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MATH 2243: **Linear Algebra** & **Differential Equations**

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3.1 (and 3.2): Intro to Linear Systems

Say you have the following system of equations:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \\ a_{41}x + a_{42}y + a_{43}z &= b_4 \end{aligned} \qquad \text{For example:} \qquad \begin{aligned} 1x + 5y + 8z &= 0 \\ 2x + 6y + 9z &= 2 \\ 3x + 7y + 11z &= 4 \\ 6x + 14y + 22z &= 8 \end{aligned}$$

Solve it by putting it into a matrix, then transforming it into an **Echelon Form** Matrix by creating pivot columns (the columns with leading 1s):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & | & b_1 \\ a_{21} & a_{22} & a_{23} & | & b_2 \\ a_{31} & a_{32} & a_{33} & | & b_3 \\ a_{41} & a_{42} & a_{43} & | & b_4 \end{bmatrix}, \text{ transforms into: } \begin{bmatrix} 1 & a'_{12} & a'_{13} & | & b'_1 \\ 0 & 1 & a'_{23} & | & b'_2 \\ 0 & 0 & 1 & | & b'_3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Continuing our example:
$$\begin{bmatrix} 1 & 5 & 8 & | & 0 \\ 2 & 6 & 9 & | & 2 \\ 3 & 7 & 11 & | & 4 \\ 6 & 14 & 22 & | & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 8 & | & 0 \\ 0 & 1 & \frac{7}{4} & | & -\frac{1}{2} \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}.$$

Method of Elimination

Know how to do matrix row manipulations!

- ♦ Multiply one row by a nonzero constant.
- ♦ Interchange two rows.
- ♦ Add a constant multiple of one row to another row.

Pro-tips: **Always Be Adding**
AVOID FRACTIONS!!!

When solving a system of linear equations there will be 0, 1, or ∞ solutions.

Problem: **#18** Use the method of elimination to determine whether the given linear system is **consistent** (has at least one solution) or **inconsistent** (has no solution). If the system is consistent, and if its solution is unique, provide it. Otherwise, describe the infinite solution set in terms of an arbitrary parameter *t*.

$$x + 5y + 6z = 3$$

$$5x + 2y - 10z = 1$$
$$8x + 17y + 8z = 5$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 6 & 3 \\ 5 & 2 & -10 & 1 \\ 8 & 17 & 8 & 5 \end{array}\right]$$

$$\Rightarrow R_2 + (-5R_1) \text{ and } R_3 + (-8R_1) \Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 6 & 3 \\ 0 & -23 & -40 & -14 \\ 0 & -23 & -40 & -19 \end{array}\right]$$

$$\Rightarrow R_3 + (-R_2) \Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 6 & 3 \\ 0 & -23 & -40 & -14 \\ 0 & 0 & 0 & -5 \end{array}\right] \qquad !?!?$$

The system is inconsistent, there are no solutions.

Problem: **#21** Use the method of elimination to determine whether the given linear system is **consistent** (has at least one solution) or **inconsistent** (has no solution). If the system is consistent, and if its solution is unique, provide it. Otherwise, describe the infinite solution set in terms of an arbitrary parameter *t*.

$$2x + 2y - 2z = 10$$
$$3x + y + 3z = 11$$
$$5z + 4x + y = 14$$

$$\Rightarrow \tfrac{1}{2}R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 3 & 1 & 3 & 11 \\ 4 & 1 & 5 & 14 \end{array}\right]$$

$$\Rightarrow R_2 + (-3R_1) \text{ and } R_3 + (-4R_1) \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & -2 & 6 & -4 \\ 0 & -3 & 9 & -6 \end{array}\right]$$

$$\Rightarrow -\tfrac{1}{2}R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{array}\right] \Rightarrow R_3 + 3R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -3 & 2 \end{array}\right]$$

Is there a solution? How do we write it?

We can choose for the variable z from the "free column" to be arbitrary: $z \to t$.

It follows that $y = 3t + 2$, and also that…

$x = -y + t + 5 = -(3t + 2) + t + 5 = -2t + 3.$

So the infinite solution set is: $(x, y, z) = \left(-2t + 3,\ 3t + 2,\ t\right)$, for every $t \in \mathbb{R}$.

What does this mean?

Problem: **#28** Given: $y'' - 10y' + 21y = 0$, and $y(x) = Ae^{3x} + Be^{7x}$, determine the constants A and B, so as to find a solution of the differential equation that satisfies the initial conditions: $y(0) = 15,\ y'(0) = 13.$

$y(0) = Ae^{3\cdot 0} + Be^{7\cdot 0} = 15,$ So, $A + B = 15.$

$y' = 3Ae^{3x} + 7Be^{7x}$ and $13 = 3Ae^{3\cdot 0} + 7Be^{7\cdot 0},$ So, $3A + 7B = 13.$

We could use the symbolic substitution method. Or, using our new matrix technique:

$$\left[\begin{array}{cc|c} 1 & 1 & 15 \\ 3 & 7 & 13 \end{array}\right]$$

$$\Rightarrow R_2 - 3R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 15 \\ 0 & 4 & -32 \end{array}\right]$$

$$\Rightarrow \tfrac{1}{4}R_2 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 15 \\ 0 & 1 & -8 \end{array}\right]$$

With $B = -8$ and $A = 15 - B = 15 - (-8) = 23$.

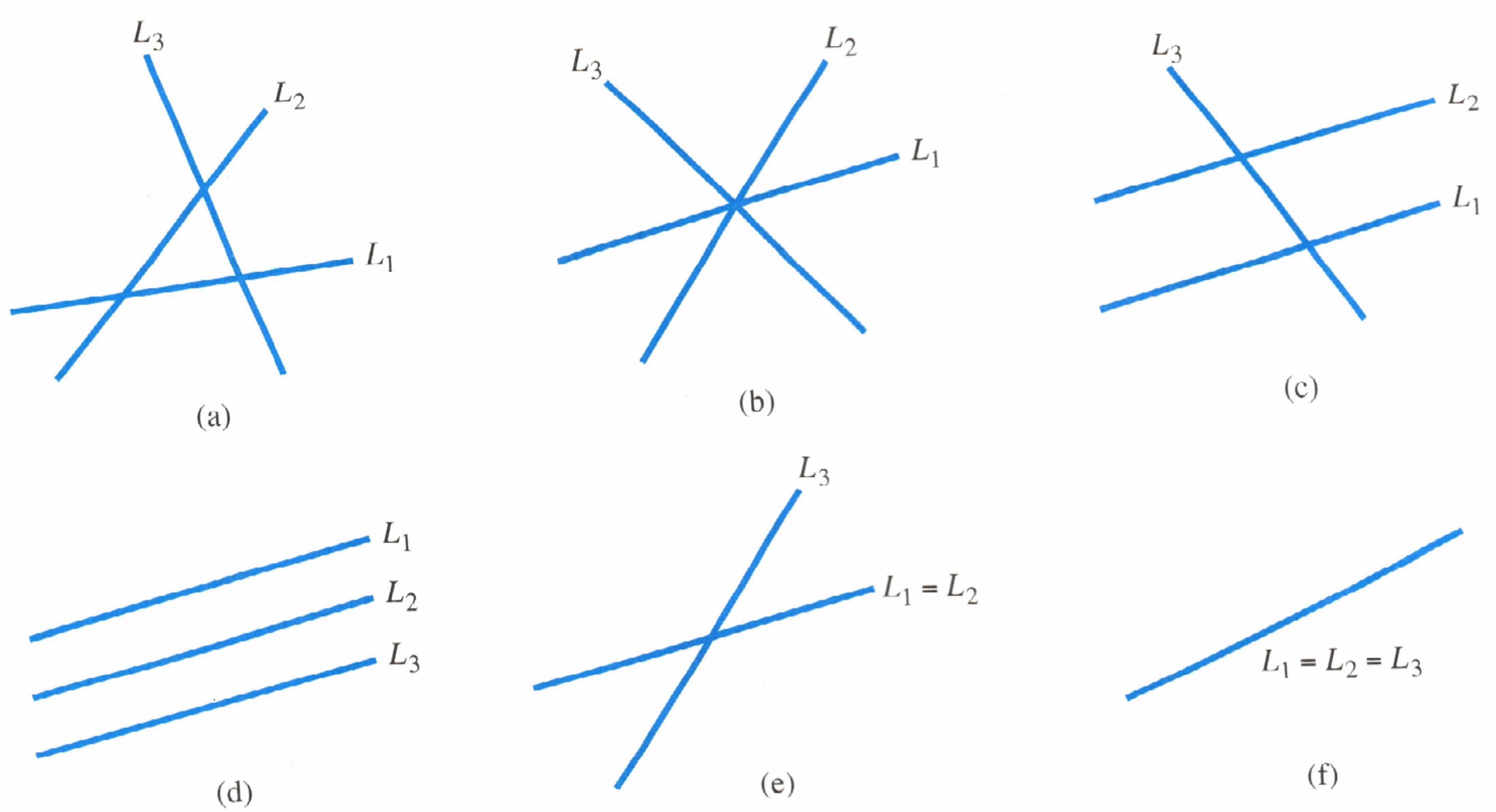
Thus the solution of the differential equation is: $y(x) = 23e^{3x} - 8e^{7x}$.

Problem: **#33**

The linear system:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ a_3x + b_3y &= c_3 \end{aligned}$$

of three equations in two unknowns (x, y) represents three lines L_1, L_2, and L_3 in the xy-plane. The figures below show six possible configurations of these 3 lines. In each case, describe the solution set of the system:



3.2: Matrices and Gaussian Elimination

If you can convert matrix $\mathbf{A}$ into matrix $\mathbf{B}$ through a sequence of elemental row operations, the two matrices are considered "**row equivalent**."

Row equivalent matrices have the **same solution set** (if $\vec{x} = (x, y, z)$ satisfies $\mathbf{A}\vec{x} = \vec{b}$, then it also satisfies $\mathbf{B}\vec{x} = \vec{b}$).

Back Substitution:

- ♦ Set each free variable equal to an arbitrary parameter ($x_4 = s,\ x_3 = t,\ \ldots$),
- ♦ Solve the lowest equation (of the echelon form) for its leading variable,
- ♦ Substitute into the line above, solve for that leading variable,
- ♦ Continue until all variables' values have been determined.

Problem: **#8** This linear system is in **echelon form**. Solve it by back substitution.

$$x_1 - 10x_2 + 3x_3 - 13x_4 = 5$$

$$x_3 + 3x_4 = 10$$

If we set $x_2 = s$ and $x_4 = t$, then the second equation gives …

$x_3 = 10 - 3t$, and the first equation gives …

$$x_1 - 10s + 3(10 - 3t) - 13t = 5.$$

$$x_1 = 10s - 3(10 - 3t) + 13t + 5 = -25 + 10s + 22t.$$

So the infinite solution set is: $(x_1, x_2, x_3, x_4) = \left(-25 + 10s + 22t,\ s,\ 10 - 3t,\ t\right)$, for every $s, t \in \mathbb{R}$.

Problem: **#22** Use elementary row operations to transform the following in a coefficient matrix to **echelon form**. Then solve the system by back substitution.

$$4x_1 - 2x_2 - 3x_3 + x_4 = 3$$

$$2x_1 - 2x_2 - 5x_3 = -10$$

$$4x_1 + x_2 + 2x_3 + x_4 = 17$$

$$3x_1 + x_3 + x_4 = 12$$

Turn the system into: $\left[\begin{array}{cccc|c} 4 & -2 & -3 & 1 & 3 \\ 2 & -2 & -5 & 0 & -10 \\ 4 & 1 & 2 & 1 & 17 \\ 3 & 0 & 1 & 1 & 12 \end{array}\right]$

$$\Rightarrow R_1 + (-R_4) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 2 & -2 & -5 & 0 & -10 \\ 4 & 1 & 2 & 1 & 17 \\ 3 & 0 & 1 & 1 & 12 \end{array}\right] \Rightarrow R_2 + (-2R_1) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 2 & 3 & 0 & 8 \\ 4 & 1 & 2 & 1 & 17 \\ 3 & 0 & 1 & 1 & 12 \end{array}\right]$$

$$\Rightarrow R_3 + (-4R_1) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 2 & 3 & 0 & 8 \\ 0 & 9 & 18 & 1 & 53 \\ 3 & 0 & 1 & 1 & 12 \end{array}\right] \Rightarrow R_4 + (-3R_1) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 2 & 3 & 0 & 8 \\ 0 & 9 & 18 & 1 & 53 \\ 0 & 6 & 13 & 1 & 39 \end{array}\right]$$

$$\Rightarrow R_2 \leftrightarrow R_3 \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 9 & 18 & 1 & 53 \\ 0 & 2 & 3 & 0 & 8 \\ 0 & 6 & 13 & 1 & 39 \end{array}\right] \Rightarrow R_3 + (-4R_3) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 1 & 6 & 1 & 21 \\ 0 & 2 & 3 & 0 & 8 \\ 0 & 6 & 13 & 1 & 39 \end{array}\right]$$

$$\Rightarrow R_3 + (-2R_3) \text{ and } R_4 + (-6R_2) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 1 & 6 & 1 & 21 \\ 0 & 0 & -9 & -2 & -34 \\ 0 & 0 & -23 & -5 & -87 \end{array}\right]$$

$$\Rightarrow -\tfrac{1}{9}R_3 \text{ and } R_4 + 23R_3 \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 1 & 6 & 1 & 21 \\ 0 & 0 & 1 & \frac{2}{9} & \frac{34}{9} \\ 0 & 0 & 0 & -5 + \frac{46}{9} & -87 + \frac{23 \cdot 34}{9} \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 1 & 6 & 1 & 21 \\ 0 & 0 & 1 & \frac{2}{9} & \frac{34}{9} \\ 0 & 0 & 0 & \frac{1}{9} & -\frac{1}{9} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 1 & 6 & 1 & 21 \\ 0 & 0 & 1 & \frac{2}{9} & \frac{34}{9} \\ 0 & 0 & 0 & 1 & -1 \end{array}\right].$$ **What is next**?

$x_4 = -1$

$x_3 = -\frac{2}{9}x_4 + \frac{34}{9} = \frac{2}{9} + \frac{34}{9} = 4$

$x_2 = -6x_3 - x_4 + 21 = -6(4) - (-1) + 21 = -2$

$x_1 = 2x_2 + 4x_3 - 9 = 2(-2) + 4(4) - 9 = 3.$

The (only) solution is: $\vec{x} = (3, -2, 4, -1)$.

Problem: **#26** Determine for what values of k the following system has

a) a unique solution: *b*) no solution; *c*) infinitely many solutions.

$3x + 2y = 1$

$7x + 5y = k$

$$\begin{bmatrix} 3 & 2 & 1 \\ 7 & 5 & k \end{bmatrix}$$ (Note, I left the " | " column out this time)

$$\Rightarrow R_2 + (-2R_1) \Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & k-2 \end{bmatrix}$$

$$\Rightarrow R_1 \leftrightarrow R_2 \Rightarrow \left[\begin{array}{ccc} 1 & 1 & k-2 \\ 3 & 2 & 1 \end{array}\right]$$

$$\Rightarrow R_2 + (-3R_1) \Rightarrow \left[\begin{array}{ccc} 1 & 1 & k-2 \\ 0 & -1 & -3k+7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc} 1 & 1 & k-2 \\ 0 & 1 & 3k-7 \end{array}\right]$$

Unique solution for any k!

$y = 3k - 7, \quad x = -y + (k-2) = -(3k-7) + k - 2 = -2k + 5,$

$\vec{v} = \begin{bmatrix} -2k+5 & 3k-7 \end{bmatrix}.$

"**For what values of k does the system have**: **a**) **a unique solution**: **b**) **no solution**; **c**) **infinitely many solutions**."

Since no free variables (and no $0 \cdot y = 1$ equations), it follows that the given system has a unique solution for every the value of k.

If our system, at the end, had instead looked like: $\left[\begin{array}{ccc} 1 & 1 & k-2 \\ 0 & 0 & 0 \end{array}\right]$, we would have concluded that

there was an infinite number of solutions.

If our system, at the end, had looked like: $\left[\begin{array}{ccc} 1 & 1 & k-2 \\ 0 & 0 & 3k-7 \end{array}\right]$, we would have concluded that

there were no solutions, except for when $k = \frac{7}{3}$, in which case there would be an infinite number of solutions.