MATH 2243: **Linear Algebra** & **Differential Equations**

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3.5: Inverses of Matrices

Theory: For each $\mathbf{A}^{m\times n}$, the identity matrix $\mathbf{I}^{n\times n}$ behaves like the multiplicative identity 1 in regular algebra ($a \cdot 1 = a$).

Proof: To show this, we first obtain two other results:

Fact 1: We can notate $\mathbf{A}$ as $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ (where we've notated the columns as vectors)

and for any n-vector $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$, we have: $\mathbf{A}\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$.

Let's try it for a small matrix: $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}\vec{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1a_{11} + x_2a_{12} \\ x_1a_{21} + x_2a_{22} \end{bmatrix}$

$= \begin{bmatrix} x_1a_{11} \\ x_1a_{21} \end{bmatrix} + \begin{bmatrix} x_2a_{12} \\ x_2a_{22} \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2$, and similarly for $n > 2$.

Fact 2: for non-square matrices, $\mathbf{A}^{m\times n}$ and $\mathbf{B}^{n\times p} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_p \end{bmatrix}$,

we have: $\mathbf{AB} = \begin{bmatrix} \mathbf{A}\vec{b}_1 & \mathbf{A}\vec{b}_2 & \dots & \mathbf{A}\vec{b}_p \end{bmatrix}$.

Let's try it for small matrices: $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$.

Observe that: $\mathbf{A}\vec{b}_1 = a_{11}b_{11} + a_{12}b_{21}$, $\mathbf{A}\vec{b}_2 = a_{11}b_{12} + a_{12}b_{22}$, and $\mathbf{A}\vec{b}_3 = a_{11}b_{13} + a_{12}b_{23}$.

And, $\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \end{bmatrix}$

$= \begin{bmatrix} \mathbf{A}\vec{b}_1 & \mathbf{A}\vec{b}_2 & \mathbf{A}\vec{b}_3 \end{bmatrix}$, and similarly for larger matrices.

To show $\mathbf{AI} = \mathbf{A}$, note that $\mathbf{I} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_n \end{bmatrix}$, where the $\vec{e}_j$ are the j^{th} standard unit vector:

$$\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ entry.}$$

Now if $\mathbf{A} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$, then by **Fact 1**:

$$\mathbf{A}\vec{e}_j = 0 \cdot \vec{a}_1 + \ldots + 1 \cdot \vec{a}_j + \ldots + 0 \cdot \vec{a}_n = \vec{a}_j.$$

Therefore, from **Fact 2**:

$$\mathbf{AI} = \mathbf{A}\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}\vec{e}_1 & \mathbf{A}\vec{e}_2 & \dots & \mathbf{A}\vec{e}_n \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix} = \mathbf{A}.$$

■

Recall from previous mathematics that if $a \neq 0$,

then there is a number $b = a^{-1} = \frac{1}{a}$ such that $ab = ba = 1$.

We call this number its **inverse**, and we say these nonzero numbers are **invertible**.

Does such a property exist for matrices? Kind of, but we must change the assumption a bit. Instead of $\mathbf{A} \neq 0$, we need something called the **determinant** of $\mathbf{A}$ to be nonzero.

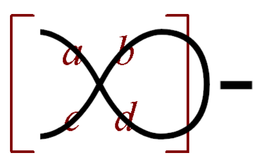
We say $\mathbf{A}$ is **invertible** (also called **non-singular**) if there exists $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

If $\mathbf{B}$ exists, then $\mathbf{B}$ is $\mathbf{A}'s$ *unique* inverse.

2×2 Matrices:

Given: $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$\mathbf{A}$ is invertible if $ad - bc$ (its **determinant**) is nonzero.



In which case (in this 2×2 example) we have **inverse**: $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(One-over-Determinant, Swap, then Signs)

Finding $\mathbf{A}^{-1}$ (with larger matrices)…

Convert $[\,\mathbf{A} \mid \mathbf{I}\,] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array}\right]$ into…

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & a'_{11} & a'_{12} & \dots & a'_{1n} \\ 0 & 1 & 0 & 0 & a'_{21} & a'_{22} & \dots & a'_{2n} \\ 0 & 0 & 1 & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & a'_{n1} & a'_{n2} & \dots & a'_{nn} \end{array}\right] = [\mathbf{I} \mid \mathbf{A}^{-1}],$$ using elementary row operations.

Matrix Exponents

For positive integer n, we define:

$\mathbf{A}^0 := \mathbf{I}, \quad \mathbf{A}^1 := \mathbf{A},$

$\mathbf{A}^{n+1} := \mathbf{A}^n\mathbf{A}$, and $\mathbf{A}^{-n} := (\mathbf{A}^{-1})^n$.

The **laws of exponents** also are consistent with prior experience:

$\mathbf{A}^r\mathbf{A}^s = \mathbf{A}^{r+s}, \quad (\mathbf{A}^r)^s = \mathbf{A}^{rs}.$

Inverse Manipulations

If $\mathbf{A}$ is invertible, then…

- ♦ $\mathbf{A}^{-1}$ (the inverse of $\mathbf{A}$) is also invertible, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

  Proof for $n = 2$:

- ♦ If n is a positive integer, then $\mathbf{A}^n = \mathbf{A}\mathbf{A}\dots\mathbf{A}$ (multiply n times) is also invertible, and $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$.

  Proof for $n = 2$:

If $\mathbf{A}, \mathbf{B}$ are equal size and invertible, then…

- ♦ The product $\mathbf{AB}$ is also invertible, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

  (observe how they switched positions!)

  Proof:

Solving Systems with Matrices

Theorem: If $\mathbf{A}^{n\times n}$ is invertible, then for the n-vector $\vec{b}$,

we have that $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution: $\vec{x} = \mathbf{A}^{-1}\vec{b}$.

Proof: First we show $\vec{x} = \mathbf{A}^{-1}\vec{b}$ is a solution.

Substituting it into the left-hand side of the equation:

$\mathbf{A}\left(\mathbf{A}^{-1}\vec{b}\right) = (\mathbf{A}\mathbf{A}^{-1})\vec{b} = \mathbf{I}\vec{b} = \vec{b}$, so $\vec{x}$ is a solution.

Next, we must show it is the only solution. Here we will use *proof by contradiction*.

Let us assume the opposite of what we believe is true, that is, that there is another *distinct* solution $\vec{x}_1$.

So we have: $\mathbf{A}\vec{x}_1 = \vec{b}$.

Multiplying both sides of this equation by $\mathbf{A}^{-1}$, we have:

$\mathbf{A}^{-1}\mathbf{A}\vec{x}_1 = \mathbf{A}^{-1}\vec{b} \quad \Rightarrow \quad \mathbf{I}\vec{x}_1 = \mathbf{A}^{-1}\vec{b}$

$\Rightarrow \quad \vec{x}_1 = \vec{x} \qquad$ (since $\vec{x} = \mathbf{A}^{-1}\vec{b}$).

But this is a contradiction of our assumption that $\vec{x}_1$ was *distinct* from $\vec{x}$.

Therefore, that assumption must have been incorrect, and in fact there is only the one solution, namely $\vec{x}$.

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Invertible Matrices and Row Operations Theorem: The $n \times n$ matrix $\mathbf{A}$ is invertible if and only if it is row equivalent to the identity matrix $\mathbf{I}$.

Corollary: $\mathbf{A}$ is invertible if and only if $\mathbf{A}\vec{x} = \vec{0}$ has only the trivial solution.

(trivial solution is: $x_1 = x_2 = \ldots = x_n = 0$).

Properties of Non-Singular Square Matrices.

Theorem: The following are equivalent (TFAE):

- ♦ $\mathbf{A}$ is invertible
- ♦ $\mathbf{A}$ is row equivalent to the identity matrix $\mathbf{I}$.
- ♦ $\mathbf{A}\vec{x} = \mathbf{0}$ has only the trivial solution.
- ♦ For every n-vector $\vec{b}$, the system $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution.
- ♦ For every n-vector $\vec{b}$, the system $\mathbf{A}\vec{x} = \vec{b}$ is consistent.

Video Tutorial (visually rich and intuitive): https://youtu.be/kYB8IZa5AuE

Exercises

Problem: #7 Find $\mathbf{A}^{-1}$, then use $\mathbf{A}^{-1}$ to solve the system $\mathbf{A}\vec{x} = \vec{b}$.

$$\mathbf{A} = \begin{bmatrix} 7 & 9 \\ 5 & 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\mathbf{A}^{-1} = -\tfrac{1}{10}\begin{bmatrix} 5 & -9 \\ -5 & 7 \end{bmatrix};$$

$$\vec{x} = -\tfrac{1}{10}\begin{bmatrix} 5 & -9 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \tfrac{1}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Problem: #18 Use the identity matrix to find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{R2+(-3R1)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R3+(-R1)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \overset{R2+(-5R3)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & 2 & 1 & -5 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{R3+(-R2)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & 2 & 1 & -5 \\ 0 & 0 & 5 & -3 & -1 & 6 \end{array}\right] \overset{R2+R3}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 5 & -3 & -1 & 6 \end{array}\right]$$

$$\overset{R1+2R3}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 5 & -3 & -1 & 6 \end{array}\right] \overset{\frac{1}{5}R3}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{5} & -\frac{1}{5} & \frac{6}{5} \end{array}\right]$$

(notice how I am avoiding fractions until the last possible moment)

$$\overset{R1+(-2R2)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{5} & -\frac{1}{5} & \frac{6}{5} \end{array}\right].$$

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & -2 \\ -5 & 0 & 5 \\ -3 & -1 & 6 \end{bmatrix}.$$

Problem: **#26** Find a matrix **X** (matrix FULL of unknowns!!!) such that $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A} = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & -2 \\ 1 & 7 & 2 \end{bmatrix}$,

and $\mathbf{B} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Since $\mathbf{AX} = \mathbf{B}$, If we knew that **A** had an inverse,

we could multiply both sides of the equation (on the left) by $\mathbf{A}^{-1}$ and get:

$(\mathbf{A}^{-1})\mathbf{AX} = (\mathbf{A}^{-1})\mathbf{B}$

The left hand side becomes: $(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$.

So, calculate $\mathbf{A}^{-1} = \begin{bmatrix} -16 & 3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix}$;

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -16 & 3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -21 & 9 & 6 \\ 8 & -3 & -2 \\ -17 & 6 & 5 \end{bmatrix}.$$

Problem: **#35**. Let **A** be an $n \times n$ matrix with either a row or a column consisting only of zeros.
Show that **A** is not invertible.

$$\mathbf{A}\vec{x} = \begin{bmatrix} a'_{11} & a'_{12} & \dots & a'_{1n} \\ 0 & 0 & 0 & 0 \\ a'_{31} & a'_{32} & \dots & a'_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \dots & a'_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}. \qquad ?$$

$\mathbf{A}\vec{x} = \mathbf{0}$ gives us $n-1$ equations (observe that the second equation in the system is not an information giving equation) in n unknowns $\{x_1, \dots, x_n\}$, so infinitely many solutions. However, recall from above that for invertible matrices: "$\mathbf{A}\vec{x} = \mathbf{0}$ has only the trivial solution."

How about:

$$\mathbf{A}\vec{x} = \begin{bmatrix} a_{11} & 0 & a_{13} & \dots & a_{1n} \\ a_{21} & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}. \qquad ?$$

Recall from above, that for invertible matrices, we had: "$\mathbf{A}\vec{x} = \vec{0}$ has only the trivial solution."

However, observe that:

$$\mathbf{A}\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$$
$$= x_1\vec{a}_1 + x_2\vec{0} + \ldots + x_n\vec{a}_n = \vec{0}.$$

But this means that x_2 can be anything and still satisfy the above equation, not just zero.

So, the trivial solution is not the only one.

(actually, infinately many more)