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7.4 - Integration by Partial Fractions

Review:

Improper Rational Function: If $\deg(P) \geq \deg(Q)$ for some polynomials $P(x)$ and $Q(x)$, use polynomial division to find… $f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$ where $\deg(R) < \deg(Q)$.

Since $S(x)$ is easily integrated, the problem reduces down to integrating $\frac{R(x)}{Q(x)}$

Case I: The denominator is a product of distinct linear factors: $Q = (ax+b)(cx+d)\ldots(ex+f)$, where no factor is repeated, and no factor is a constant multiple of another).

1. Set: $\frac{R(x)}{Q(x)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \ldots + \frac{C}{ex+f}$
2. Multiply the equation through by $Q(x)$
3. Collect like terms on the right-hand side of the equation, and generate a system of equations by equating the coefficients of equal powers of x.
4. Solve the system of equations, and replace A, B, C in the equation above with the resulting values.
5. Integrate this simpler expression.

Example: $\frac{3x+1}{(7x+2)(4x+13)} = \frac{A}{7x+2} + \frac{B}{4x+13}$

$3x + 1 = A(4x+13) + B(7x+2)$

$= 4Ax + 13A + 7Bx + 2B = (4A+7B)x + (13A+2B)$

$\Rightarrow$ $4A + 7B = 3$ and $13A + 2B = 1$.

The last two steps above for this example are elementary.

Case II: Some of the linear factors in the above example are repeated…

For example: $\frac{R(x)}{Q(x)} = \frac{3x+1}{x^3(7x+2)^2(4x+13)}$.

♦ Set: $\frac{3x+1}{x^3(7x+2)^2(4x+13)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{7x+2} + \frac{E}{(7x+2)^2} + \frac{F}{4x+13}$

♦ Continue with the procedure for Case 1, from step 2.

Case III: The denominator contains irreducible quadratic factors, none of which are repeated.
(This is the case when the discriminant of the quadratic ($b^2 - 4ac$) is less than zero.)

For example: $\frac{R(x)}{Q(x)} = \frac{3x+1}{(x^2+2)(4x+13)}$.

♦ Set: $\frac{3x+1}{(x^2+2)(4x+13)} = \frac{Ax+B}{x^2+2} + \frac{C}{4x+13}$

♦ Continue with the procedure for Case 1, from step 2.

Case IV: The denominator contains a repeated irreducible quadratic factor

For example: $\frac{R(x)}{Q(x)} = \frac{3x+1}{(x^2+2)^2(4x+13)}$.

♦ Set: $\frac{3x+1}{(x^2+2)^2(4x+13)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{E}{4x+13}$

♦ Continue with the procedure for Case 1, from step 2.

When integrating the resulting terms from Case III, you will find the need to integrate terms in the form: $\frac{1}{x^2+a^2}$, in these instances, a useful integration formula is…

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

Rationalizing Substitutions:

When you have a function that contains a radical ($\sqrt[n]{g(x)}$), you can often transform the function into a rational function for which you can apply the above cases. You can test out if this is possible by substituting $u = \sqrt[n]{g(x)}$, and seeing if this results in a rational function.

Problem #8 Evaluate $\int \frac{3t-2}{t+1}dt$.

We hope for the following equality $\frac{3t-2}{t+1} = \frac{A}{1} + \frac{B}{t+1}$ for some A, B.

If so, then: $3t-2 = A(t+1)+B = At+(A+B)$.

$A = 3$ and $A+B = -2$.

So: $\int \frac{3t-2}{t+1}dt = \int\left(3 - \frac{5}{t+1}\right)dt = 3t - 5\ln|t+1| + C$.

Problem #32 Evaluate $\int_0^1 \frac{x}{x^2+4x+13}dx$.

Your first impulse might be to complete the square in the denominator $x^2+4x+13 = (x+2)^2+9$. And we will use this! However, we need one other tool. Since we know that $(\ln y)' = \frac{y'}{y}$, one method for integrating is to see if you can make the numerator look like the derivative of the denominator. Because then we have $\int \frac{y'}{y} = \ln y$. In this case, the derivative of the denominator is $2x+4$. To make the numerator look like this, we must multiply it by 2, and add 4. However, we must then do the inverse operations so we don't change our answer. Observe:

$$\int_0^1 \frac{x}{x^2+4x+13}dx = \int_0^1 \frac{\frac{1}{2}(2x+4)-2}{x^2+4x+13}dx = \frac{1}{2}\int_0^1 \frac{2x+4}{x^2+4x+13}dx - \int_0^1 \frac{2}{x^2+4x+13}dx$$

We are now in position to complete the first integral as described above where $y = x^2+4x+13$, $dy = (2x+4)dx$.

$$\frac{1}{2}\int_0^1 \frac{2x+4}{x^2+4x+13}dx = \frac{1}{2}\int_{y=13}^{y=18}\frac{dy}{y} = \frac{1}{2}[\ln y]_{13}^{18} = \frac{1}{2}\ln\frac{18}{13}.$$

With the second integral, we will complete the square

$$-\int_0^1 \frac{2}{x^2+4x+13}dx = -2\int_0^1 \frac{1}{9+(x+2)^2}dx.$$

Now from this, hopefully you can see the connection to $(\tan^{-1}f(x))' = \int \frac{f'}{1+f^2}dx$.

However, to get the 9 transformed into a one, were going to have to pull a 9 out of the squared term in the denominator. In other words, let $x+2 = 3u$, and $dx = 3du$, and observe:

$$-2\int_{x=0}^{x=1}\frac{1}{9+(x+2)^2}dx = -2\int_{u=\frac{2}{3}}^{u=1}\frac{3}{9+9u^2}du = -\frac{2}{3}\int_{\frac{2}{3}}^{1}\frac{1}{1+u^2}du$$

$$= -\frac{2}{3}\left[\tan^{-1}u\right]_{\frac{2}{3}}^{1} = -\frac{2}{3}\left(\frac{\pi}{4} - \tan^{-1}\frac{2}{3}\right) = -\frac{\pi}{6} + \frac{2}{3}\tan^{-1}\frac{2}{3}.$$

Finally, putting the first and second in together we have:

$$\int_0^1 \frac{x}{x^2+4x+13}dx = \left(\frac{1}{2}\ln\frac{18}{13}\right) + \left(-\frac{\pi}{6} + \frac{2}{3}\tan^{-1}\frac{2}{3}\right).$$

Problem #43 Make a substitution to express $\int \frac{x^3}{\sqrt[3]{x^2+1}}dx$ as a rational function and then evaluate the integral.

Let $u = \sqrt[3]{x^2+1}$.

Your first impulse might then be to take the derivative of both sides, but you might observe in advance that this will be messy. To get rid of the radical sign, it is helpful to cube both sides first. Then, $x^2 = u^3 - 1$, and $2xdx = 3u^2du$.

So: $\int \frac{x^3}{\sqrt[3]{x^2+1}}dx = \int \frac{(u^3-1)\frac{3}{2}u^2}{u}du = \frac{3}{2}\int (u^4 - u)du$

$= \frac{3}{10}u^5 - \frac{3}{4}u^2 + C = \frac{3}{10}(x^2+1)^{\frac{5}{3}} - \frac{3}{4}(x^2+1)^{\frac{2}{3}} + C.$