1. Let $f : [\alpha, \beta] \to \mathbb{R}$ be increasing. Let

$$L(x) = \sup \{f(y) : y < x\} \quad \text{and} \quad U(x) = \inf \{f(y) : y > x\}$$

for each $x \in (\alpha, \beta)$.

Let $x_0 \in (\alpha, \beta)$. Prove that

$$L(x_0) \leq f(x_0) \leq U(x_0).$$

**Hint:** Let $A = \{f(y) : y < x_0\}$ and let $B = \{f(y) : y > x_0\}$ so that $L(x_0) = \sup A$ and $U(x_0) = \inf B$. You may prove the statement in two ways:

(i) Show that $f(x_0)$ is an upper bound for $A$ and a lower bound for $B$.

(ii) Prove by contradiction. Assume $f(x_0) < L(x_0)$. Using the fact that $L(x_0)$ is the supremum of a certain set, derive a contradiction (A standard argument leads to contradicting the fact that $f$ is increasing). Similarly, assume $f(x_0) > U(x_0)$ and then derive a contradiction using the fact that $U(x_0)$ is the infimum of $B$.

**Solution:**

Let me prove the set of inequalities using BOTH the methods discussed in the hint.

**Method 1:** (Showing that $f(x_0)$ is an upper/lower bound for the appropriate sets)

First let us show that $f(x_0)$ is an upper bound for $A$. Let $s \in A$. Our goal is to show that $s \leq f(x_0)$. By definition of $A$, if $s \in A$, then $s = f(y)$ for some $y < x_0$. Now, since $f$ is increasing, we must have that

$$y < x_0 \implies f(y) \leq f(x_0) \implies s \leq f(x_0).$$

Thus, we have achieved our goal and we have that $f(x_0)$ is an upper bound for $A$. Now, since $L(x_0) = \sup A$, it is the least upper bound for $A$. So it must be less than or equal to any other upper bound, and therefore, $L(x_0) \leq f(x_0)$.

In an analogous way, let us now show that $f(x_0)$ is a lower bound for $B$. (I will be more concise than the previous paragraph). Let $s \in B$. Then $s = f(y)$ for some $y > x_0$, by definition of $B$. Then since $f$ is increasing we have
\[ x_0 < y \implies f(x_0) \leq f(y) = s \]

and thus, \( f(x_0) \) is a lower bound for \( B \). Hence, \( f(x_0) \leq \inf B = U(x_0) \).

Combining the result of the two paragraphs above, we have that

\[ L(x_0) \leq f(x_0) \leq U(x_0). \]

**Method 2:** (Proof by contradiction)

First suppose \( L(x_0) > f(x_0) \), that is, \( f(x_0) < L(x_0) \). Then \( f(x_0) < \sup A \), and hence \( f(x_0) \) is NOT an upper bound for \( A \). So there is some \( s \in A \) such that \( f(x_0) < s \). Since \( s \in A \), \( s = f(y) \) for some \( y < x_0 \), by definition of \( A \). Thus we have \( y < x_0 \) and \( f(x_0) < s = f(y) \). In other words, we have

\[ y < x_0 \quad \text{but} \quad f(y) > f(x_0), \]

contradicting the fact that \( f \) is increasing. Therefore, our assumption that \( f(x_0) < L(x_0) \) is incorrect. We must have that \( L(x_0) \leq f(x_0) \).

In an analogous way, let us argue that \( f(x_0) \leq U(x_0) \) (again I will be more concise than the previous paragraph). Suppose not. That is, assume \( f(x_0) > U(x_0) \). Then \( f(x_0) > \inf B \), and hence \( f(x_0) \) is NOT a lower bound for \( B \). So there is some \( s \in B \) such that \( f(x_0) > s \). That is, there is some \( y > x_0 \) such that \( f(x_0) > f(y) \). But then, we have

\[ x_0 < y \quad \text{and} \quad f(x_0) > f(y), \]

again contradicting the fact that \( f \) is increasing. Hence, we must have that \( f(x_0) \leq U(x_0) \).

Combining the result of the two paragraphs above, we have that

\[ L(x_0) \leq f(x_0) \leq U(x_0). \]

2. Let \( E \subseteq \mathbb{R} \) and let \( f : E \to \mathbb{R} \) and let \( x_0 \in E \). Suppose that \( x_0 \) is NOT an accumulation point of \( E \). Prove that \( f \) is continuous at \( x_0 \).

**Solution:**

We discussed an example of this scenario in class. The key idea is simply the definition (meaning) of \( x_0 \) being NOT an accumulation point of \( E \).

Recall that \( x_0 \) is an accumulation point of \( E \) if and only if every neighborhood of \( x_0 \) has infinitely many points of \( E \), and this happens if and only if every deleted (punctured) neighborhood of \( x_0 \) has at least one point of \( E \) (see Lemma following the definition of an accumulation point given in page 39), that is, for every neighborhood \( U \) of \( x_0 \), the set \( (U \setminus \{x_0\}) \cap E \) is nonempty.
(Let me provide a proof of this lemma at the end of this worksheet).

Therefore, we deduce that \( x_0 \) is NOT an accumulation point of \( E \) if and only if there exists some neighborhood \( U \) of \( x_0 \) such that \((U \setminus \{x_0\}) \cap E = \emptyset\), by negating the last statement in the previous paragraph. The proof follows immediately from this deduction.

**Proof:**

If \( x_0 \) is NOT an accumulation point of \( E \), then there is a neighborhood \( U \) of \( x_0 \) such that \((U \setminus \{x_0\}) \cap E = \emptyset\). Since \( U \) is a neighborhood, this means that there is a \( \delta > 0 \) such that \((x_0 - \delta, x_0 + \delta) \subseteq U\), and hence \((x_0 - \delta, x_0 + \delta) \cap E = \{x_0\}\). In other words, if \( x \in E \) and \(|x - x_0| < \delta\), then \( x = x_0 \).

Now let \( \epsilon > 0 \) be given. We choose \( \delta \) as described in the previous paragraph. Then if \( x \in E \) and \(|x - x_0| < \delta\), then \( x = x_0 \), and hence we have

\[
|f(x) - f(x_0)| = |f(x_0) - f(x_0)| = 0 < \epsilon.
\]

Thus \( f \) is continuous at \( x_0 \) whenever \( x_0 \) is NOT an accumulation point of \( E \).

3. Let \( f : \mathbb{N} \to \mathbb{R} \) be a function. By the previous problem, deduce that \( f \) is continuous.

**Solution:**

The domain, \( \mathbb{N} \) has NO accumulation points. Then by the previous problem, \( f \) is continuous at every point in the domain, \( \mathbb{N} \). That is, \( f \) is continuous.

**Lemma 0.1.** Let \( E \) be a subset of \( \mathbb{R} \). Then \( x \) is an accumulation point of \( E \) if and only if every neighborhood of \( x \) contains a member of \( E \) different from \( x \), that is, for every neighborhood \( U \) of \( x \), \((U \setminus \{x\}) \cap E \neq \emptyset\).

**Proof.**

( \( \implies \) )

Suppose \( x \) is an accumulation point of \( E \). Let \( U \) be a neighborhood of \( x \). Then \( U \) has infinitely many points of \( E \). In particular there is some element \( y \in E \) such that \( y \neq x \) and \( y \in U \). Thus \( y \in (U \setminus \{x\}) \cap E \), and therefore \((U \setminus \{x\}) \cap E \neq \emptyset\).

( \( \impliedby \) )

Now suppose \( x \) satisfies the given condition that every neighborhood of \( x \) contains a member of \( E \) different from \( x \). And suppose for the sake of contradiction that \( x \) is NOT an accumulation point of \( E \). Then there is a neighborhood of \( x \) that has only finitely many points of \( E \). Let us call this neighborhood \( U \). And let \( U \cap E = \{x_1, x_2, ..., x_n\} \) for some \( n \in \mathbb{N} \) and \( U \cap E \) may or may not contain \( x \). Set \( S = (U \cap E) \setminus \{x\} \) (so that we “remove” \( x \) from the finite list of points).

Now set \( \delta = \min\{|y-x| : y \in S\} \). Observe that \( \delta > 0 \) since \( x \notin S \). Now set \( W = (x-\delta, x+\delta) \). Then \( W \) is a neighborhood of \( x \) that contains NO points of \( E \) other than possibly \( x \) itself. This contradicts the hypothesis that every neighborhood of \( x \) contains a member of \( E \) different from \( x \). Therefore \( x \) must be an accumulation point of \( E \).