Math 4603: Advanced Calculus I, Summer 2016
Worksheet 1 - Solutions

1. Let \( A_n = \left[ \frac{1}{n}, 1 \right) \) for \( n \in \mathbb{N} \). What is \( A_1 \)? Find \( \bigcap_{n=2}^{\infty} A_n \). Then prove that \( \bigcup_{n=1}^{\infty} A_n = (0, 1) \).

Solution:

We first observe that \( A_1 = \emptyset \) because \( A_1 = \{ x \in \mathbb{R} : 1 \leq x < 1 \} \) but there is no real number \( x \) which is both less than 1 and greater than or equal to 1.

Then we observe that \( \bigcap_{n=2}^{\infty} A_n = \left[ \frac{1}{2}, 1 \right) \); in other words, the intersection is simply the set \( A_2 \). This is because \( A_2 \) is a proper subset of \( A_n \) for every natural number \( n \geq 3 \). Do you see that? One can also formally prove the claim as follows. We must show that \( \bigcap_{n=2}^{\infty} A_n \subseteq \left[ \frac{1}{2}, 1 \right) \) and that \( \left[ \frac{1}{2}, 1 \right) \subseteq \bigcap_{n=2}^{\infty} A_n \).

On one hand, if \( x \in \bigcap_{n=2}^{\infty} A_n \), then \( x \in A_n \) for all \( n \in \mathbb{N} \) with \( n \geq 2 \), that is, \( x \in \left[ \frac{1}{n}, 1 \right) \) for all \( n \in \mathbb{N} \) with \( n \geq 2 \). In particular, setting \( n = 2 \), we see that \( x \in \left[ \frac{1}{2}, 1 \right) \). Hence we have \( \bigcap_{n=2}^{\infty} A_n \subseteq \left[ \frac{1}{2}, 1 \right) \).

On the other hand, if \( x \in \left[ \frac{1}{2}, 1 \right) \), then \( \frac{1}{2} \leq x < 1 \). Now since \( \frac{1}{n} < \frac{1}{2} \) for all natural numbers \( n > 2 \), from \( \frac{1}{n} < \frac{1}{2} \leq x < 1 \) it follows that \( \frac{1}{n} \leq x < 1 \) for all natural numbers \( n > 2 \), and thus \( x \in \left[ \frac{1}{n}, 1 \right) \) for all natural numbers \( n > 2 \). Therefore, \( x \in \bigcap_{n=2}^{\infty} A_n \). Hence we have \( \left[ \frac{1}{2}, 1 \right) \subseteq \bigcap_{n=2}^{\infty} A_n \) as well.

Finally, let us prove that \( \bigcup_{n=1}^{\infty} A_n = (0, 1) \). We must show that \( \bigcup_{n=1}^{\infty} A_n \subseteq (0, 1) \) and that \( (0, 1) \subseteq \bigcup_{n=1}^{\infty} A_n \).
On one hand, if \( x \in \bigcup_{n=2}^{\infty} A_n \), then \( x \in A_{n_0} \) for some \( n_0 \in \mathbb{N} \). Then we have,

\[
\frac{1}{n_0} \leq x < 1.
\]

But since \( 0 < \frac{1}{n_0} < x \), it follows that \( 0 < x < 1 \), and thus \( x \in (0,1) \). Hence, \( \bigcup_{n=1}^{\infty} A_n \subseteq (0,1) \).

On the other hand, if \( x \in (0,1) \) then \( 0 < x < 1 \). Then by the Archimedean Property of the real numbers, there exists a natural number \( n_0 \) such that \( \frac{1}{n_0} < x \). It follows that \( \frac{1}{n_0} < x < 1 \) implying that \( x \in \left[ \frac{1}{n_0}, 1 \right) \), that is, \( x \in A_{n_0} \). Since there is some \( n_0 \) such that \( x \in A_{n_0} \), it follows that \( x \in \bigcup_{n=1}^{\infty} A_n \). Hence, \( (0,1) \subseteq \bigcup_{n=1}^{\infty} A_n \) as well.

2. Describe each of the following sets as the empty set, as \( \mathbb{R} \), or in interval notation, as appropriate.

(a) \[
\bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right)
\]

(b) \[
\bigcup_{n=1}^{\infty} (-n, n)
\]

(c) \[
\bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right)
\]

(d) \[
\bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, 2 + \frac{1}{n} \right)
\]

Solution (without proofs):

(a) The singleton set, \( \{0\} \).
(b) The set of all real numbers, \( \mathbb{R} \)
(c) The closed interval, \( [0,1] \).
(d) The open interval, \( (-1,3) \).
Proofs of parts (c) and (d):

(c) We must show that
\[ \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) \subseteq [0, 1] \] and that \([0, 1] \subseteq \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right). \]

On one hand, if \( x \in \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) \), then \( x \in \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) \) for all \( n \in \mathbb{N} \). We now claim that \( x \geq 0 \) and that \( x \leq 1 \). Let us prove this claim by contradiction. Suppose that the claim is not true. Then either \( x < 0 \) or \( x > 1 \).

Case 1: \( x < 0 \)

Then \( -x > 0 \). By the Archimedean Property of the real numbers, there is some natural number \( n_0 \) such that \( \frac{1}{n_0} < y \). Multiplying this inequality by \(-1\) on both sides, we get \( -\frac{1}{n_0} > (-y) \), that is, \( -\frac{1}{n_0} > x \) for the natural number \( n_0 \).

But then \( x \notin \left( -\frac{1}{n_0}, 1 + \frac{1}{n_0} \right) \) contradicting our assumption. Thus, Case 1 can not hold.

Case 2: \( x > 1 \)

In this case, observe that \( x - 1 > 0 \). Set \( y = x - 1 \). By the Archimedean Property of the real numbers, there is some natural number \( n_0 \) such that \( \frac{1}{n_0} < y \). But then again \( x \notin \left( -\frac{1}{n_0}, 1 + \frac{1}{n_0} \right) \) contradicting our assumption that \( x \in \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) \).

Hence, our claim must hold true. The claim implies that \( x \in [0, 1] \) proving that \( \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) \subseteq [0, 1] \).

Conversely, suppose that \( x \in [0, 1] \). Then \( 0 \leq x \leq 1 \). And since for every \( n \in \mathbb{N} \) we have that
\[ -\frac{1}{n} < 0 \leq x \leq 1 < 1 + \frac{1}{n}, \]

we see that \( x \in \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) \) for all \( n \in \mathbb{N} \). Therefore, \( x \in \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) \), proving that \([0, 1] \subseteq \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) \) as well.

(d) We must show that \( \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, 2 + \frac{1}{n} \right) \subseteq (-1, 3) \) and that \((-1, 3) \subseteq \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, 2 + \frac{1}{n} \right) \).

First suppose \( x \in \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, 2 + \frac{1}{n} \right) \). Then \( x \in \left( -\frac{1}{n_0}, 2 + \frac{1}{n_0} \right) \) for some \( n_0 \in \mathbb{N} \).

So for this \( n_0 \), we have \( -\frac{1}{n_0} < x < 2 + \frac{1}{n_0} \). Then since \( n_0 \geq 1 \), we have
\[ -1 \leq -\frac{1}{n_0} < x < 2 + \frac{1}{n_0} \leq 3 \]
from which it follows that \( x \in (-1, 3) \). Hence, \( \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, 2 + \frac{1}{n} \right) \subseteq (-1, 3) \).
Conversely, suppose $x \in (-1, 3)$. Then $-1 < x < 3$. Then setting $n_0 = 1$, clearly, $x \in \left(\frac{-1}{n_0}, 2 + \frac{1}{n_0}\right) = (-1, 3)$. Since we have found some $n_0$ (namely, $n_0 = 1$) such that $x \in \left(\frac{-1}{n_0}, 2 + \frac{1}{n_0}\right)$, it follows that $x \in \bigcup_{n=1}^{\infty} \left(\frac{-1}{n}, 2 + \frac{1}{n}\right)$. Hence, $(-1, 3) \subseteq \bigcup_{n=1}^{\infty} \left(\frac{-1}{n}, 2 + \frac{1}{n}\right)$ as well.