MATH 2243: **Linear Algebra** & **Differential Equations**

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Big idea: Relationship between # of irredundant equations, # of unknowns (columns), and # of linearly independent solutions of homogeneous systems.

4.5: Row and Column Spaces

Gaussian reduction of homogeneous systems reveals redundant equations.

$$\begin{aligned} x - 2y + 2z &= 0 \\ x + 4y + 3z &= 0 \\ 2x + 2y + 5z &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 5 \end{bmatrix} \quad \overset{\text{Add } R_1 \text{ and } R_2 \text{ to } R_3}{\Rightarrow} \quad \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x - 2y + 2z &= 0 \\ x + 4y + 3z &= 0 \end{aligned}$$

What is the domain and codomain of a matrix $\mathbf{A}^{m \times n}$, when thought of as an operator?

$\mathbb{R}^n$ is the domain, and $\mathbb{R}^m$ is the codomain of $\mathbf{A}^{m \times n}$.

Row Space and Row Rank

Row Vectors of A: Given $\mathbf{A}^{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$, the row vectors are $\vec{r}_1 = (a_{11}, a_{12})$, $\vec{r}_2 = (a_{21}, a_{22})$, and $\vec{r}_3 = (a_{31}, a_{32})$, which exist in $\mathbb{R}^2$ (the domain of **A**).

The subspace of $\mathbb{R}^2$ spanned by $\{\vec{r}_1, \vec{r}_2, \vec{r}_3\}$ is called the **row space** of the matrix **A** or $\mathbf{Row}(\mathbf{A})$.

The dimension of the row space $\dim(Row(\mathbf{A}))$ is called the **row rank** of the matrix **A**.

The solution subspace for a system is contained in the same vector space (the domain of **A**) as contains the row space.

Given any **A**, transform to echelon (**A** → **E**), and we have:

Row Space of an Echelon Matrix Theorem: The non-zero row vectors of an echelon matrix **E** are linearly independent and therefore form a basis of the row space of **E**.

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 5 \end{bmatrix} \quad \Rightarrow \quad \mathbf{E} = \begin{bmatrix} 3 & 0 & 7 \\ 0 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Proof: Let the non-zero rows of **E** be of the form:

$$\vec{r}_1 = \begin{bmatrix} e_{11} & \ldots & e_{1p} & \ldots & e_{1q} & \ldots \end{bmatrix},$$
$$\vec{r}_2 = \begin{bmatrix} 0 & \ldots & e_{2p} & \ldots & e_{2q} & \ldots \end{bmatrix},$$

$$\vec{r}_3 = \begin{bmatrix} 0 & \dots & 0 & \dots & e_{3q} & \dots \end{bmatrix}.$$

We need to show that $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are linearly independent.

Therefore, the equation $c_1\vec{r}_1 + c_2\vec{r}_2 + \ldots + c_k\vec{r}_k = \vec{0}$ must imply $c_i = 0$ for all i.

But if we look at this equation component-wise, we find:

$c_1e_{11} = 0, \quad c_1e_{1p} + c_2e_{2p} = 0, \quad c_1e_{1q} + c_2e_{2q} + c_3e_{3q} = 0,$ etc.

From the first equation, we conclude $c_1 = 0$. Substituting this into the second equation, we conclude $c_2 = 0$.

Continuing this way, we see that $c_i = 0$ for all i, and the row vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are linearly independent. ■

Row Space of Equivalent Matrices Theorem: If two matrices **A** and **B** are (row) equivalent, then they have the same row space.

Proof: Because **A** becomes **B** by row operations, it follows that each row vector of **B** is a linear combination of the row vectors of **A**.

This further implies that each vector in Row(**B**) is also a linear combination of the row vectors of **A**.

Therefore, Row(**A**) contains Row(**B**).

Now recall that row operations are reversible. So, **B** can be transformed into **A** with row operations.

So similar to above, we can conclude that Row(**A**) contains Row(**B**).

But these two statements can only be true if Row(**A**) = Row(**B**). ■

Using the previous two theorems, we have:

Algorithm - Basis for the Row Space: To find a basis for the row space $Row(\mathbf{A})$, use elementary row operations to reduce **A** to an echelon matrix **E**. Then the non-zero row vectors of **E** form a basis for $Row(\mathbf{A})$.

Column Space and Column Rank

Column Vectors of $\mathbf{A}^{m \times n}$: Given: $\mathbf{A}^{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$, the column vectors of **A** are the vectors $\vec{c}_1 = (a_{11}, a_{21}, a_{31})$, and $\vec{c}_2 = (a_{12}, a_{22}, a_{32})$ existing in $\mathbb{R}^3$ (the co-domain of **A**).

The subspace of $\mathbb{R}^3$ spanned by $\{\vec{c}_1, \vec{c}_2\}$ is called the **column space** of the matrix **A** or $Col(\mathbf{A})$.

The dimension of the column space $\dim(Col(\mathbf{A}))$ is called the **column rank** of the matrix **A**.

The range of a matrix **A** is contained in the same vector space (the co-domain of **A**) as contains the column space.

After transforming a matrix **A** into an echelon matrix **E**, the columns containing the leading entries are called the **pivot columns** of **E**.

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 5 \end{bmatrix} \quad \Rightarrow \quad \mathbf{E} = \begin{bmatrix} \mathbf{3} & \mathbf{0} & 7 \\ \mathbf{0} & \mathbf{6} & 1 \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}$$

Basis for the Column Space Algorithm: To find a basis for the column space of a matrix $\mathbf{A}$, use elementary row operations to reduce **A** to an echelon matrix **E**. Then the column vectors of **A** (NOT **E** !!!) that correspond to the pivot columns of **E** form a basis for $Col(\mathbf{A})$. (Proof is in book)

We can conclude from above that the column vectors in **A** that do not correspond to the pivot columns in **E** are linear combinations of the pivot columns.

Rank and Dimension

Equality of Row/Column Rank Theorem: The row rank and column rank of any matrix are equal.

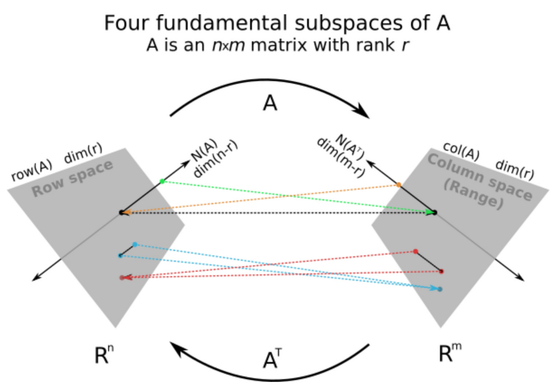
So instead of the row rank or column rank of a matrix, we usually just refer to the **rank of a matrix**.

To solve linear systems (homogeneous, or not), we will first need to solve the associated homogeneous equation. Therefore, the subspace of these solutions is of particular interest, and is called the *null* of **A** or $Null(\mathbf{A})$.

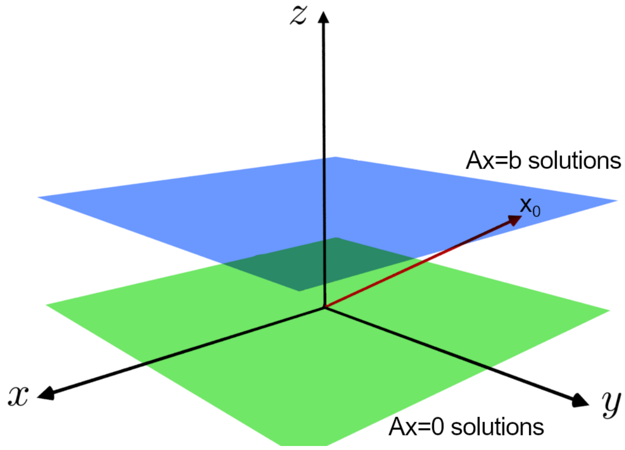
Null Space of A: The solution space of the homogeneous system $\mathbf{A}\vec{x} = \vec{0}$ is called the null of $\mathbf{A}$, denoted by $Null(\mathbf{A})$.

For $\mathbf{A}^{m \times n}$, we have: $rank(\mathbf{A}) + \dim(Null(\mathbf{A})) = n$.

(# of irredundant eqs) + (# of linearly independent sols) = (# of unknowns (columns)) = dim(domain)



Non-Homogeneous Linear Systems



If we can find a particular solution $\vec{x}_0$ of the non-homogeneous system $\mathbf{A}\vec{x} = \vec{b}$, then we can solve the system by first solving the *homogeneous* system $\mathbf{A}\vec{x} = \vec{0}$, where we find solutions $\vec{x}_h := c_1\vec{x}_1 + \ldots + c_r\vec{x}_r$, with basis $\{\vec{x}_1, \ldots, \vec{x}_r\}$.

Then the general solution to the original *non-homogeneous* system is: $\vec{x} = c_1\vec{x}_1 + \ldots + c_r\vec{x}_r + \vec{x}_0 + \quad = \vec{x}_h + \vec{x}_0$.

To make sense of this, let's restrict ourselves to $\mathbb{R}^3$. Imagine our solution space of the homogeneous system to be a subspace of $\mathbb{R}^3$, maybe a plane (intersecting the origin since we have a homogeneous equation). So when $c_1 = \ldots c_r = 0$, we have $\vec{x} = \vec{0}$, a solution to $\mathbf{A}\vec{x} = \vec{0}$.

However, for this plane to be situated correctly to be the solution for $\mathbf{A}\vec{x}_0 = \vec{b}$, we move (translate) this plane so that when $c_1 = \ldots c_r = 0$, we have $\mathbf{A}\vec{x}_0 = \vec{b}$. To ensure our subspace (plane) includes $\vec{x}_0$, we can simply add $\vec{x}_0$ to our homogeneous solution, as this will move the $\vec{0}$ solution to $\vec{x}_0$. This has the effect of moving the plane in $\mathbb{R}^3$ away from the origin, and to the proper location intersecting $\vec{x}_0$.

In particular, imagine we have found the homogeneous solutions to be $\vec{x}_h$, and we have a particular solution $\vec{x}_0$. We are asserting that all the solutions to the nonhomogeneous system are in $\vec{x}_0 + \vec{x}_h$. To see this is true, we multiply $\vec{x}_0 + \vec{x}_h$ by $\mathbf{A}$, and find: $\mathbf{A}(\vec{x}_0 + \vec{x}_h) = \mathbf{A}\vec{x}_0 + \mathbf{A}\vec{x}_h$. But we know that $\mathbf{A}\vec{x}_h = 0$, and we were given that $\mathbf{A}\vec{x}_0 = \vec{b}$, so we have $\mathbf{A}(\vec{x}_0 + \vec{x}_h) = \vec{b}$ for all of the linear combinations in $\vec{x}_h$.

Exercises

Problem 8: Find both a basis for the row space and also a basis for the column space of:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -3 & -5 \\ 1 & 4 & 9 & 2 \\ 1 & 3 & 7 & 1 \\ 2 & 2 & 6 & -3 \end{bmatrix}.$$

$$\Rightarrow \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The row basis is the first three row vectors of **E**. $Row(\mathbf{A}) = span\{r_1, r_2, r_3\}$

The column basis is the first, second, and fourth column vectors of **A**. $Col(\mathbf{A}) = span\{c_1, c_2, c_4\}$

Problem 15: Let $\vec{v}_1 = (3,2,2,2)$, $\vec{v}_2 = (2,1,2,1)$, $\vec{v}_3 = (4,3,2,3)$, and $\vec{v}_4 = (1,2,3,4)$. Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. Find a subset of S that forms a basis for the subspace of $\mathbb{R}^4$ spanned by S.

$$\text{Define } \mathbf{A} := \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix}.$$

Calculating the echelon matrix, we get:

$$\Rightarrow \quad \mathbf{E} = \begin{bmatrix} 3 & 2 & 4 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Linearly independent: $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_4$.

Problem 18: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$. Then a basis T for $\mathbb{R}^n$ that contains S can be found by applying the method of Example 5 in the book to the vectors $\vec{v}_1, \ldots, \vec{v}_k, \vec{e}_1, \ldots, \vec{e}_n$.
Using this technique, find a basis T for $\mathbb{R}^3$ that contains the vectors $\vec{v}_1 = (3,2,-1)$ and $\vec{v}_2 = (2,-2,1)$.

Calculating the echelon matrix of $\mathbf{A} = \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_k & \vec{e}_1 & \ldots & \vec{e}_n \end{bmatrix}$, we get:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \quad \mathbf{E} = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 0 & 10 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

The basis vectors are $\vec{v}_1$, $\vec{v}_2$, $\vec{e}_2$.