MATH 2243: **Linear Algebra & Differential Equations**

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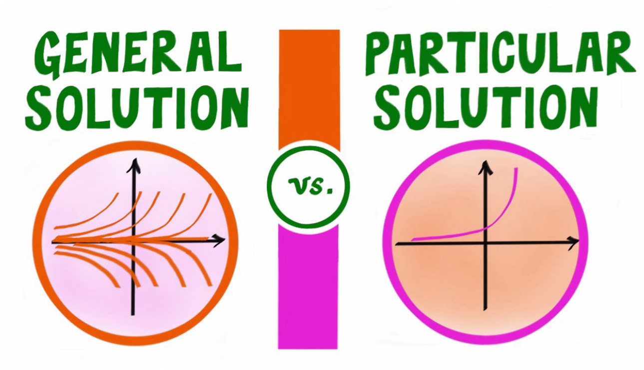
1.2: Integrals as General and Particular Solutions



General Solutions:

Recall for $y' = ky$, our general solution is $y = Ce^{kx}$, where we can choose any constant C.

For $y'' = -4y$ from last time, the general solution is ...

$y = c_1 \sin 2x + c_2 \cos 2x$, where c_1, c_2 are arbitrary constants (we can choose them as we please).

So, in addition to $y = \cos 2x$ being a solution as we verified,

so is $y = 5\cos 2x$, $y = e \sin 2x$, and $y = \sin 2x - \pi \cos 2x$.

You should verify this is true (on your own) by plugging the second derivatives

of these equations into $y'' = -4y$.

The Pattern:

Our 1st order DEQs each had only "one" **general solution**. Although, infinitely many **particular solutions** were present and we were free to arbitrarily choose the constant C. So, this "one" solution is actually a **family** of solutions.

Our 2nd-order DEQ had "two" general solutions, that we added in a **linear combination**.

Although, again we had the freedom to choose from an infinite number of **particular solutions** for each term since they each were each multiplied by an arbitrary constant. So, this also was a family of solutions.

DEQs which have the form $\frac{d^2y}{dx^2} = g(x)$,   (RHS depends only on x)

allow for easy solving. Rewrite as $\frac{d^2y}{dx^2} = \frac{dv}{dx} = g(x)$, where $v := \frac{dy}{dx}$.

We transformed it into a 1st order DEQ!

Once we have solved for $v = G(x)$, then solve $v = \frac{dy}{dx} = G(x)$, as another 1st order DEQ to find y.

Position $= x(t)$, Velocity $= \frac{dx}{dt}$, Acceleration $= \frac{dv}{dt} = \frac{d^2x}{dt}$.

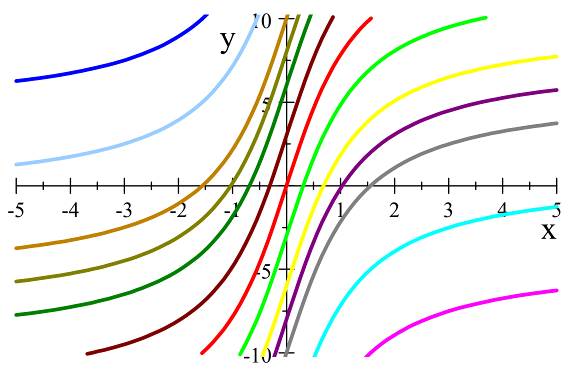
Weight $=$ mass • gravity $(F = ma)$.

Exercises

Problem: #7 Find the function $y = f(x)$ satisfying the differential equation $\frac{dy}{dx} = \frac{10}{x^2+1}$; with **initial condition** $y(0) = 0$.

$y(x) = \int \frac{10}{x^2+1} dx$

$= 10 \tan^{-1}x + C.$ (If need be, review your **inverse trigonometric derivatives**)

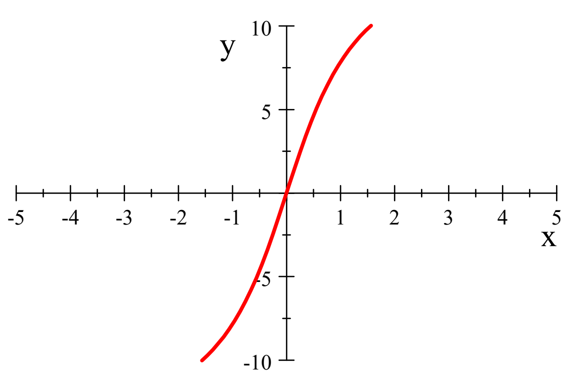


$10 \tan^{-1}x + C,$ various values of C

Then, the substitution of init cond $x = 0,\ y = 0$ gives ...

$0 = 10 \cdot 0 + C,$

so $y(x) = 10 \tan^{-1}x.$



$10 \tan^{-1}x$

Problem: #17

Find the position function $x(t)$ of a moving particle with the given acceleration $a(t) = \frac{1}{(t+1)^3}$,

initial position $x_0 = x(0) = 0$, and **initial velocity** $v_0 = v(0) = 0$.

$v(t) = \int (t+1)^{-3} dt$

Recall u-substitution: $u = t + 1 \quad \Rightarrow \quad du = dt \quad \Rightarrow \quad v(t) = \int u^{-3} du$

$= -\frac{1}{2}u^{-2} + C$

$= -\frac{1}{2}(t+1)^{-2} + C.$ Now what?

$0 = -\frac{1}{2}(0+1)^{-2} + C = -\frac{1}{2} + C$

$v(t) = -\frac{1}{2}(t+1)^{-2} + \frac{1}{2}.$ And then ...

$x(t) = \int \left[-\frac{1}{2}(t+1)^{-2} + \frac{1}{2} \right] dt$

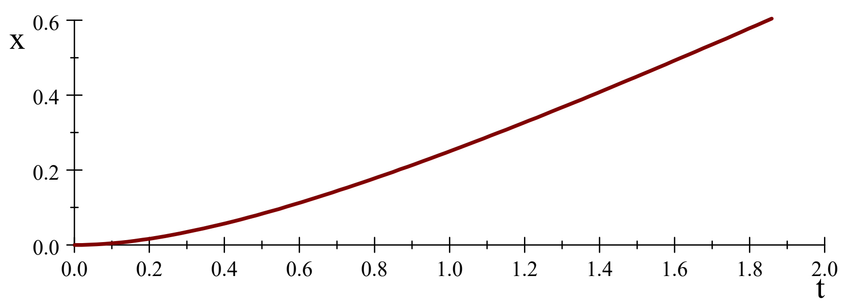
u –substitution: $u = t + 1 \quad \Rightarrow \quad du = dt \quad \Rightarrow \quad x(t) = \int [-\frac{1}{2}u^{-2} + \frac{1}{2}] du$

$= \frac{1}{2}u^{-1} + \frac{1}{2}u + C_1 = \frac{1}{2}(t+1)^{-1} + \frac{1}{2}(t+1) + C_1$ And then ...

Since $x(0) = 0$,

$0 = \frac{1}{2}(0+1)^{-1} + \frac{1}{2} \cdot (0+1) + C_1 = 1 + C_1$ and $C_1 = -1$.

So, $x(t) = \frac{1}{2}(t+1)^{-1} + \frac{1}{2}(t-1).$



$\frac{1}{2}(t+1)^{-1} + \frac{1}{2}(t-1)$

Problem: #36 Suppose a woman has enough "spring" in her legs to jump from the ground (on earth) to a height of 2.25 ft.

Assume she jumps straight upward with the same initial velocity on the moon, where the surface gravitational acceleration is $5.3\ ft/s^2$.
How high above the surface will she rise?

Eventually, we will need to solve an equation like: $x_m(t) = \int\int a_m(t)dtdt = Ct^2 + v_0 t + x_0$, where x_m, and a_m are position and velocity functions on the moon and where $x_0 = 0$ (the ground), and v_0 is the "same initial velocity" on both the Earth and the moon. So we need to find out what v_0 is.

We know: $a_e = g_e = -32\ ft/s^2$.

$v_e(t) = -\int 32dt = -32t + v_0.$

$x_e(t) = \int v_e dt = \int(-32t + v_0)dt = -16t^2 + v_0 t + x_0 = -16t^2 + v_0 t.$

Since we know something about when $x_e(t) = 2.25$, let's solve for t in the previous equation…

$$2.25 = -16t^2 + v_0 t \qquad \Rightarrow \qquad t = \frac{v_0 \pm \sqrt{v_0^2 - 64(2.25)}}{32} = \frac{v_0 \pm \sqrt{v_0^2 - 144}}{32}$$

Plugging this back into our velocity equation, and realizing that the velocity is zero at the top of the jump…

$$0 = -32\left(\frac{v_0 \pm \sqrt{v_0^2 - 144}}{32}\right) + v_0 = -v_0 \pm \sqrt{v_0^2 - 144} + v_0 = \pm\sqrt{v_0^2 - 144}.$$

And so, $v_0 = 12\ ft/\sec.$

We know: $a_m = -g_m = -5.3\ ft/s^2$.

$v_m(t) = -\int 5.3dt = -5.3t + v_0 = -5.3t + 12.$

When $v_m = 0,\ t = \frac{12}{5.3} = 2.2642$ sec.

$$x(t) = \int v_m dt = \int(-5.3t + 12)dt = -2.65t^2 + 12t + x_0$$
$$= -2.65(2.2642)^2 + 12(2.2642) \approx 13.58\ ft.$$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