Research Statement

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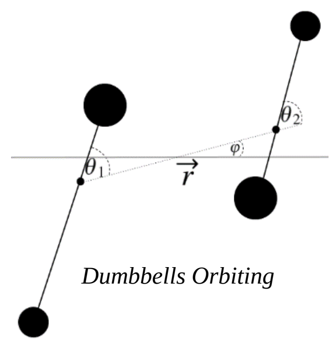
In the urban landscape where I grew up, the city lights obscured much of the sky. Nonetheless, I found some clear nights in which to pierce the haze with my little telescope and found fascination in staring at the stars and planets, speculating about the inner workings of the universe. Now I research celestial gravity in two of its manifestations: orbital mechanics and black hole gravity waves.

Orbital Mechanics

I wrote my dissertation on the two-body problem. That is, the problem of predicting the motions of two bodies that are interacting freely due to gravity (imagine two asteroids out in space). Newton solved a simplified version of the problem, where one assumes that each body is merely a point mass. However, the full two-body problem (where we do not make such an assumption) is still an open problem.

Since this describes a dynamical system, our first task is to identify equilibria of the system (configurations where the bodies can remain motionless). However, because the bodies are orbiting each other, we don't expect a proper equilibrium solution. But we might be able to find "relative equilibria" (RE), where the bodies are moving (orbiting each other), but they maintain a constant radius, and they do not rotate relative to each other. Another task is to determine whether these RE are stable, that is, whether such a configuration, when perturbed, will remain close to the RE.

In my research I approximate the full two-body problem by modeling each body not as a point mass (as Newton did), but by two point-masses connected with a massless rod, a "dumbbell" (see the figure).



Dumbbells Orbiting

My research has uncovered:

- Nonlinearly stable colinear RE, generalizing previous results.
- The existence of several families of symmetric RE, nonsymmetric RE (when the dumbbell masses are pairwise equal), and determined their linear and nonlinear stability.
- A generalized Conley's Perpendicular Bisector Theorem that places restrictions on the RE configurations allowed with a dumbbell body and second body consisting of discretized masses.
- Unexpected linearly stable RE when the bodies are parallel, and perpendicular.

To discover these RE, one generates a Lagrangian:

$$\mathcal{L}(r, \dot{r}, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \dot{\phi}) = K - U$$

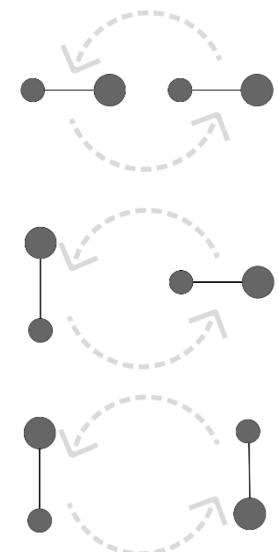
from the kinetic and potential energies of the configuration. Then the equations of motion result from using the Euler-Lagrange equation $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$, for each degree of freedom $q_i \in \{r, \varphi, \theta_1, \theta_2\}$. Next, we note that for a RE, the system rotation angle φ has symmetry. As a result, this variable does not make an appearance in our Lagrangian, it is a "cyclic variable." So, by Noether [1], we have a conserved quantity (angular momentum in our case) obtained by: $L := \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$.

We are now free to solve for $\dot{\phi}$ in the above equation and use this to eliminate $\dot{\phi}$ from our Lagrangian and equations of motion. The resulting reduced Lagrangian includes a reduced kinetic and potential energy:

$$\mathcal{L}_{red}(r, \dot{r}, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = K_{red} - U_{red}.$$

Smale, showed in [2] that after such a reduction, the RE can then be found as critical points of the reduced potential U_{red}. Then if one can find solutions to the system $\partial_{\theta_1} U_{red} = 0$, $\partial_{\theta_2} U_{red} = 0$, in terms of particular θ_1, θ_2, then the requirement $\partial_r U_{red} = 0$ (which can be rearranged into $L(r) = f(r; \theta_1, \theta_2)$ allows us (for any angular momentum L) to solve for the radii r of the RE. Using this technique, I discovered symmetric RE (configurations in which the two bodies are colinear, perpendicular).

Another likely RE configuration is when the masses on the dumbbells are pairwise equal. And while this is insufficient by itself to satisfy the RE requirements, setting the dumbbells to be parallel then satisfies the requirements.



Symmetric RE.
Colinear, Perpendicular & Parallel

Next, I performed a bifurcation analysis on each of the symmetric RE, using radius as my bifurcation parameter. Numerically, one observes pitchfork-like bifurcations for small radii. And is easily verified that $\partial_{\theta_1} U_{red}$ and $\partial_{\theta_2} U_{red}$ are odd functions of θ_1, θ_2 (a requirement for pitchforks). Then, for each symmetric configuration $(\theta_1, \theta_2) = (\theta_1^*, \theta_2^*)$, I identified the radius r of any pitchforks by solving $\left| D\left(\partial_{\theta_1} U_{red}, \partial_{\theta_2} U_{red}\right)_{(\theta_i^*; r)} \right| = 0$. I then used the Implicit Function Theorem (after a change of variables) to approximate solutions to the system. These are asymmetric RE bifurcating from the symmetrical ones.

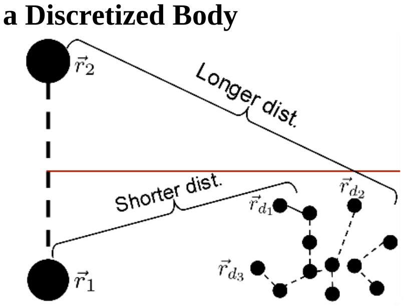
A bonus to this amended potential method of locating RE, is that if these critical points are strict minima, Smale showed that they are nonlinearly stable. Determining whether these RE are strict minima requires analyzing whether the U_{red} Hessian is positive definite. This analysis led to the theorem:

Stability for Two Dumbbells in Colinear Orbit
RE of the two dumbbell colinear configuration with radii large enough such that $\partial_r L^2 > 0$ are stable.

But practically speaking, linear stability can also be of use. So, I linearized the equations of motion, and required purely imaginary eigenvalues from the resulting system matrix A, evaluated at RE. This leads to three requirements on the coefficients of this matrix's characteristic polynomial. Satisfying these requirements for my symmetric RE led to the identification of linearly stable radii for the perpendicular configuration, as well as stability for the parallel configuration for small radii.

For a more general theory, I also analyzed the accelerations involved in a model which allows one of the orbiting bodies to consist of many discretized masses. This led to the theorem:

Perpendicular Bisector Theorem for RE of a Dumbbell and a Discretized Body
Let a dumbbell and a discretized body be in a planar relative equilibrium. The dumbbell point masses can be colinear with the center of mass of the discretized body. Alternatively, consider the four open quadrants determined by the lines through the dumbbell's rod and that rod's perpendicular bisector. If the point masses of the discretized body are located solely in the open quadrants, they cannot be located only in one quadrant, or only in diagonal quadrants.



But why do we care about RE of the two-body problem? Practically speaking, the finite resources humans use on Earth can often be found in asteroids, many of which are irregular in shape. The shapes of these asteroids and the probes we'll be sending there can be modeled by masses connected by rods. Calculations a spacecraft will need to make to find a static orbit or RE around these asteroids can benefit from gravitational models that consider their irregular shapes. Finding these RE is the aim of my research.

Black Hole Gravity Waves

As an undergraduate I conducted research into environmental science with my mathematical modeling professor, Rikki Wagstrom. Knowing of my interest in celestial objects, upon my graduation (in 2015) she suggested I join a research team working on black holes. Since then, I have conducted research with mathematician Michael Green and physicist Ramin Daghigh at Metropolitan State University, St. Paul. Together, we have published two papers in Physical Review D, and have a third paper in preparation [11].

Einstein taught us that when the universe gives birth to a black hole, it cries out with gravity waves. My colleagues and I calculate the "sound" that different types of black holes make at birth, or when otherwise perturbed. A black hole's gravity waves are analogous to an earthquake's seismographic waves; they can be read as curvy lines on a page (called waveforms). It was only in 2017 that, for the first time, gravity waves were verifiably detected at the LIGO and Virgo interferometers. A natural question arises: what can we learn about black holes from looking at these waveforms? Can we determine the type or size of the black hole?

Mathematical models can be generated in the form of Schrödinger-type wave equations that represent black hole oscillations of different types. One research goal is to calculate the gravity waves predicted by these models. As we develop more sophisticated models that more closely mirror real black holes, the gravity waves they produce should match those found by the gravity wave detectors. That way, when one detects gravity waves from space, they can compare those waveforms to ones calculated from this type of research, and thereby determine the type of black hole that generated the waves. Additionally, since there is still much debate about the exact structure of black holes, we can compare the waveforms predicted by competing theories to determine which theory is correct.

Gravity waves have frequencies that occur along a spectrum. Each black hole has its own resonant frequencies due to its geometry. The sum of these gravity wave frequencies constitutes its normal modes. However, after a perturbation of the black hole, energy is dissipated through the propagation of gravity waves, and the normal modes decay over time. Because of this dissipation, we refer to these decaying modes as quasi-normal modes (QNMs). QNMs relationship to waveforms is analogous to the relationship individual notes have to a musical chord. QNMs are indivisible components which, once added together, produce the waveform.

I research high-damping QNMs (HQNM) of regular black holes. A *regular* black hole is one for which there is no singularity. Instead, the central mass is compact, but finite. Depending upon the quantum theory of gravity used, regular black holes can have different geometries. Particularly, I have been studying Bardeen regular black holes. This gravitational theory allows for regular black holes due to the assumption that, in the presence of very dense mass, space-time becomes "de Sitter-like" [30]. That is, its dynamics are that of flat space under the influence of dark energy. In other words, space time expands, and creates an outward pressure to balance the gravitational pressure of the black hole mass.

While much research has found the low-frequency/overtone QNMs of regular black hole gravity waves [13-29], HQNMs have seen less research. Once perturbed, a black hole starts producing gravity waves, but HQNMs very quickly relax to an equilibrium. This makes these QNMs harder to detect. However, HQNMs potentially provide information about a black hole that low-overtone QNMs cannot.

For example, it is known that the small-scale structure of an oscillating medium (think of a guitar string or flute) is audibly detectable by the HQNMs given off by its oscillation. This is what differentiates the sound (timbre) of one type of instrument (guitar) from another (flute), even if they're playing the same note. In a similar way, the small-scale structure of a gravitationally oscillating black hole may be revealed through analysis of its HQNMs. In fact, [31,32] had attempted to make a connection between the HQNMs of black holes and quantum gravity, and then later it was established in 2011 [3]. This connects QNMs with the small-scale structure of black hole space-times.

HQNMs have been calculated for Kerr black holes (rotating, but without electric charge, and with a singularity) in 1985 [4], and for Schwarzschild black holes (non-rotating, without electric charge, and with a singularity) in 1993 [5]. However, HQNMs become increasingly difficult to calculate as the model becomes more complicated.

In our effort to calculate HQNMs, I reviewed [4] and [5], and generated a Mathematica script which recovered results from both papers and serves as a simultaneous generalization of the procedures therein. In 2020, we used my script to publish [6] in Physical Review D, which introduced a technique to simplify the calculations involved in the process of finding the waveforms.

Some mathematical models lend themselves more easily to waveform calculation. For the more difficult models, we showed that one can replace the Regge-Wheeler potential appearing in the wave equation with a simplified potential allowing for more easy calculations, while still approximating the waveform to any accuracy desired. Specifically, we examined how dependent the calculated QNMs are on a smooth model of the gravitational potential. This examination led to the realization that the gravitational waveform can be approximated to arbitrary precision using either step functions or a piecewise linear function in place of the smooth Regge-Wheeler potential.

Later that year, we published [7] in Physical Review D in collaboration with Gabor Kunstatter (Physics, University of Winnipeg), where we once again used the methods referenced above to investigate a regular fourth dimensional black hole (nonrotating, and without charge). Being a regular black hole, the size of the central mass is measured by k, where $k = 0$ is the singular Schwartzchild case [5]. We looked at the black hole in the presence of a scalar field and examined the QNMs generated when the scalar field undergoes a perturbation, see Figures 1,2.

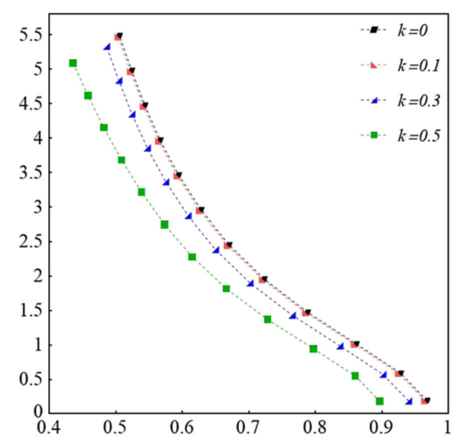


Figure 1: QNMs in the real/(negative)imaginary-plane for Schwartzchild (k=0) and regular BHs with various k.

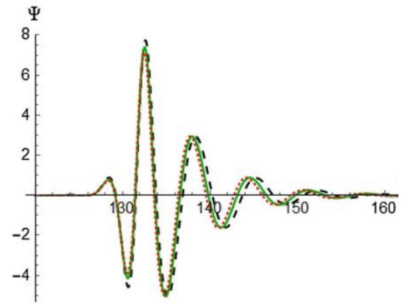


Figure 2: Gravity waveforms in the time/displacement-plane for Schwartzchild (k=0) and regular BHs with various k.

We found the QNMs had negative imaginary parts, which results in damping (the amplitudes of the waves diminish over time). This further implies the black hole will not gravitationally radiate away all its mass, i.e., it is stable.

Currently, our team is putting the finishing touches on research into the Reissner Nordstrom (RN) black hole (charged, but nonrotating and singular). I have been producing QNMs for various amounts of electric charge. We have produced QNMs not yet reported elsewhere [11].

To find QNMs, one starts with a version of the wave equation which models the gravity waves of the type of black hole you're studying. One then applies boundary conditions to ensure that the gravity waves radiate only inward at the black hole event horizon, and only outward at spatial infinity. Baber and Hasse [9] discovered the form of an ansatz (series solution) for this type of problem, which is then adjusted to be consistent with the boundary condition requirements. Upon substituting this ansatz, one then obtains an *m*-term recurrence relation: $a_0 = 1,\ a_{-1} = 0,\ c_0 a_n + c_1 a_{n-1} + \cdots + c_{m-1} a_{n-m-1} = 0$, where the c_i are functions of n, ω; and the ω are the (complex) QNMs to be solved for.

The specifics of the recurrence relation are worked out for each black hole model. For Schwarzschild, one finds a 3-term recurrence. This is good, because if $m = 3$, then Gautschi in [10] showed that for each positive integer n, the desired solutions satisfy the continued fraction relation:

$$0 = c_1(n+1,\omega) - c_0(n+1,\omega)\frac{c_2(n+2,\omega)}{c_1(n+2,\omega) - c_0(n+2,\omega)\dfrac{c_2(n+3,\omega)}{c_1(n+3,\omega) - \cdots}} \tag{1}$$

And accuracy of the ω roots is gained as the depth of the continued fraction increases.

However, my team observed that the more complicated black hole models generally have an *m*-term recurrence relation, with $m > 3$. For example, we found a 4-term recurrence relation was generated when evaluating a RN black hole. However, an *m*-term relation can be reduced to a 3-term relation using Gaussian reduction. After this reduction, one can then apply the Baber and Hasse technique to find the QNMs.

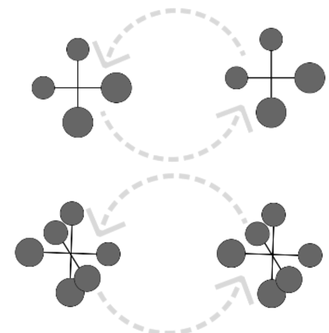
A difficulty with this process, however, is that the convergence for HQNMs occurs slowly, and computation times become unreasonable. A technique to manage this was found by Nollert in [5]. It involves generating an estimate for the tail end of the continued fraction.

I have observed that computational demands also increase as *m* increases. For the Gaussian reduction, the size of the associated coefficient matrix may become quite large, in the hundreds or thousands of rows. Calculations of this complexity cannot be done by hand and is instead done using Mathematica. Indeed, I have made use of the supercomputer at the University of Minnesota in these computations.

Once the QNMs are calculated, we can consider stability. Stability in this context has to do with whether the black hole will radiate away all of its mass upon being perturbed. In the ansatz solution used to solve these equations, we always have a factor like $e^{-i\omega r}$, where r is the radius from the black hole. Mathematically, to ensure this factor doesn't grow without bound as the wave moves away from the black hole (taking the black hole's energy with it), observe we need the imaginary part of the ω to be negative. So, this becomes the criteria for stability.

Future Research

I have plans to expand my two-body problem research to include bodies consisting of two dumbbells perpendicular to each other, connected along their rods, a "pinwheel." This way, each body is two-dimensional, instead of the one-dimensional dumbbell. The end goal would be eventually to model each body as a three-dimensional object, with three dumbbells connected along their rods, a "jack." This model would provide a lot of flexibility to better approximate real-world objects (as you modify each body's six mass parameters, and three rod length parameters).



Pinwheels and Jacks

Regarding gravitational waves, my colleagues and I are planning to study other theoretical models for "regular" black holes, possibly using the approximation technique described in our earlier paper to produce gravitational waveforms and checking for stability. In particular, we are next going to examine a regular charged black hole, and then expand on this with a regular charged black hole which has some small rotation. Results from this research will help to reveal which theoretical black hole models represent real black holes, and which models can be eliminated.

I am lucky to have been able to pursue my childhood fascination. Not only have I had the chance to learn about those motions in the night sky, but also to contribute to the scientific conversation. I look forward to providing undergraduates with a similar opportunity. There are many extensions of my two-body problem work that an undergraduate could pursue with my guidance. In addition, some of the research I did as an undergraduate into automobile fossil fuel consumption and electoral voting systems could be updated and extended by an undergraduate researcher. I look forward to making these opportunities available.

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