Math 4603: Advanced Calculus I, Summer 2016
Worksheet 3 - Solutions

1. Consider the sequences, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ whose $n^{th}$ terms are given by $a_n = 4 + \frac{1}{n}$ and $b_n = 3 - \frac{2}{n}$.

(a) Let $\epsilon = 0.1$. Find positive integers $N_1$ and $N_2$ such that $|a_n - 4| < \epsilon$ for all $n \geq N_1$ and $|b_n - 3| < \epsilon$ for all $n \geq N_2$.

Solution:
We want $N_1$ such that $\frac{1}{n} < \frac{1}{10}$ whenever $n \geq N_1$ and similarly we want $N_2$ such that $\frac{2}{n} < \frac{1}{10}$ whenever $n \geq N_2$. We may choose $N_1 = 100$ and $N_2 = 2000$ for example.

(b) Now improve your answers in part (a) by finding the smallest such $N_1$ and $N_2$ respectively. Is there any reason for these (smallest) $N_1$ and $N_2$ to be equal? Could you think of two different sequences for which these (smallest) $N_1$ and $N_2$ coincide?

Solution:
We want $N_1$ such that $\frac{1}{n} < \frac{1}{10}$ whenever $n \geq N_1$, if and only if, $n > 10$ for all $n \geq N_1$. So the smallest such $N_1$ is 11. Similarly, for $b_n$, the smallest such $N_2$ is 21. There is no reason for these $N_1$ and $N_2$ to be equal in general, for two arbitrary sequences. However, consider the sequence, $\{c_n\}_{n=1}^{\infty}$ given by $c_n = 5 - \frac{1}{n}$. Let $N_3$ be the smallest natural number such that $|c_n - 5| < \epsilon$ for all $n \geq N_3$. Then observe that $N_1 = N_3 = 11$. Do you see how? And do you observe why they are equal?

(c) Now let $\epsilon = 0.01$. Repeat parts (a) and (b) for this $\epsilon$. Observe how the smallest $N_1$ and $N_2$ have grown in size for this new value of $\epsilon$.

Solution:
For the new value of $\epsilon$, the smallest $N_1$ will be 101 and similarly, the smallest $N_2$ will be 201.

(d) Can you repeat part (a) for any given positive $\epsilon$, no matter how small it is? What can you conclude about the convergence of the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$?

Solution:
Yes you can repeat this for every positive $\epsilon$. Both the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge.

2. Determine whether the given subsets of $\mathbb{R}$ are neighbourhoods of 2.

(a) $A = [1, 3]$
(b) $B = (1.999, 2.001)$
(c) $C = [2, 3]$
(d) $D = (1, 3) \cap \mathbb{Q}$
(e) $E = (\infty, 0) \cup (1, 3) \cup \{\pi, 46\}$
Solution:
The sets $A, B$ and $E$ are neighbourhoods of 2, while the sets $C$ and $D$ are not.

3. Is the sequence $\left\{ \frac{3n + 7}{n} \right\}_{n=1}^{\infty}$ bounded? If so find upper and lower bounds.

Solution:
Yes, the given sequence is bounded. On one hand $\frac{3n + 7}{n} = 3 + \frac{7}{n} \leq 3 + 7 = 10$ for all $n \in \mathbb{N}$. On the other hand, $3 + \frac{7}{n} \geq 3$ for all $n \in \mathbb{N}$. Thus, 10 is an upper bound for the sequence and 3 is a lower bound for the sequence.

4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Prove that if the sequence converges to a real number, $A$, then it is bounded. Is the converse of this statement true?

Solution:
The textbook’s proof:
Choose $\epsilon = 1$. Then there is a positive integer $N$ such that for all $n \geq N$, $|a_n - A| < 1$, that is, $A - 1 < a_n < A + 1$. Let $S = \min\{a_1, a_2, ..., a_{N-1}, A - 1\}$ and let $M = \max\{a_1, a_2, ..., a_{N-1}, A + 1\}$. Then for all $n$, we have $S \leq a_n \leq M$. Thus $\{a_n\}_{n=1}^{\infty}$ is bounded.

The proof that I gave in class:
Choose $\epsilon = 1$. Then there is a positive integer $N$ such that for all $n \geq N$, $|a_n - A| < 1$. Then for all $n \geq N$, we have (by the triangle inequality)

$$|a_n| = |(a_n - A) + A|$$
$$\leq |a_n - A| + |A|$$
$$< 1 + |A|$$

Now let $M = \max\{|a_1|, |a_2|, ..., |a_{N-1}|, 1 + |A|\}$. Then for all $n \in \mathbb{N}$ we have $|a_n| \leq M$ and thus $\{a_n\}_{n=1}^{\infty}$ is bounded.

Why are both these proofs equivalent? Think about the following statement given in page 36 (and prove it!).

A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded if and only if there are real numbers $P$ and $M$ such that $P \leq a_n \leq M$ for all $n$ or, equivalently, if and only if there is a real number $S$ such that $|a_n| \leq S$ for all $n$. Draw a picture first to see what is going on and then write down a formal proof.

Finally note that the converse statement: “If a sequence is bounded, then it is convergent” is NOT TRUE. For example, consider the sequence, $\{a_n\}_{n=1}^{\infty}$, where $a_n = (-1)^n$. Then $\{a_n\}_{n=1}^{\infty}$ is bounded, but not convergent (why?).