MATH 1272: Calculus II

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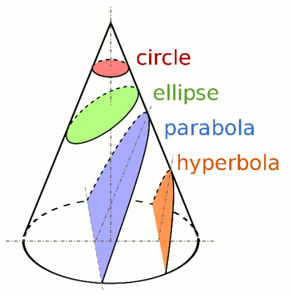
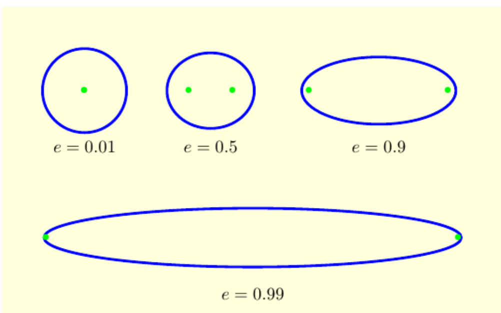
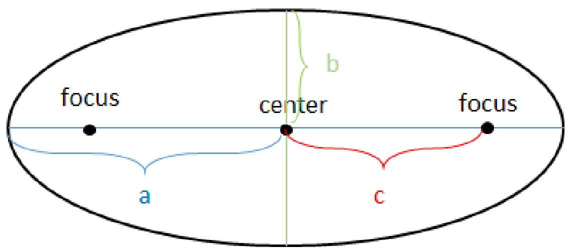
10.5 Conic Sections

Review:



Ellipse



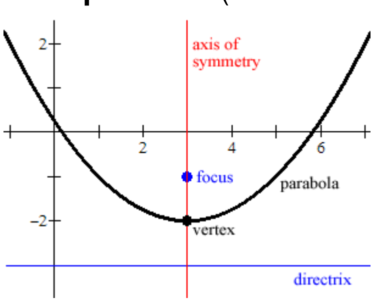


Let the distance between points P and the **focus** F_1 be labeled $|PF_1|$, and similarly $|PF_2|$ for the **focus** F_2. Then, an ellipse is the set of points where the difference $||PF_1| + |PF_2||$ is constant. The **major axis** is the largest line segment connecting opposite sides. We label the length of this line segment $2a$. We call the points where the major axis and the ellipse intersect its **vertices**. The **minor axis** is the smallest line segment connecting opposite sides. We labeled the length of this line segment $2b$.

The equation is simplest when we locate the center at the origin $(0,0)$. Then, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $a \geq b > 0$ has foci $F_1 = (c,0), F_2 = (-c,0)$ where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

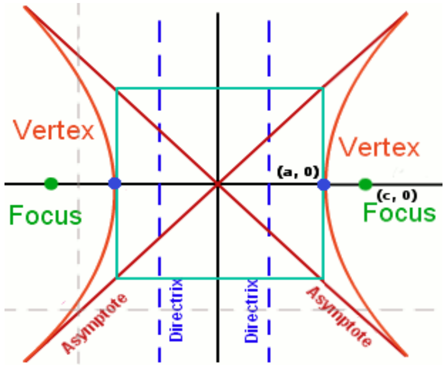
Parabola

A parabola is the set of points that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**) (see above). The point halfway between the focus and the directrix lies on the parabola, and is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis of the parabola** (or axis of symmetry).



An equation of the parabola with focus $(0,p)$ and directrix $y = -p$ is $x^2 = 4py$. If we write $a = \frac{1}{4p}$, than the previous equation becomes $y = ax^2$ (possibly a more familiar form), and the parabola opens upward if $p > 0$ and downward if $p < 0$.

Hyperbola



Let the distance between points P and the **focus** F_1 be labeled $|PF_1|$, and similarly $|PF_2|$ for the **focus** F_2. Then, a hyperbola is the set of all points where the difference $||PF_1| - |PF_2||$ is constant. The need to study a hyperbolas often occurs in chemistry, physics, biology, and economics.

As before, the equations simplify if we locate the **center** of the hyperbola at the origin. In this case, $|PF_1| - |PF_2| = \pm 2a$ for some constant a. The **vertices** are located at $(\pm a, 0)$, the hyperbola asymptotically approaches **asymptotes** $y = \pm \frac{b}{a}x$, and the **foci** are located at $F_1 = (c, 0)$ and $F_2 = (-c, 0)$, where $c^2 = a^2 + b^2$. This simplified equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

For all of the equations for shapes above, we can exchange the roles of x and y to reflect the shapes about the diagonal line $y = x$.

Shifted Conics

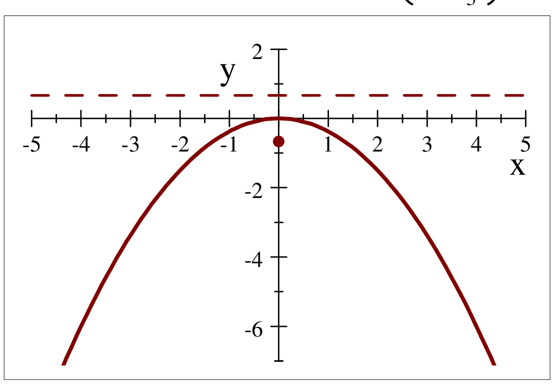
How do you construct a conic not located at the origin, but starting with the simple origin equations above? Shift them either horizontally or vertically in order to locate it elsewhere in the plane. To shift it to the right by h, substitute x with $x - h$ (or $x + h$ to shift to the left) in the equation. Similarly, substitute y with $y - k$ to shift upwards by k, or $y + k$ to shift downwards.

Problem #4 Find the vertex, focus, and directrix of the parabola $3x^2 + 8y = 0$, and sketch its graph.

$3x^2 + 8y = 0 \quad \Rightarrow \quad 3x^2 = -8y \quad \Rightarrow \quad x^2 = -\frac{8}{3}y.$

$4p = -\frac{8}{3} \quad \Rightarrow \quad p = -\frac{2}{3}.$

The vertex is $(0, 0)$, the focus is $\left(0, -\frac{2}{3}\right)$, and the directrix is $y = \frac{2}{3}$.



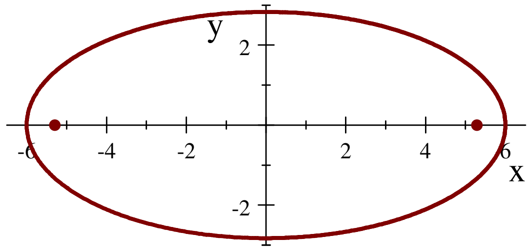
Problem #12 Find the vertices and foci of the ellipse $\frac{x^2}{36} + \frac{y^2}{8} = 1$, and sketch its graph.

$a = \sqrt{36} = 6, \qquad b = \sqrt{8},$

$c = \sqrt{a^2 - b^2} = \sqrt{36 - 8} = \sqrt{28} = 2\sqrt{7}.$

The ellipse is centered at $(0, 0)$, with vertices at $(\pm 6, 0)$.

The foci are $\left(\pm 2\sqrt{7}, 0\right)$.

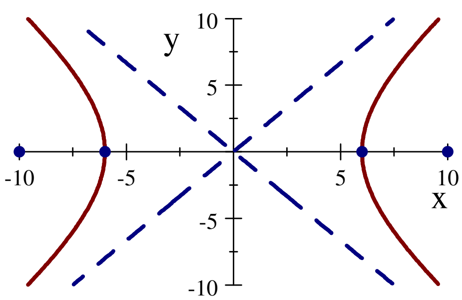


Problem #20 Find the vertices, foci, and asymptotes of the hyperbola $\frac{x^2}{36} - \frac{y^2}{64} = 1$, and sketch its graph.

$a = 6,\ b = 8,$

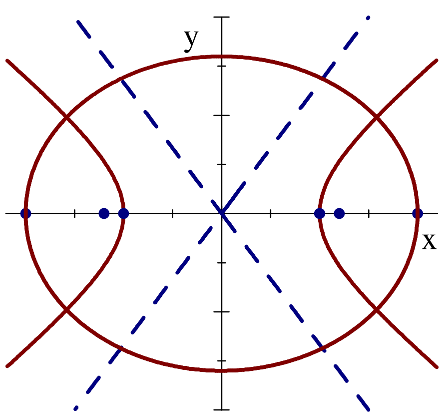
$c = \sqrt{36+64} = 10$

Center at $(0,0)$, vertices at $(\pm 6, 0)$, foci at $(\pm 10, 0)$ asymptotes at $y = \pm\frac{8}{6}x = \pm\frac{4}{3}x$.



Problem #58 Show that if an ellipse and a hyperbola have the same foci, then their tangent lines at each point of intersection are perpendicular.

Observe that if two lines are perpendicular where they intersect, then the slopes are negative reciprocals, and therefore the product of the slopes is -1. We would like to show that this is the case here. So our task is to find the slopes at the intersections.



Without a loss of generality, let the ellipse, hyperbola, and foci $(\pm c, 0)$ be as shown in the figure.
So we have the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the hyperbola $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$.

So, we find the slopes of the tangent lines of the curves through implicit differentiation. For the the ellipse we have:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0,$$

$$\frac{yy'}{b^2} = -\frac{x}{a^2} \quad \Rightarrow \quad y'_E = -\frac{b^2}{a^2}\frac{x}{y}.$$

And for the hyperbola we have: $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 \quad \Rightarrow \quad \frac{2x}{A^2} - \frac{2yy'}{B^2} = 0,$

$$\frac{yy'}{B^2} = \frac{x}{A^2} \quad \Rightarrow \quad y'_H = \frac{B^2}{A^2}\frac{x}{y}.$$

The product of the slopes at (x_0, y_0) is $y'_E y'_H = -\frac{b^2B^2x_0^2}{a^2A^2y_0^2}$.

So to determine this, we need to know what x_0^2 and y_0^2 are. Notice from our equations that we have two equations with two unknowns. So we can solve for x^2 and y^2 in terms of our other parameters.

In order to isolate y^2 in each equation, multiply by b^2 and B^2 respectively. Then adding the two equations, we get:
$\left(\frac{B^2}{A^2}x^2 - y^2\right) + \left(\frac{b^2}{a^2}x^2 + y^2\right) = B^2 + b^2$

We therefore have: $x^2\left(\frac{B^2}{A^2} + \frac{b^2}{a^2}\right) = B^2 + b^2$ or $x^2 = \frac{B^2+b^2}{\frac{a^2B^2+b^2A^2}{A^2a^2}} = \frac{A^2a^2\left(B^2+b^2\right)}{a^2B^2+b^2A^2}$.

Similarly, $y^2 = \frac{B^2b^2\left(a^2-A^2\right)}{b^2A^2+a^2B^2}$.

The product of the slopes at (x_0, y_0) is then $y'_E y'_H = -\frac{b^2B^2x_0^2}{a^2A^2y_0^2} = \frac{b^2B^2\left[\frac{A^2a^2\left(B^2+b^2\right)}{a^2B^2+b^2A^2}\right]}{a^2A^2\left[\frac{B^2b^2\left(a^2-A^2\right)}{b^2A^2+a^2B^2}\right]} = -\frac{B^2+b^2}{a^2-A^2}$.

Since in the definitions of the ellipse and hyper below we have $a^2 - b^2 = c^2$ and $A^2 + B^2 = c^2$, we can conclude that $a^2 - b^2 = A^2 + B^2 \quad \Rightarrow \quad a^2 - A^2 = b^2 + B^2$, so the product of the slopes is -1, and hence, the tangent lines at each point of intersection are perpendicular. ■