MATH 1271: Calculus I

Discussion Instructor: Jodin Morey
moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

2.3 - Calculating Limits Using the Limit Laws

Review:

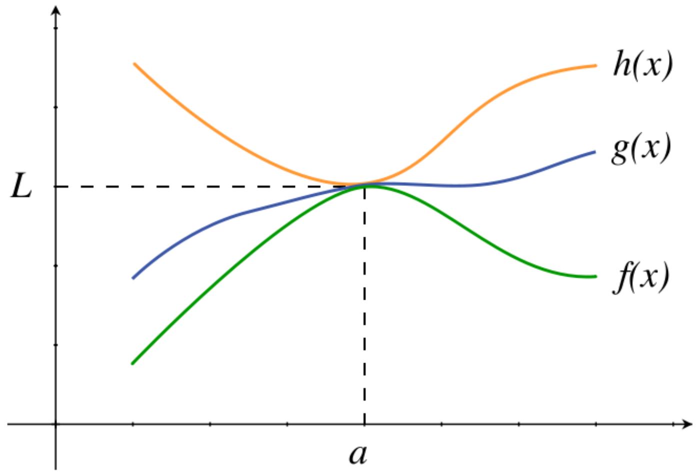
Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

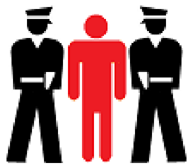
- $\lim_{x\to a} [f \pm g] = \lim_{x\to a} f \pm \lim_{x\to a} g$
- $\lim_{x\to a} [cf] = c \lim_{x\to a} f$
- $\lim_{x\to a} [f \cdot g] = \lim_{x\to a} f \cdot \lim_{x\to a} g$
- $\lim_{x\to a} \frac{f}{g} = \frac{\lim_{x\to a} f}{\lim_{x\to a} g}$, if $\lim_{x\to a} g \neq 0$
- $\lim_{x\to a} [f]^n = \left[\lim_{x\to a} f\right]^n$, where n is a positive integer.
- $\lim_{x\to a} c = c$
- $\lim_{x\to a} x = a$
- $\lim_{x\to a} x^n = a^n$, where n is a positive integer.
- $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer (when n is even, we also need $a \geq 0$ for real numbers)
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer.
  (when n is even, we also need $\lim_{x\to a} f(x) > 0$ for real numbers)

Direct Substitution Property: If f is a polynomial or a rational function (i.e. $\frac{\text{polynomial}}{\text{polynomial}}$),
and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.
(this is because these types of functions are continuous on their domain)

- If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limits exist.
- $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$. (two sided limit)
- If $f(x) \leq g(x)$ when x is near a (except possibly at a) and
  the limits of f and g both exist as $x \to a$, then: $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

The Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$, when x is near a (except possibly at a),
and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then: $\lim_{x\to a} g(x) = L$.



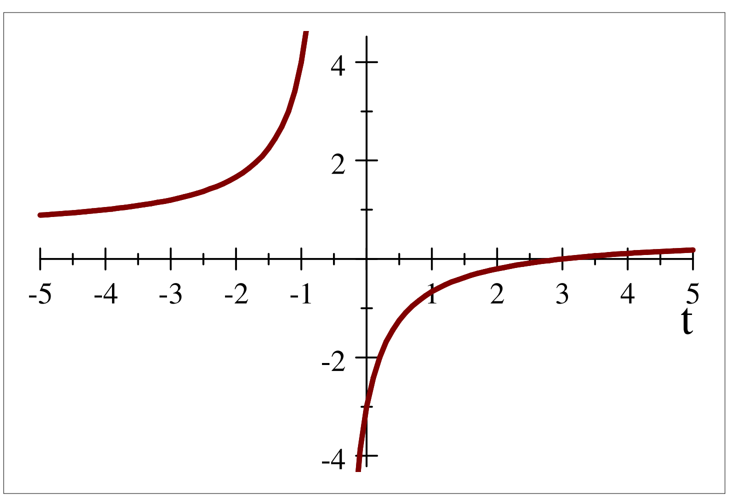
A.K.A.: The Two Policemen Theorem: 

Problem 15. Evaluate the limit, if it exists: $\lim\limits_{t \to -3} \frac{t^2-9}{2t^2+7t+3}$

$$= \lim_{t \to -3} \frac{(t+3)(t-3)}{(2t+1)(t+3)}$$

$$= \lim_{t \to -3} \frac{t-3}{2t+1}$$

$$= \frac{-3-3}{2(-3)+1} = \frac{-6}{-5} = \frac{6}{5}.$$



$$\frac{t^2-9}{2t^2+7t+3}$$

Problem 19. Evaluate the limit, if it exists: $\lim\limits_{x \to -2} \frac{x+2}{x^3+8}$.

Does denominator have factor of $x+2$? Polynomial division!

$$x^3+8 = (x+2)x^2 + (-2x^2+8) \qquad (*)$$

$$-2x^2+8 = (x+2)(-2x) + (4x+8)$$

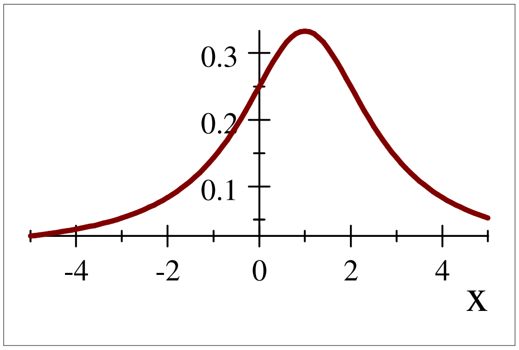
$$= (x+2)(-2x) + 4(x+2) = (x+2)(-2x+4).$$

Substituting back into $(*)$. $\quad x^3+8 = (x+2)x^2 + (x+2)(-2x+4) = (x+2)(x^2-2x+4).$

So, $\displaystyle \lim_{x\to -2} \frac{x+2}{x^3+8} = \lim_{x\to -2} \frac{x+2}{(x+2)(x^2-2x+4)}$

$$= \lim_{x\to -2} \frac{1}{x^2-2x+4} \qquad \text{(-2 is in the domain of this rational function)}$$

$$= \frac{1}{4+4+4} = \frac{1}{12}.$$



$\frac{x+2}{x^3+8}$

Problem 21. Evaluate the limit, if it exists: $\displaystyle \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h}.$

$$= \lim_{h\to 0} \left(\frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} \right)$$

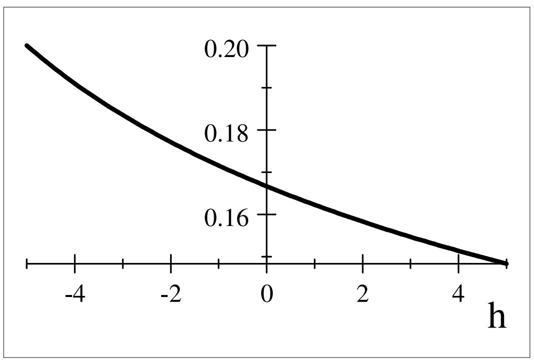
$$= \lim_{h\to 0} \frac{\left(\sqrt{9+h}\right)^2 - 3^2}{h\left(\sqrt{9+h}+3\right)} \qquad \text{(inverse "difference of squares": } (a-b)(a+b) = (a^2-b^2))$$

$$= \lim_{h\to 0} \frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{9+h}+3\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3}$$

$$= \frac{1}{3+3} = \frac{1}{6}.$$



$$\frac{\sqrt{9+h}-3}{h}$$

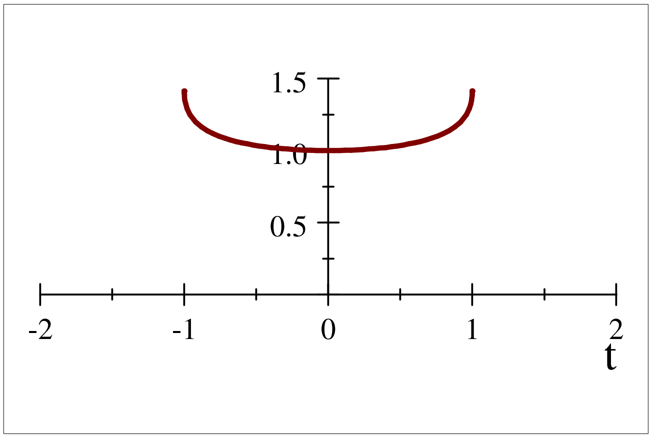
Problem 25. Evaluate the limit, if it exists: $\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

$$= \lim_{t\to 0} \left(\frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} \right)$$

$$= \lim_{t\to 0} \frac{\left(\sqrt{1+t}\right)^2-\left(\sqrt{1-t}\right)^2}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)} = \lim_{t\to 0} \frac{(1+t)-(1-t)}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)} = \lim_{t\to 0} \frac{2t}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0} \frac{2}{\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{2} = 1.$$



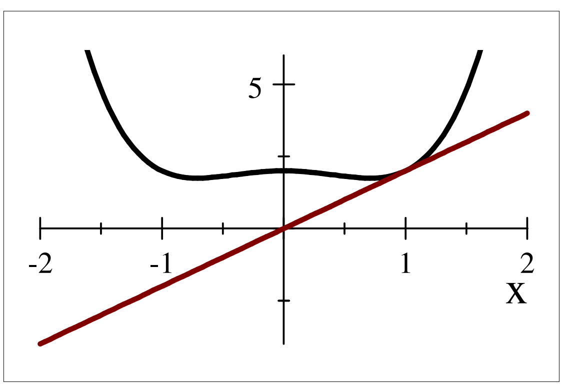
$\frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

Problem 38. If $2x \le g(x) \le x^4 - x^2 + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

$\lim_{x\to 1} (2x) = 2(1) = 2.$

and $\lim_{x\to 1} (x^4 - x^2 + 2) = 1^4 - 1^2 + 2 = 2.$

$\lim_{x\to 1} g(x) = 2$ by the squeeze theorem.



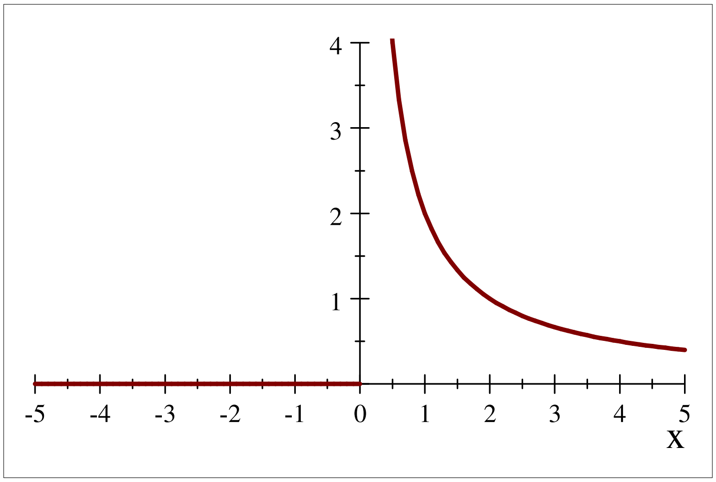
$2x$ and $x^4 - x^2 + 2$

Problem 45. Evaluate: $\lim_{x\to 0^-} \left(\frac{1}{x} + \frac{1}{|x|}\right)$

Since $|x| = -x$ for $x < 0$,

$$\lim_{x \to 0^-} \left(\frac{1}{x} + \frac{1}{|x|} \right) = \lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{x} \right)$$

$$= \lim_{x \to 0^-} 0 = 0.$$



$$\frac{1}{x} + \frac{1}{|x|}$$