MATH 1272: **Calculus II**

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7.7 - Approximate Integration

Review:

Left and Right Endpoint Approximation:

- $\int_a^b f(x)dx \approx L_n := \sum_{i=1}^n f(x_{i-1})\Delta x$
- $\int_a^b f(x)dx \approx R_n := \sum_{i=1}^n f(x_i)\Delta x$

Midpoint Rule: $\int_a^b f(x)dx \approx M_n = \Delta x[f(\overline{x}_1) + f(\overline{x}_2) + \ldots + f(\overline{x}_n)]$, where $\Delta x = \frac{b-a}{n}$ and $\overline{x}_i = \frac{x_{i-1}+x_i}{2}$ = midpoint of $[x_{i-1}, x_i]$.

Trapezoidal Rule:
$\int_a^b f(x)dx \approx T_n := \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$, where $x_i = a + i\Delta x$.

Error Bounds: Suppose $\left|f''(x)\right| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then $|E_T| \leq \frac{K(b-a)^3}{12n^2}$ and $|E_M| \leq \frac{K(b-a)^3}{24n^2}$.

Simpson's Rule: $\int_a^b f(x)dx \approx S_n := \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$, where n is even.

Error Bound for Simpson's Rule: Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_S is the error involved in using Simpson's Rule, then $|E_S| \leq \frac{K(b-a)^5}{180n^4}$.

Problem #6 Use the Midpoint Rule and Simpson's Rule to approximate $\int_0^\pi x\cos x dx$, with $n = 4$. (Round your answers to six decimal places.) Compare your results to the actual value to determine the error in each approximation.

$\Delta x = \frac{b-a}{n} = \frac{\pi}{4}$.

$M_4 = \frac{\pi}{4}\left[f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right)\right]$

$= \frac{\pi}{4}\left(\frac{\pi}{8}\cos\frac{\pi}{8} + \frac{3\pi}{8}\cos\frac{3\pi}{8} + \frac{5\pi}{8}\cos\frac{5\pi}{8} + \frac{7\pi}{8}\cos\frac{7\pi}{8}\right) \approx -1.945744$

$S_4 = \frac{\pi}{4 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{2\pi}{4}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi)\right]$

$= \frac{\pi}{12}\left(0 + \pi\cos\frac{\pi}{4} + \pi\cos\frac{\pi}{2} + 3\pi\cos\frac{3\pi}{4} + \pi\cos\pi\right) \approx -1.985611$

Exact area under the curve: ??

$\int_0^\pi x\cos x dx$

$= [x\sin x]_0^\pi - \int_0^\pi \sin x dx = [x\sin x + \cos x]_0^\pi$

$= (0 + (-1)) - (0 + 1) = -2.$

Errors: $E_M = \text{exact} - M_4 \approx (-2) - (-1.945744) = -0.054256,$

$E_S = \text{exact} - S_4 \approx (-2) - (-1.985611) = -0.014389.$

Problem #12 Use of the Trapezoidal Rule, the Midpoint Rule, and Simpson's rule to approximate $\int_1^3 e^{\frac{1}{x}}dx$, with $n = 8$. (Round your answers to six decimal places.)

$\Delta x = \frac{3-1}{8} = \frac{1}{4}$.

$$T_8 = \frac{1}{4\cdot 2}\left[e + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + 2f(2) + 2f\left(\frac{9}{4}\right) + 2f\left(\frac{5}{2}\right) + 2f\left(\frac{11}{4}\right) + f(3)\right]$$

$$= \frac{1}{8}\left[e + 2e^{\frac{4}{5}} + 2e^{\frac{2}{3}} + 2e^{\frac{4}{7}} + 2e^{\frac{1}{2}} + 2e^{\frac{4}{9}} + 2e^{\frac{2}{5}} + 2e^{\frac{4}{11}} + e^{\frac{1}{3}}\right] \approx 3.534934.$$

$$M_8 = \frac{1}{4}\left[f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right) + f\left(\frac{17}{8}\right) + f\left(\frac{19}{8}\right) + f\left(\frac{21}{8}\right) + f\left(\frac{23}{8}\right)\right]$$

$$= \frac{1}{4}\left[e^{\frac{8}{9}} + e^{\frac{8}{11}} + e^{\frac{8}{13}} + e^{\frac{8}{15}} + e^{\frac{8}{17}} + e^{\frac{8}{19}} + e^{\frac{8}{21}} + e^{\frac{8}{23}}\right] \approx 3.515248.$$

$$S_8 = \frac{1}{4\cdot 3}\left[f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + 2f(2) + 4f\left(\frac{9}{4}\right) + 2f\left(\frac{5}{2}\right) + 4f\left(\frac{11}{4}\right) + f(3)\right]$$

$$= \frac{1}{12}\left[e + 4e^{\frac{4}{5}} + 2e^{\frac{2}{3}} + 4e^{\frac{4}{7}} + 2e^{\frac{1}{2}} + 4e^{\frac{4}{9}} + 2e^{\frac{2}{5}} + 4e^{\frac{4}{11}} + e^{\frac{1}{3}}\right] \approx 3.522375.$$

Problem #21

a) Find the approximations T_{10}, M_{10}, and S_{10} for $\int_0^\pi \sin x\,dx$ and the corresponding errors E_T, E_M, and E_S.

$$T_{10} = \frac{\pi}{10\cdot 2}\left[f(0) + 2f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + \ldots + 2f\left(\frac{9\pi}{10}\right) + f(\pi)\right]$$

$$= \frac{\pi}{20}\left[\sin 0 + 2\sin\frac{\pi}{10} + 2\sin\frac{2\pi}{10} + 2\sin\frac{2\pi}{10} + \ldots + 2\sin\frac{9\pi}{10} + \sin\pi\right] \approx 1.983524.$$

$$M_{10} = \frac{\pi}{10}\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{3\pi}{20}\right) + f\left(\frac{5\pi}{20}\right) + \ldots + f\left(\frac{19\pi}{20}\right)\right]$$

$$= \frac{\pi}{10}\left[\sin\frac{\pi}{20} + \sin\frac{3\pi}{20} + \sin\frac{5\pi}{20} + \ldots + \sin\frac{19\pi}{20}\right] \approx 2.008248.$$

$$S_{10} = \frac{\pi}{10\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + 4f\left(\frac{3\pi}{10}\right) + \ldots + 4f\left(\frac{9\pi}{10}\right) + f(\pi)\right]$$

$$= \frac{\pi}{30}\left[\sin 0 + 4\sin\frac{\pi}{10} + 2\sin\frac{2\pi}{10} + 4\sin\frac{3\pi}{10} + \ldots + 4\sin\frac{9\pi}{10} + \sin\pi\right] \approx 2.000110.$$

Exact area under the curve: $\int_0^\pi \sin x\,dx = [-\cos x]\int_0^\pi = 1 - (-1) = 2.$

So, $E_T = \text{exact} - T_{10} \approx 2 - 1.983524 = 0.016476,$

$E_M \approx 2 - 2.008248 = -0.008248$, and $E_S \approx 2 - 2.000110 = -0.000110.$

b) Compare the actual errors in part a) with the error estimates given by 3. and 4.

Since $f(x) = \sin x$, we have $\left|f^{(n)}(x)\right| \leq 1,$ so we can use $K = 1$ for all our (upper) error estimates.

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{1(\pi-0)^3}{12(10)^2} = \frac{\pi^3}{1200} \approx 0.025839$$

$$|E_M| \leq \frac{|E_T|}{2} = \frac{\pi^3}{2400} \approx 0.012919.$$

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{1(\pi-0)^5}{180(10)^4} = \frac{\pi^5}{1{,}800{,}000} \approx 0.000170.$$

The actual error is about 64% of the error estimate in all three cases.

c) How large do we have to choose n so that the approximations T_n, M_n, and S_n to the integral in part a) are accurate to within 0.00001?

$|E_T| \le 0.00001$ gives us: $\frac{\pi^3}{12n^2} \le \frac{1}{10^5}$ or $n^2 \ge \frac{10^5\pi^3}{12} \quad \Rightarrow \quad n \ge 508.3$.
So we can take $n = 509$ for T_n.

$|E_M| \le 0.00001$ gives us: $\frac{\pi^3}{24n^2} \le \frac{1}{10^5}$ or $n^2 \ge \frac{10^5\pi^3}{24} \quad \Rightarrow \quad n \ge 359.4$.
So we can take $n = 360$ for M_n.

$|E_S| \le 0.00001$ gives us: $\frac{\pi^5}{180n^4} \le \frac{1}{10^5}$ or $n^4 \ge \frac{10^5\pi^5}{180} \quad \Rightarrow \quad n \ge 20.3$.
So we can take $n = 22$ for S_n (since n must be even).

Problem #28 Given $\int_1^4 \frac{1}{\sqrt{x}}dx$, find the approximations T_n, M_n, and S_n for $n = 6$ and $n = 12$. Then compute the corresponding errors E_T, E_M, and E_S. (Round your answers to six decimal places.) What observations can you make? In particular, what happens to the errors when n is doubled?

Exact: $\int_1^4 \frac{1}{\sqrt{x}}dx = [2\sqrt{x}]_1^4 = 4 - 2 = 2, \qquad \Delta x = \frac{4-1}{n} = \frac{3}{n}$.

For $n = 6$:

$$T_6 = \frac{3}{6\cdot 2}\left\{f(1) + 2\left[f\left(\tfrac{3}{2}\right) + f\left(\tfrac{4}{2}\right) + f\left(\tfrac{5}{2}\right) + f\left(\tfrac{6}{2}\right) + f\left(\tfrac{7}{2}\right)\right] + f(4)\right\}$$

$$= \frac{1}{4}\left\{1 + 2\left[\sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{2}{5}} + \sqrt{\tfrac{2}{6}} + \sqrt{\tfrac{2}{7}}\right] + \tfrac{1}{2}\right\} \approx 2.008966.$$

$$M_6 = \frac{3}{6}\left[f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right) + f\left(\tfrac{9}{4}\right) + f\left(\tfrac{11}{4}\right) + f\left(\tfrac{13}{4}\right) + f\left(\tfrac{15}{4}\right)\right]$$

$$= \frac{1}{2}\left[\sqrt{\tfrac{4}{5}} + \sqrt{\tfrac{4}{7}} + \sqrt{\tfrac{4}{9}} + \sqrt{\tfrac{4}{11}} + \sqrt{\tfrac{4}{13}} + \sqrt{\tfrac{4}{15}}\right] \approx 1.995572.$$

$$S_6 = \frac{3}{6\cdot 3}\left[f(1) + 4f\left(\tfrac{3}{2}\right) + 2f\left(\tfrac{4}{2}\right) + 4f\left(\tfrac{5}{2}\right) + 2f\left(\tfrac{6}{2}\right) + 4f\left(\tfrac{7}{2}\right) + f(4)\right]$$

$$= \frac{1}{6}\left[1 + 4\sqrt{\tfrac{2}{3}} + 2\sqrt{\tfrac{1}{2}} + 4\sqrt{\tfrac{2}{5}} + 2\sqrt{\tfrac{1}{3}} + 4\sqrt{\tfrac{2}{7}} + \tfrac{1}{2}\right] \approx 2.000469.$$

$E_T = \text{exact} - T_6 \approx 2 - 2.008966 = -0.008966,$

$E_M \approx 2 - 1.995572 = 0.004428,$

$E_S \approx 2 - 2.000469 = -0.000469.$

For $n = 12$:

$$T_{12} = \frac{3}{12\cdot 2}\left\{f(1) + 2\left[f\left(\tfrac{5}{4}\right) + f\left(\tfrac{6}{4}\right) + f\left(\tfrac{7}{4}\right) + \ldots + f\left(\tfrac{15}{4}\right)\right] + f(4)\right\}$$

$$= \frac{1}{8}\left\{1 + 2\left[\sqrt{\tfrac{4}{5}} + \sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{4}{7}} + \ldots + \sqrt{\tfrac{4}{15}}\right] + \tfrac{1}{2}\right\} \approx 2.002269,$$

$$M_{12} = \frac{3}{12}\left[f\left(\tfrac{9}{8}\right) + f\left(\tfrac{11}{8}\right) + f\left(\tfrac{13}{8}\right) + \ldots + f\left(\tfrac{31}{8}\right)\right]$$

$$= \frac{1}{4}\left[\sqrt{\tfrac{8}{9}} + \sqrt{\tfrac{8}{11}} + \sqrt{\tfrac{8}{13}} + \ldots + \sqrt{\tfrac{8}{31}}\right] \approx 1.998869,$$

$$S_{12} = \frac{3}{12\cdot 3}\left[f(1) + 4f\left(\tfrac{5}{4}\right) + 2f\left(\tfrac{6}{4}\right) + 4f\left(\tfrac{7}{4}\right) + 2f\left(\tfrac{8}{4}\right) + \ldots + 4f\left(\tfrac{15}{4}\right) + f(4)\right]$$

$$= \frac{1}{12}\left[1 + 4\sqrt{\tfrac{4}{5}} + 2\sqrt{\tfrac{2}{3}} + 4\sqrt{\tfrac{4}{7}} + 2\sqrt{\tfrac{1}{2}} + \ldots + 4\sqrt{\tfrac{4}{15}} + \tfrac{1}{2}\right] \approx 2.000036.$$

$E_T = \text{exact} - T_{12} \approx 2 - 2.002269 = -0.002269,$

$E_M \approx 2 - 1.998869 = 0.001131,$

$E_S \approx 2 - 2.000036 = -0.000036.$

Putting these results into tables:

n	T_n	M_n	S_n
6	2.008966	1.995572	2.000469
12	2.002269	1.998869	2.000036

n	E_T	E_M	E_S
6	-0.008966	0.004428	-0.000469
12	-0.002269	0.001131	-0.000036

Observations:
1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.

2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and E_S seems to decrease by a factor of about 16 as n is doubled.