rank is exactly equal to 2, from which we also deduce that the set $T = \{u_1, u_2\}$ is in fact, a basis for 2-dimensional column space of $B$ (why?).

(c) From the previous part along with the remark, we have established that the column space is a 2-dimensional subspace of $\mathbb{R}^3$ with a basis given by the set $T = \{u_1, u_2\}$. Now observe that for any vector $v$ that is in the column space of $B$, the determinant of the matrix, $B_v$ whose columns are $u_1, u_2$ and $v$ must be zero (Why? This is
because then the vectors $u_1, u_2$ and $v$ are linearly dependent). This tells us that if we find any vector $v$ for which the determinant of $B_v$ is nonzero, then that vector must not be in the column space of $B$.

With this idea, we may arrive at any vector, for example, say $v = (0, 0, 1)$ that is not in the column space of $B$ simply because $\det B_v \neq 0$.

(d) One can do this part, with or without the knowledge of the previous parts in the question.

First let us suppose that we do have the knowledge from the previous parts. Then from the remark following part (b), we established that $T = \{u_1, u_2\}$ is a basis for the column space of $B$. Also, from part (d), we found a vector, $v$ that is not in the column space of $B$ and we saw that $\det B_v \neq 0$. Then, it follows that the set $U = \{u_1, u_2, v\}$ is a basis for $\mathbb{R}^3$ containing a basis (namely, $T$) for the column space of $B$ (Why?).

Now suppose that you do not have any of the knowledge from the previous parts. Then your first step is to find a basis for the column space of $B$ (by the standard method), and next extend this basis (call it $T$ for example) to a basis for (all of) $\mathbb{R}^3$, by the other standard method (as in the homework problems #17, #19 in section 4.5). Or, you might use the idea of part (c) to extend the basis for the column space of $B$ to a basis for $\mathbb{R}^3$.

**Step 1:** Starting with $B$, one can arrive at an echelon form

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which would show that the first two columns $u_1, u_2$ are the pivot columns of $B$ and thus the set $T = \{u_1, u_2\}$ is a basis for the column space of $B$.

**Step 2:** Either follow the idea of part (c) or follow the procedure in the homework problems #17, #19 in section 4.5. With the second method, one starts with the matrix whose columns are $u_1, u_2, e_1, e_2, e_3$ where $e_1, e_2, e_3$ are the standard basis vectors of $\mathbb{R}^3$ and arrives at an echelon form

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{bmatrix}$$

which shows that the pivot columns of the original matrix, $u_1, u_2$ and $e_2$ form a basis for $\mathbb{R}^3$ and this basis contains a basis for the column space of $B$.

2. In general, such a matrix may have rank equalling 3, which would be the case when all three columns of $M$ are linearly independent.

On the other hand, also observe that at least two rows of $M$ are linearly independent (for example, the first two rows of $M$ are linearly independent - why?). This would mean that the rank of $M$ is at least 2.

(Why? Imitate the same argument as in the remark in the previous problem).

3. (a) Using a similar argument as with the problem #1(b), we first observe that the rank of $A$ is less than or equal to 2. Again, the key idea is that any spanning set for a
vector space contains a basis for the vector space. Here the vector space we are concerned with is the column space of the matrix $A$ (let us call it $Col(A)$) which is a subspace of $\mathbb{R}^4$. Obviously, the set $S = \{v_1, v_2, v_3\}$ consisting of the columns of $A$ is a spanning set for $Col(A)$. Hence, it contains a basis for $Col(A)$. But then $S$ itself is not a basis since it is linearly dependent. It follows that any basis for $Col(A)$ has less than or equal to 2 elements. In other words, the rank of $A$ is less than or equal to 2.

And once again, by an argument similar to the remark in problem #1 (or by the argument in the second paragraph of the solution to problem #2), one can deduce that the rank of $A$ is at least 2.

The above two observations imply that the rank of $A$ is exactly equal to 2.

Finally, observe that the matrix $A$ is just one special kind of a matrix, $M$ that is described in problem #2 where the first three rows coincide with those of $B$ in problem #1. However, note that in general, the columns of such a matrix $M$ may be linearly independent. However, in this special case, when $M = A$, it happens that the columns are linearly dependent. Observe how in this case the rank is reduced.

(b) As usual we consider the homogenous linear system of equations

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

which translates to $Ac = 0$, where $c = (c_1, c_2, c_3)$. To solve this we reduce the matrix, $A$ (the augmented matrix has all zeros on the last column and row operations do nothing to these zeros, hence we don’t bother to write down the augmented matrix) to an echelon form. Starting with $A$, one can arrive at an echelon form,

$$E = \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

From this, the existence of a free variable shows that the vectors are linearly dependent.

(c) From the echelon matrix in the previous part, we see that the first two columns of $A$ are the pivot columns. Hence, the set $T = \{v_1, v_2\}$ is a basis for the column space of $A$.

(d) Looking at the matrix $A$ as the augmented matrix of the (non-homogeneous) system of equations

$$d_1 v_1 + d_2 v_2 = v_3$$

we see that solving this system amounts to arriving at the same echelon matrix as in part (b). And from this, first we deduce that $v_3$ is indeed in $\text{span}\{v_1, v_2\}$ since the system of equations is consistent, and next we can actually “read off” the solutions:

$$d_2 = 2$$
$$d_1 = 1 - 2d_2 = 1 - 4 = -3$$

Hence, we have that $v_3 = -3v_1 + 2v_2$. 

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From the previous part, we have a linear dependence relation

\[3v_1 - 2v_2 + v_3 = 0\]

From this we can “solve for” \(v_2\) as follows:

\[v_2 = \left(\frac{3}{2}\right)v_1 + \left(\frac{1}{2}\right)v_3\]

This shows that \(v_2\) is indeed in \(\text{span}\{v_1, v_3\}\).

In part (c), we already established that the column space of \(A\) is 2-dimensional. Hence we just need to find two linearly independent vectors in \(\text{Col}(A)\) that span \(\text{Col}(A)\). But recall that any linearly independent subset of \(\text{Col}(A)\) that has two elements is automatically a basis for \(\text{Col}(A)\), and similarly any spanning set for \(\text{Col}(A)\) that has two elements is automatically a basis for \(\text{Col}(A)\) (Why?).

Method 1:
Observe that the set \(U = \{v_1, v_3\}\) is a linearly independent subset of \(\text{Col}(A)\) (Why?). Hence \(U\) is a basis for the 2-dimensional vector space \(\text{Col}(A)\).

Method 2:
Observe that the set \(U = \{v_1, v_3\}\) is a spanning set for \(\text{Col}(A)\) by part (e) and by the pruning lemma (Exercise #31 in section 4.4). Hence \(U\) is a basis for the 2-dimensional vector space \(\text{Col}(A)\).

And \(U \neq T\).

The statement is not always true. For example, consider the following vectors in \(\mathbb{R}^3\).

Let \(v_1 = 0, v_2 = e_2, v_3 = e_3\), where \(e_1, e_2, e_3\) are the standard basis vectors for \(\mathbb{R}^3\).

Then the vectors \(v_1, v_2, v_3\) are linearly dependent (why?). But then neither \(v_3\) is in \(\text{span}\{v_1, v_2\}\) nor \(v_2\) is in \(\text{span}\{v_1, v_3\}\) (why?).

However, note that if \(v_1, v_2, v_3\) are linearly dependent and if \(v_1, v_2\) are linearly independent then one can show that \(v_3\) is in \(\text{span}\{v_1, v_2\}\).

And similarly, if \(v_1, v_2, v_3\) are linearly dependent and if \(v_1, v_3\) are linearly independent then one can show that \(v_2\) is in \(\text{span}\{v_1, v_3\}\); and so on.

4. (a) FALSE.
A \(3 \times 4\) matrix is a “fat” matrix \((n = 4 > 3 = m)\) and hence the homogeneous system \(Ax = 0\) always has infinitely many solutions (there is at least one free variable).

(b) FALSE.
Consider, for example, the \(3 \times 4\) matrix

\[A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\]

Then the column space of \(A\) is \textbf{not} (all of) \(\mathbb{R}^3\). For example, the vector \(e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\) can not be written as a linear combination of the columns of \(A\).
(c) TRUE.
If \( A \) is a nonsingular matrix, then \( A \) is row equivalent to the identity matrix, which means that \( A \) has no all-zero rows in any of its echelon forms (that is, \( A \) has full row rank) and this means that the system of equations \( Ax = b \) is always consistent.
Also, in fact, in this case, all the columns of \( A \) are pivot columns (that is, \( A \) has full column rank) and hence there are no free variables. Thus the system of equations \( Ax = b \) has a unique solution given by \( x = A^{-1}b \).

(d) FALSE.
Consider, for example, the \( 4 \times 3 \) matrix
\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
Then the system of equations \( Ax = b \) is not consistent when \( b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \) for example.

5. We do this in two steps.

Step 1: (Find a basis for the null space of \( A \))

The null space of \( A \) is the solution space of the system of equations, \( Ax = 0 \). By doing just one row operation, the matrix \( A \) can be brought to its reduced echelon form,
\[
R = \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix}
\]
From this we see that there are two free variables, \( x_3 \) and \( x_4 \). Setting \( x_3 = s, x_4 = t \), we solve for \( x_1 \) and \( x_2 \) and hence we obtain the solution space (that is, the null space of \( A \)) to be the set
\[
Null(A) = \{(s, t, s, t) : s, t \in \mathbb{R}\}
\]
\[
= \left\{ s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}
\]
\[
= \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}
\]
Clearly, the vectors in the spanning set are linearly independent (use the determinant criterion). So the set \( U = \{v_1, v_2\} \) where \( v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \) is a basis for the null
Step 2: (Extending to a basis for $\mathbb{R}^4$)

We now extend the set $U$ to a basis for $\mathbb{R}^4$ by finding a basis for the column space of the matrix whose columns are $v_1, v_2, e_1, e_2, e_3, e_4$ (recall the procedure followed in problem #1(d)). Let $M$ be this matrix, so that

$$M = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

This matrix reduces to an echelon form,

$$E = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 & 0
\end{bmatrix}$$

from which we see that the first four columns of $M$ are its pivot columns. These form a basis for the column space of $M$ which is $\mathbb{R}^4$. Thus the set $\{v_1, v_2, e_1, e_2\}$ is the required basis for $\mathbb{R}^4$ that contains a basis for the null space of the matrix $A$.

6. (a) FALSE.
   The rank of $A$ is 4. So the row space of $A$ is a 4-dimensional subspace of $\mathbb{R}^9$.

(b) FALSE.
   The rank of $A$ is 4. So the row space of $A$ is a 4-dimensional subspace of $\mathbb{R}^9$.

(c) TRUE.
   The rank of $A$ is 4. Since there are 4 pivot columns out of a total of 9 columns, there are $9 - 4 = 5$ free variables in the homogeneous system, $Ax = 0$.

   In other words, the rank-nullity theorem implies that the dimension of the null space of $A$ is the number of columns minus the rank, that is, $9 - 4 = 5$.

(d) TRUE.
   The rank of $A$ is 4. So the column space of $A$ is a 4-dimensional subspace of $\mathbb{R}^7$.

(e) FALSE.
   Since $A$ does not have full row rank, that is the rank is less than the number of rows, there are all-zero rows in any echelon form. Then one can always find some $b$ in $\mathbb{R}^4$ for which the system of equations $Ax = b$ is inconsistent.

(f) FALSE.
   This is not always true. For example, consider the $7 \times 9$ matrix,

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$
Then $A$ has rank 4 (why?) and yet the last column is NOT in the subspace spanned by the remaining columns (why?).

(g) FALSE.
Since the rank of $A$ is 4, the column space is 4 dimensional. Hence, any collection of more than 5 vectors in the column space is linearly dependent.

(h) FALSE.
Since the rank of $A$ is 4, the column space is 4 dimensional. And the set of all columns of $A$ is a spanning set for the column space of $A$. And since any spanning set contains a basis, there is a basis for the column space consisting of some 4 of the columns of $A$. Then the set of these 4 columns is linearly independent.

Thus it is not true that at most 3 columns of $A$ are linearly independent.

(i) FALSE.
Since the rank of $A$ is 4, and since the columns form a spanning set for the column space of $A$, there is a basis consisting of 4 columns of $A$. And this basis (consisting of only 4 elements) is sufficient to span the column space of $A$.

(j) FALSE.
Since the rank of $A$ is 4, any spanning set for the column space of $A$ must have at least 4 elements (since any spanning set contains a basis and any basis consists of 4 elements).

(k) TRUE.
Any spanning set for the column space contains a basis for the column space. And any basis has exactly 4 elements. It follows that any spanning set for the column space has at least 4 elements. In particular, any collection of less than 4 columns of $A$ is not sufficient to span the column space of $A$.

For row space, the argument is similar.

(l) TRUE.
The column space is a 4-dimensional space. Hence, any collection of more than 4 vectors in this space is not linearly independent. In particular, any collection of more than 4 columns of $A$ is linearly dependent.

For row space, the argument is similar.

7. The first subset is a subspace of $P$ while the second subset is not a subspace of $P$. The arguments are standard. I will leave them to you.