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MATH 2243: **Linear Algebra** & **Differential Equations**

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10.2: Laplace Transformation of IVPs

Laplace transformations can be used to solve linear diff. eqs with constant coefficients. Note that because of the linearity of Laplace transformations, given $ax''(t) + bx'(t) + cx(t) = f(t)$ we can apply a Laplace transform to both sides of the equation in the following way: $a\mathcal{L}\{x''(t)\} + b\mathcal{L}\{x'(t)\} + c\mathcal{L}\{x(t)\} = \mathcal{L}\{f(t)\}$.

How does this help us solve the differential equation?
If the result of applying each Laplace transform above leaves us with an algebraic equation, then we can use our regular tools of algebra to solve the equation, and then use the inverse Laplace transforms we learned in the last section to solve for $x(t)$.

But how do we apply a Laplace transform to a derivative?
Well, if $x(t)$ is continuous and piecewise smooth (piecewise continuously differentiable) for $t \geq 0$ and is of exponential order as $t \to \infty$, then $\mathcal{L}\{x'(t)\}$ exists and $\mathcal{L}\{x'(t)\} = s\mathcal{L}\{x(t)\} - x(0)$.
Similarly, $\mathcal{L}\{x''(t)\} = s\mathcal{L}\{x'(t)\} - x'(0) = s\{s\mathcal{L}\{x(t)\} - x(0)\} - x'(0)$
$= s^2\mathcal{L}\{x(t)\} - sx(0) - x'(0)$.
Therefore, if we know the initial conditions $x(0)$ and $x'(0)$, we can take the transform of the derivatives.

Solving a System:

Given: $x'' = f(y,x,t)$ and $y'' = g(y,x,t)$, apply the Laplace transform to both equations as indicated above. The resulting set of equations will be just an algebraic system which you can solve with our regular methods (Gaussian Reduction, Cramer's Rule, etc.).

Equation of Motion: Recall $mx'' + cx' + kx = f(t)$.

Here's a new notation, instead of $\mathcal{L}(x(t))$, let's write $X(s)$.
Applying the transform: $m[s^2X(s) - sx(0) - x'(0)] + c[sX(s) - x(0)] + kX(s) = F(s)$.

Or: $(ms^2 + cs + k)X(s) - mx(0)s - cx(0) - mx'(0) = F(s)$

$$\Rightarrow X(s) = \frac{mx(0)s+cx(0)+mx'(0)+F(s)}{ms^2+cs+k} = \frac{mx(0)s+cx(0)+mx'(0)}{ms^2+cs+k} + \frac{F(s)}{ms^2+cs+k} = \frac{I(s)}{Z(s)} + \frac{F(s)}{Z(s)},$$

Note: $\frac{I(s)}{Z(s)}$ is a function of the **initial conditions** of the system,
and $\frac{F(s)}{Z(s)}$ is a function of the **external force**.
So, this should remind you of $x(t) = x_{tr}(t) + x_{sp}(t)$.

Recall that $x_{tr}(t)$ is the transient solution of the system (complementary), while $x_{sp}(t)$ is the steady periodic solution (particular) which remains once the transient solution dissipates (due to damping).

Therefore, apparently: $\mathcal{L}\{x(t)\} = \mathcal{L}\{x_{sp}(t)\} + \mathcal{L}\{x_{tr}(t)\} = \frac{F(s)}{Z(s)} + \frac{I(s)}{Z(s)}$.

Additional Transform Techniques:

$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$, $\qquad \mathcal{L}\{t\sin kt\} = \frac{2ks}{(s^2+k^2)^2}$.

Transforms of Integral Functions:

Making our usual assumption that $f(t)$ is piecewise continuous for $t \geq 0$, and is of exponential order ($|f(t)| \leq Me^{ct}$), then:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{1}{s}F(s), \qquad \text{for } s > c.$$

Equivalently, $\mathcal{L}^{-1}\{\frac{1}{s}F(s)\} = \int_0^t f(\tau)d\tau$.

Problem: **#10** Use Laplace transforms to solve the initial value problem…

$$x'' + 3x' + 2x = t, \qquad x(0) = 0, \qquad x'(0) = 2.$$

$$\mathcal{L}\{x'' + 3x' + 2x\} = \mathcal{L}\{t\}$$

Things I'm going to need:

$$\mathcal{L}\{x'\} = sX(s) - x(0) = sX(s).$$

$$\mathcal{L}\{x''\} = s\mathcal{L}\{x'(t)\} - x'(0) = s(s\mathcal{L}\{x(t)\} - x(0)) - 2$$

$$= s^2X(s) - 2.$$

$$[s^2X(s) - 2] + 3[sX(s)] + 2[X(s)] = \mathcal{L}\{t\} = \ ?$$

$$(s^2 + 3s + 2)X(s) = 2 + \frac{1}{s^2} = \frac{2s^2+1}{s^2}$$

$X(s) = \frac{2s^2+1}{s^2(s^2+3s+2)}$ $\qquad$ Then what?

In order to apply the inverse Laplace transform, we must first use partial fractions!

$$\frac{2s^2+1}{s^2(s+1)(s+2)} = \left(\frac{A}{s} + \frac{B}{s^2}\right) + \frac{C}{s+1} + \frac{D}{s+2},$$

Multiplying both sides of the equation by $s^2(s+1)(s+2)$:

$$2s^2 + 1 = As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1)$$

Simplifying and collecting like terms:

$$= (As^2 + As)(s+2) + (Bs + B)(s+2) + (Cs^3 + 2Cs^2) + Ds^3 + Ds^2$$

$$= (As^3 + As^2) + (2As^2 + 2As) + (Bs^2 + Bs) + (2Bs + 2B) + (Cs^3 + 2Cs^2) + Ds^3 + Ds^2$$

$$= (A + C + D)s^3 + (3A + B + 2C + D)s^2 + (2A + 3B)s + 2B.$$

Therefore, $A + C + D = 0, \quad 3A + B + 2C + D = 2, \quad 2A + 3B = 0, \quad 2B = 1.$

Solving four equations in four unknowns:

$B = \frac{1}{2},\ A = \frac{1}{2}(-\frac{3}{2}) = -\frac{3}{4},\ (-\frac{3}{4}) + C + D = 0,\ C = \frac{3}{4} - D,$

$3(-\frac{3}{4}) + (\frac{1}{2}) + 2(\frac{3}{4} - D) + D = 2,\ \ -\frac{9}{4} + \frac{1}{2} + \frac{6}{4} - D = 2,\ D = -\frac{9}{4}.$

$C = \frac{3}{4} - (-\frac{9}{4}) = 3.$

So, $X(s) = \frac{2s^2+1}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} = \frac{(-\frac{3}{4})}{s} + \frac{(\frac{1}{2})}{s^2} + \frac{(3)}{s+1} + \frac{(-\frac{9}{4})}{s+2}.$

And, $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\left(\frac{-3}{s} + \frac{2}{s^2} + \frac{12}{s+1} - \frac{9}{s+2}\right)\right\} =$

$$\frac{1}{4}\left(-3\mathcal{L}^{-1}\{\tfrac{1}{s}\} + 2\mathcal{L}^{-1}\left\{\tfrac{1}{s^2}\right\} + 12\mathcal{L}^{-1}\{\tfrac{1}{s+1}\} - 9\mathcal{L}^{-1}\{\tfrac{1}{s+2}\}\right)$$

$$\mathcal{L}^{-1}\{X(s)\} = \frac{1}{4}(-3 + 2t + 12e^{-t} - 9e^{-2t}).$$

Problem: ≈**22** Use the theorem for the Laplace transform of an integral to find the inverse Laplace transform of $F(s) = \frac{1}{s(s^2+9)}$.

Recall that:

If $f(t)$ is piecewise continuous for $t \geq 0$, and satisfies the condition of exponential order, then:

$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{1}{s}F(s)$ for $s > c$.

Equivalently, $\mathcal{L}^{-1}\{\frac{1}{s}F(s)\} = \int_0^t f(\tau)d\tau.$

Also:

$$\mathcal{L}(\sin kt) = \frac{k}{s^2+k^2} \text{ for } s > |k|$$

First observe that:

$$\frac{1}{s(s^2+9)} = \frac{1}{s} \cdot \frac{1}{s^2+9} \text{ and } \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{1}{3}\sin(3t).$$

So, from the definition above…

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{s^2+9}\right\} = \frac{1}{3}\int_0^t \sin(3\tau)d\tau$$

$$= \left[-\frac{1}{9}\cos 3\tau\right]_{\tau=0}^{t} = \left(-\frac{1}{9}\cos 3t + \frac{1}{9}\cos(0)\right) = \frac{1}{9}(1 - \cos 3t).$$