MATH 2243: **Linear Algebra & Differential Equations**

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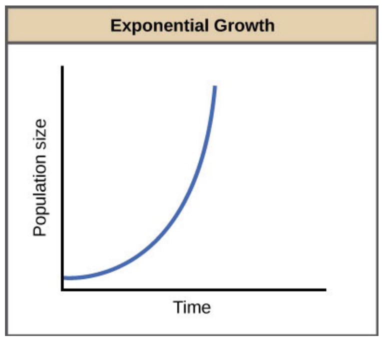
2.1: Mathematical Models and Numerical Methods

Population Models

Our previous (exponential) population model was: $\frac{dP}{dt} = kP$, with $P(t) = P_0 e^{kt}$.
where the birth and death rates were constant.



However, the following model is more realistic, with variable death and birth rates.
(but still assumes no immigration or emigration)

The change in population over some time period Δt is:
$$\Delta P = [In] - [Out] = [Born] - [Die].$$

To get the average rate of change per unit time:
$$\frac{\Delta P}{\Delta t} = \frac{[Born]}{\Delta t} - \frac{[Die]}{\Delta t}.$$

Taking the limit as $\Delta t \to 0$ (Calculus!) gives us the **General Population Equation**:
$$\frac{dP(t)}{dt} = \beta(t)P(t) - \delta(t)P(t) = (\beta(t) - \delta(t))P(t),$$
where $\beta(t)$ and $\delta(t)$ is a continuous approximation of the number of births and deaths (respectively) per person (or animal), occuring instantaneously at time t.

Notice that when β and δ are constant in this equation we retrieve: $\frac{dP}{dt} = kP$, where $k = \beta - \delta$.

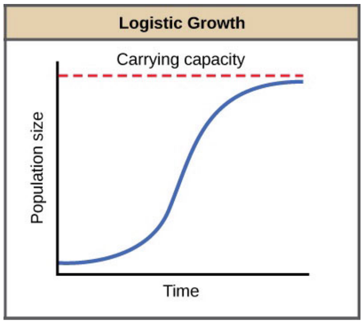
Bounded Populations

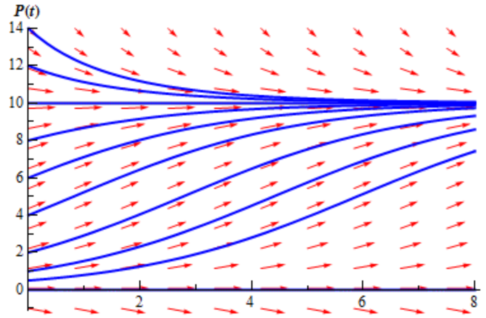
A common population model (called the **logistic equation**) occors when β is linearly decreasing with respect to P (which means: $\beta = c - kP$, with $c, k > 0$), and the death rate δ is a constant:

$$\frac{dP}{dt} = (\beta - \delta)P = (c - kP - \delta)P = kP\left(\frac{c-\delta}{k} - P\right) = kP(M - P), \text{ where } M = \frac{c-\delta}{k}.$$

The reason we notate it this way is to isolate an important quantity M, which we can see on a graph, and is the "**limiting population**" or "**carrying capacity**."

Logistic Equation: $\frac{dP}{dt} = kP(M - P)$. (models: finite resources, interspecies competition, disease, etc.)





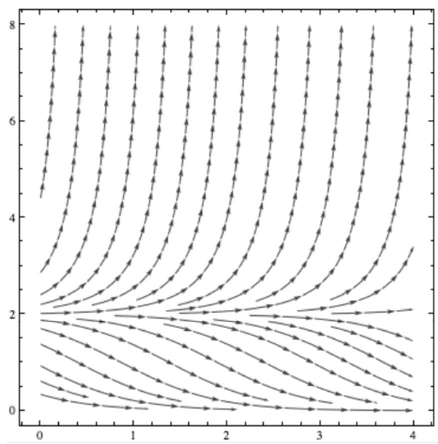
Using your newly acquired skill of separation-of-variables, you can now show that the initial value problem: $\frac{dP}{dt} = kP(M-P),\ P(0) = P_0$ has the solution $P(t) = \frac{MP_0}{P_0+(M-P_0)e^{-kMt}}$. (try this as an exercise!)

If initial population $P_0 = M$?

If initial population $P_0 > M$? If initial population $P_0 < M$?

Observe $\lim\limits_{t\to\infty} P(t) = \frac{MP_0}{P_0+0} = M$.

Explosion or Extinction Model: $\frac{dP}{dt} = kP(P-M)$. (models species near extinction)



Exercises

Problem: **#8** Separate variables and use partial fractions to solve the initial value problem:

$\frac{dx}{dt} = 7x(x-13),\ x(0) = 17.$

$\frac{1}{x(x-13)}dx = 7dt$, if $x \neq 13$ and $x \neq 0$.

Can $x(t) \equiv 13$ or $x(t) \equiv 0$, with our initial condition $x(0) = 17$?

$\int \frac{1}{x(x-13)}dx = \int 7dt$

We would prefer to integrate something more like $\int \frac{A}{x} + \frac{B}{x-13}dx = \int 7dt$.

Partial fractions:

Observe that $\frac{1}{x(x-13)} = \frac{A}{x} + \frac{B}{x-13}$, when $1 = A(x-13) + Bx$.

Collecting powers of x : $(A+B)x - 13A = 1$,

Comparing powers of x : $A + B = 0$ and $-13A = 1$,

so $A = -\frac{1}{13}$, and $B = \frac{1}{13}$.

Therefore: $\int \frac{1}{x(x-13)}dx = \int \frac{-\frac{1}{13}}{x} + \frac{\frac{1}{13}}{x-13}dx$.

And our equation becomes: $-\frac{1}{13}\int \frac{1}{x} - \frac{1}{x-13}dx = \int 7dt$

$\Rightarrow \int\left(\frac{1}{x} - \frac{1}{x-13}\right)dx = -91\int dt$

$\Rightarrow \ln|x| - \ln|x-13| = -91t + c$

$\Rightarrow \ln\left|\frac{x}{x-13}\right| = -91t + c$

$\Rightarrow \frac{x}{x-13} = Ce^{-91t}$, where $C \neq 0$. Then …?

Since $x(0) = 17$, $\frac{17}{17-13} = Ce^{-91\cdot 0} = C$, $\Rightarrow C = \frac{17}{4}$.

So, $\frac{x}{x-13} = \frac{17}{4}e^{-91t}$.

Solve explicitly for x?

$\Rightarrow 4x = 17(x-13)e^{-91t}$

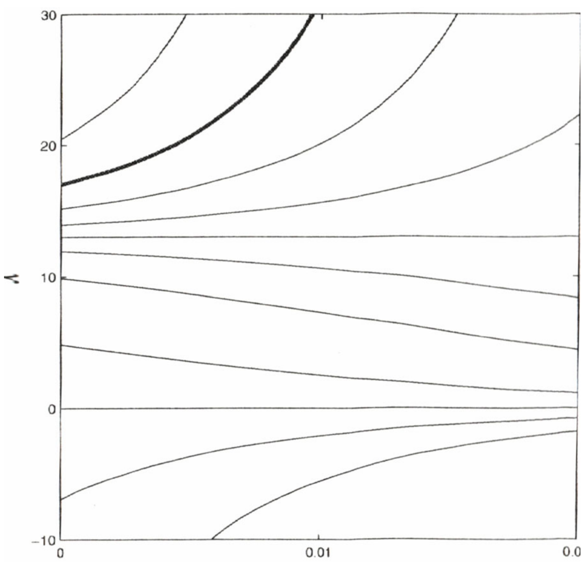
$\Rightarrow 4x = 17xe^{-91t} - 221e^{-91t}$, $\Rightarrow$ $4x - 17xe^{-91t} = -\frac{221}{e^{91t}}$

$\Rightarrow x(4 - 17e^{-91t}) = -\frac{221}{e^{91t}}$, $\Rightarrow$ $x = -\frac{221}{(4-17e^{-91t})e^{91t}}$, (when $4 - 17e^{-91t} \neq 0$, which occurs when $t \neq -\frac{1}{91}\ln\frac{4}{17} \approx 0.016$)

$\Rightarrow x(t) = \frac{221}{17-4e^{91t}}$.

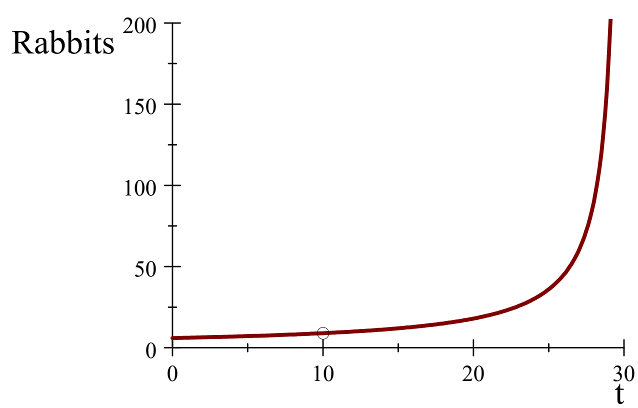
b.) Use either the exact solution or a computer-generated slope field to sketch the graphs of several solutions of the given differential equation, and highlight the indicated particular solution.

Typical solution curves…



Problem: #13 Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$.

(a) Show that $P(t) = \frac{P_0}{1-kP_0t}$, with k constant: Note that as $t \to \frac{1}{kP_0}$ we have $P(t) \to +\infty$. This is doomsday.



Initial population: 6. $k = \frac{1}{180}$

$\beta := c_1 P$ and $\delta := c_2 P$, so $\beta - \delta = (c_1 - c_2)P = kP$.

Using the equation $\frac{dP}{dt} = (\beta - \delta)P$ from the book, and substituting from above gives us: $P' = kP^2$ with k positive.

Using separation of variables: $\frac{1}{P^2}\frac{dP}{dt} = \frac{d}{dt}(-P^{-1})$, so we have:

$$\int \frac{d}{dt}(-P^{-1})dt = k\int dt \quad \Rightarrow \quad -P^{-1} = kt + C$$
$$\Rightarrow \quad P(t) = \frac{1}{C-kt}.$$

The initial condition $P(0) = P_0$ then gives $P_0 = \frac{1}{C}$ or $C = \frac{1}{P_0}$. So $P(t) = \frac{1}{\frac{1}{P_0}-kt} = \frac{P_0}{1-kP_0 t}$.

(**b**) **Suppose that** $P_0 = 6$ **and that there are nine rabbits after ten months**. **When does doomsday occur**?

If $P_0 = 6$, then $P(t) = \frac{6}{1-6kt}$.

The fact that $P(10) = 9$ implies that $9 = \frac{6}{1-60k}$, or $k = \frac{1}{180}$.

So $P(t) = \frac{6}{1-\frac{t}{30}} = \frac{180}{30-t}$. Hence, it is clear that $P \to \infty$ as $t \to 30$ months (doomsday of infinite bunnies).

Problem: **#22** Logistic Equation, $\frac{dP}{dt} = kP(M-P)$.

Suppose that at time $t = 0$, half of a "**logistic**" **population** of $100{,}000$ persons have heard a certain rumor, and that the number of those who have heard it is then increasing at the rate of $1{,}000$ persons per day. How long will it take for this rumor to spread to 80% of the population?

Logistic means we have: $P' = kP(M - P)$. We'll work in thousands of persons.

What does P stand for? What is the carrying capacity?

The word problem indicates: $P(0) = 50$ and $P'(0) = 1$.

So $M = 100$ and $P' = kP(100 - P)$

Substituting $P'(0) = 1$:

$1 = P'(0) = kP(0)(100 - P(0))$

$= 50k(100 - 50) = 2,500k$

$1 = 2,500k$ or $k = 0.0004$ and $P' = (0.0004)P(100 - P)$.

"**How long will it take for this rumor to spread to** 80% **of the population**?"

If t denotes the number of days until 80,000 people have heard the rumor, then Equation 7 in the text ($P(t) = \frac{MP(0)}{P(0)+(M-P(0))e^{-kMt}}$) gives…

$80 = \frac{100 \cdot 50}{50+(100-50)e^{-0.04t}} = \frac{5000}{50+50e^{-0.04t}} \quad \Rightarrow \quad 50 + 50e^{-0.04t} = \frac{5000}{80} = 62.5$

$\Rightarrow e^{-0.04t} = \frac{12.5}{50} = \frac{1}{4}, \quad \Rightarrow \quad -0.04t = \ln\frac{1}{4} = -\ln 4$

$\Rightarrow t = \frac{\ln 4}{0.04} \approx 34.66$ days.

Thus the rumor will have spread to 80% of the population in a little less than 35 days.

Problem: #34 If $P(t)$ satisfies the logistic equation ($P' = kP(M-P)$), use the chain rule to show that…

$P''(t) = 2k^2P(P - \frac{1}{2}M)(P-M)$.

Differentiation of both sides of the logistic equation $P' = kP \cdot (M-P)$ yields

$$P'' = \frac{dP'}{dP} \cdot \frac{dP}{dt} = [k(M-P) + kP(-1)] \cdot kP(M-P)$$

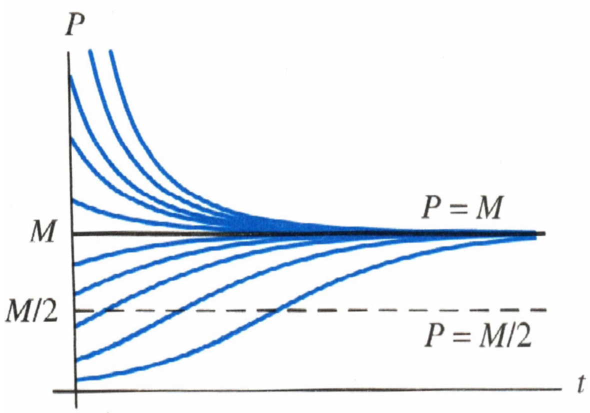
$$= k[M-P-P] \cdot kP(M-P) = k^2P(M-2P)(M-P)$$

$$= 2k^2P(P - \tfrac{1}{2}M)(P-M) \text{ as desired.}$$

The question continues…"**Conclude that**:

- $P'' > 0$ if $0 < P < \frac{1}{2}M$
- $P'' = 0$ if $P = \frac{1}{2}M$
- $P'' < 0$ if $\frac{1}{2}M < P < M$
- $P'' > 0$ if $P > M$

In particular, it follows that any solution curve that crosses the line $P = \frac{1}{2}M$ has an inflection point where it crosses that line, and therefore resembles one of the lower S shaped curves in the graph of the logistic equation below…"



Problem: **Derive the solution**: $\frac{MP_0}{P_0+(M-P_0)e^{-kMt}}$ **of the logistic initial value problem** $P' = kP(M-P),\ P(0) = P_0$. **Make it clear how your derivation depends on whether** $0 < P_0 < M$ **or** $P_0 > M$.

$$\frac{1}{P(M-P)}\frac{dP}{dt} = k \quad \Rightarrow \quad \int \frac{1}{P(M-P)}dP = \int (k)dt$$

I would prefer to integrate something like: $\frac{a}{P} + \frac{b}{M-P}$, for some $a, b \in 5$.

Solving for $\frac{1}{P(M-P)} = \frac{a}{P} + \frac{b}{M-P} \quad \Rightarrow \quad 1 = a(M-P) + bP$

$\Rightarrow \quad 1 = (aM - aP) + bP = aM + (b-a)P$

Therefore, $a = \frac{1}{M}$ and $b - a = 0 \quad \Rightarrow \quad b = \frac{1}{M}$.

So: $\frac{1}{P(M-P)} = \frac{1}{M}(\frac{1}{P} + \frac{1}{M-P})$.

$\Rightarrow \quad \int \frac{1}{M}(\frac{1}{P} + \frac{1}{M-P})dP = \int(k)dt \qquad$ or $\qquad \int(\frac{1}{P} + \frac{1}{M-P})dP = \int(Mk)dt,$

The left-hand side becomes: $\ln P - \ln|M-P| = \ln \frac{P}{|M-P|}$.

So we have: $\ln \frac{P}{|M-P|} = kMt + c \quad \Rightarrow \quad \frac{P}{|M-P|} = Ce^{kMt}. \qquad (**)$

Case: If $P_0 < M$ then $P < M$ and $|M-P| = M-P$.

Using initial conditions $t = 0,$ and $P(0) = P_0$ in $(**)$ gives $C = \frac{P_0}{M-P_0}$.

It follows that $\frac{P}{M-P} = \frac{P_0}{M-P_0}e^{kMt}$.

Case: But if $P_0 > M$ then $P > M$ and $|M-P| = P-M$, so substitution of $t = 0,\ P(0) = P_0$ in $(**)$ gives $C = \frac{P_0}{P_0-M}$,

and it follows that $\frac{P}{P-M} = \frac{P_0}{P_0-M}e^{kMt}$.

We see that the preceding two equations are equivalent (multiply both sides of either equation by -1).

So, using either yields:

$(M-P_0)P = (M-P)P_0e^{kMt} = MP_0e^{kMt} - P \cdot P_0e^{kMt}$

$\Rightarrow \quad P((M-P_0) + P_0e^{kMt}) = MP_0e^{kMt}$

$\Rightarrow P(t) = \frac{MP_0e^{kMt}}{(M-P_0)+P_0e^{kMt}} = \frac{MP_0}{P_0+(M-P_0)e^{-kMt}}. \qquad \blacksquare$