Oral Probability Questions

These are notes made in preparation for oral exams involving the following topics in probability: Random walks, Martingales, and Markov Chains. Textbook used: "Probability: Theory and Examples," Durrett.

Chapter 4

1. Define a Random Walk
   Let $X_1, X_2, \ldots$ be iid taking values in $\mathbb{R}^d$
   and let $S_n = X_1 + \ldots + X_n$. $S_n$ is a random walk.

2. Name a Random Walk Theorem
   - **RW Possibilities on R**: Four possibilities, one w/prob = 1.
     - $S_n = 0 \forall n$, (recurrent)
     - $S_n \to \pm \infty$, (transient)
     - $-\infty = \liminf S_n < \limsup S_n = \infty$ (recurrent)
   - **RW Recurrence on $\mathbb{R}^d$**:
     - $S_n$ recurrent in $d=1$ if $S_n / n \to 0$ in probability. (or SSRW)
     - $S_n$ recurrent in $d=2$ if $S_n / n$ converges in distribution to a non-deg. norm. dist. (or SSRW)
     - $S_n$ transient in $d \geq 3$ if is "truly three-dimensional"
   - **RW Equivalencies Theorem**: Let $\tau_0 = 0$ and $\tau_n = \inf \{ m > \tau_{n-1} : S_m = 0 \}$ be time of nth return to 0. Then, $P(\tau_1 < \infty) = 1 \iff P(S_m = 0 \text{ i.o.}) = 1 \iff \sum_{m=0}^\infty P(S_m = 0) = \infty$.
   - **RW Convergence/Divergence Theorem**: Convergence (divergence) of $\Sigma_n P(|S_n| < \epsilon) \forall \epsilon > 0$ is sufficient to determine transience (recurrence) of $S_n$.

3. Does (a version of 1) always have _________ property (related to 2)?
   - For iid $X_i, X_2, \ldots$, is exchangeable sigma field $\epsilon$ trivial? Yes. By Hewitt Savage 0-1. $P(A) \in \{0,1\}$ for each $A \in \epsilon$
   - Types of sets for RW recurrent values (V)? Empty set, or a closed subgroup of $\mathbb{R}^d$.
   - If V (recurrent values) is a closed subgroup, $V = \phi \iff \phi$={Possible Values}

4. Question that leads to a Counterexample/Example.
   - Are SSRW always recurrent? They are on $d < 3$.
   - Are RW on $\mathbb{R}^d$ always recurrent w/ $d < 3$? No, only w/ SSRW or w/ correct convergence (see above)
   - Will Wald’s theorem hold with a SSRW $S_n = X_1 + \cdots + X_n$, with $X_n \in \{\pm1\}$ starting at $S_0 = 0$, with a stopping time $T$ when $S_T = s \neq 0$? (Wald has $X_i$ as iid w/ $E[|X]| < \infty$ and $E[X] < \infty$)
     Note that for any SSRW, that the time $T$ to any position $S_T = s$ is finite, with probability one. However, the expected time is infinite. Therefore, it does not satisfy one of Wald’s Theorem's assumptions.
     **Proof by Contradiction**: Having conditioned on $C = \{ S_T = X_1 + \cdots + X_T = s \}$, then the conditioned expectation $E(X_T + \cdots + X_T | C) = s$ is evident; furthermore, since $X_n = \pm 1$ for all $n$ with equal probability, we easily see that $E[X] = 0$. Under these observations, assuming Wald’s Identity ($E[S_T] = \mu \cdot T$), we obtain an immediate contradiction ($s \neq 0 \cdot T$).
If $S, T$ are stopping times, then is it necessary that $(S - T)$ is a stopping time?

$S - T$ is not necessarily a stopping time. For a counterexample, consider the simple random walk $(X_n)$ on $\{\ldots, -1, 0, 1, \ldots\}$ starting at $X_0 = 0$, and let $S := \inf\{n: X_n = 1\}$ and $T := 1$. Note that \(\{S - T = 1\} = \{S = 2\}\) which is not $X_1$-measurable.

Examples of stopping times

- To illustrate some examples of random times that are stopping rules and some that are not, consider a gambler playing roulette with a typical house edge, starting with $100$ and betting $1$ on red in each game:
- Playing exactly five games corresponds to the stopping time $\tau = 5$, and is a stopping rule.
- Playing until he either runs out of money or has played 500 games is a stopping rule.
- Playing until he is the maximum amount ahead he will ever be is not a stopping rule and does not provide a stopping time, as it requires information about the future as well as the present and past.
- Playing until he doubles his money (borrowing if necessary) is not a stopping rule, as there is a positive probability that he will never double his money.
- Playing until he either doubles his money or runs out of money is a stopping rule, even though there is potentially no limit to the number of games he plays, since the probability that he stops in a finite time is $1$.

1. **Define: Stopping Time**

   $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$ a filtered prob space.

   Stopping time $T : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ is r.v. s.t. $\{T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$.

2. **Name a Stopping Time Theorem**

   - **Wald's Identity:** Let $X_0, X_1, \ldots$ be iid w/ $\mu := E[X_0] < \infty$. Set $X_0$ and let $S_i = X_i + \ldots + X_n$, and $T$ be stopping time w/ $E[T] < \infty$. Then, $E[S_T] = \mu E[T]$.
   - If $S, T, T_n$ are stopping times on $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$. Then so are:
     - $S + T$, \quad $S \wedge T := \min(S, T)$, \quad $S \vee T := \max(S, T)$
     - $\liminf_{n} T_n$ and $\inf_{n} T_n$, \quad $\limsup_{n} T_n$ and $\sup_{n} T_n$

3. **Does (a version of 1) always have __________ property (related to 2)?**
   - Are constants stopping times? Yes.

4. **Question that leads to a Counterexample/Example.**

   If stopping time $T$ and $\mathcal{F}_T$, and $X_0, X_1, X_2, \ldots$ iid, is $\{X_{T_n}\}_{n=0}^{\infty}$ independent of $\mathcal{F}_T$ for all $T$? Yes.

   **Examples of Stopping Times:**
   - Constants
   - If $X_n$ is an adapted process, and $A \in \mathcal{F}_T$, the first entry time into $A$ is a stopping time.
Chapter 5

1. Define: Martingale (or sub, or super)
   \( X_n \) on \( (\Omega, F, P, F_n) \), s.t.
   - \( X_n \) is adapted to \( F_n \).
   - \( \mathbb{E}|X_n| < \infty \) for each \( n \).
   - \( \mathbb{E}[X_{n+1}|F_n] = X_n \) a.s. \( \forall n \) (or \( \geq \), or \( \leq \) resp.)

2. Name a Martingale Theorem
   - **Stopping Time (Super)Martingale Prop**: If \( T \) is a stopping time and \( X_n \) is a (super)mart, then \( X_{T \wedge n} \) is a (super)mart.
   - **Submartingale Convergence**: Suppose that \( X_n \) is a sub-martingale with \( \sup_n \mathbb{E}[X_n] < \infty \). Then for some \( X \), we have \( X_n \rightarrow X \) a.s., where \( \mathbb{E}[X] < \infty \).
   - **Martingale Convergence**: If \( X_n \) is a martingale with \( \sup_n \mathbb{E}[X_n] < \infty \), then \( X_n \rightarrow X \) a.s. and \( \mathbb{E}[X] < \infty \).
   - **Nonnegative SuperMartingale Convergence**: If \( X_n \) is a super-martingale with \( X_{n} \geq 0 \), then \( X_n \rightarrow X \) a.s. and \( \mathbb{E}[X] \leq \mathbb{E}[X_0] \).
   - **Galton-Watson**: Let \( \xi_n, i \geq 1, n \geq 0 \) be iid nonnegative integer-valued r.v.s with a common \( \mu := \mathbb{E}[\xi_1] \in (0, \infty) \). Define \( Z_0 = 1 \) and \( Z_{n+1} = \xi_1 + \cdots + \xi_{Z_n} \) if \( Z_n > 0 \); and 0 if \( Z_n = 0 \). Then, \( (Z_n/\mu)^n \) is a martingale with respect to \( F_n = \sigma(\xi_i, n \geq 1, 0 \leq i < n) \).

3. Does (a version of 1) always have _________ property (related to 2)?
   - Do supermartingales always converge a.s.? Not necessarily, it’s guaranteed when \( X_n \) nonnegative.
   - If \( \mu < 1 \), Then \( P(\text{extinction}) = ? \quad P(\text{extinction}) = 1 \).

4. Question that leads to a Counterexample/Example.
   - When \( \mu = 1 \), is \( P(\text{extinction}) \) equal to 1? Only when \( P(\xi_1 = 1) < 1 \).
   - From Durrett Exmpl. 5.2.3: Do nonnegative martingales converge in \( L^1 \)?
   - Not always. Let \( S_n \) be a symmetric simple random walk with \( S_0 = 1 \), i.e., \( S_n = S_{n-1} + \xi_n \) where \( \xi_1, \xi_2, \ldots \) are i.i.d. with \( P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2 \). Let \( N = \inf\{n : S_n = 0\} \) and let \( X_n = S_{N \wedge n} \). Since the martingale property is closed under stopping times, \( X_n \) is a nonnegative martingale. The Nonnegative SuperMartingale Convergence Theorem implies \( X_n \) converges a.s. to a limit \( X < \infty \) that must be \( = 0 \), since convergence to \( k > 0 \) is impossible. (If \( X_n = k > 0 \) then \( X_{n+1} = k \pm 1 \).) Since \( \mathbb{E}X_n = \mathbb{E}X_0 = 1 \) for all \( n \) and \( X_n = 0 \), convergence cannot occur in \( L^1 \). \( \mathbb{E}[X_n - X_0] = \mathbb{E}[X_0] \rightarrow 1 \neq 0 \).
   - Consider the random walk \( S_n = X_1 + \cdots + X_n \) starting at zero with \( X_n \)'s having \( P(X_n = 1) = P(X_n = -1) = \frac{1}{2} \), a martingale. Now if \( T = \inf\{n \geq 0 : S_n = 1\} \). Can we bound \( T \)?
   - No. For any \( n \in \{1, 2, \ldots \} \) we have \( P(S_k \leq 0 \text{ for all } k \leq n) \geq P(X_n = \ldots = X_0 = -1) = 1/2^n \) since \( \{S_k \leq 0 \text{ for all } k \leq n\} \subseteq \{T > n\} \), this implies \( P(T > n) \geq P(S_k \leq 0 \text{ for all } k \leq n) \geq 1/2^n > 0 \). As \( n \in \mathbb{N} \) is arbitrary, this proves that \( T \) is unbounded.
   - **Do all Martingales which converge in probability, also do so in \( L^1 \)?**
   - No. Any martingale which converges almost surely but not in \( L^1 \) does the job (since a.s. conv. implies conv. in prob.); see example 5.2.3 above.
• If \( E(X_{n+1} | X_n) = X_n \) for all \( n \), must \( X_n \) be a martingale (instead of \( E(X_{n+1} | F_n) = X_n \))?

  No. Let \( (Y_i)_{i \in \mathbb{N}} \) be a sequence of iid r.v. such that \( EY_i^2 = 0 \). Fix \( N \in \{1,2,\ldots\} \) and define: \( X_n := \sum_{i=1}^n Y_i \) for all \( n \leq N \), and \( X_n := X_N + Y_1 + Y_2 - X_2 \) for all \( n > N \).

  For \( n \leq N \) and \( n > N + 1 \), the condition \( E(X_n | X_{n-1}) = X_{n-1} \) is obviously satisfied. For \( n = N + 1 \), we have \( E(X_{N+1} | X_N) = X_N + E(Y_1 | X_N) - E(Y_2 | X_N) \). Since \( (Y_i)_{i \in \mathbb{N}} \) is identically distributed and independent, we have \( E(Y_1 | X_N) = E(Y_2 | X_N) \) and therefore \( E(X_{N+1} | X_N) = X_N \). On the other hand,

  \[
  \mathbb{E}(X_{N+1} | F_N) = X_N + 2\mathbb{E}(Y_1 | F_N) - \mathbb{E} \left( Y_1 + Y_2 \bigg| F_N \right) = X_N + 2Y_1 - (Y_1 + Y_2) = X_{N+1} \neq X_N.
  \]

  So, \( X_n \) is not a martingale.

1. Define: Optional Stopping Sigma-Field

   Let \( (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P) \) and \( T \) be stopping time.

   Denote by \( \mathcal{F}_T \), the \( \sigma \)-field of "events which occur prior to time \( T \)."

   In symbols: \( \mathcal{F}_T := \{ A \in \mathcal{F} : A \cap \{ T \leq n \} \in \mathcal{F}_n, \forall n \geq 0 \}. \)

2. Name an Optional Stopping Time Theorem

   Optional Stopping Thm for SubMarts (or mart)

   If \( S,T \) are stopping times w/\( P(S \leq T < \infty) = 1 \), and \( (X_{T\wedge n})_{n \geq 0} \) is UI submart, then \( \mathbb{E}[X_T | \mathcal{F}_S] \leq X_S \) a.s.

   Consequently, \( \mathbb{E}[X_T] \leq \mathbb{E}[X_S] \). (switch to =’s for mart)

3. Does (a version of 1) always have \underline{property} (related to 2)?

   • If \( T \) is a stopping time, then is \( F_T \) a Sigma field? Yes
   
   • If \( X_n \) is UI sub-martingale and \( T \) a stopping time, is \( X_{T\wedge n} \) UI? Yes
   
   • If \( S \leq T \) are stopping times, then is \( F_S \subseteq F_T \)? No, but \( F_S \subseteq F_T \).

4. Question that leads to a Counterexample/Example.

   • If \( T \) is a stopping time, and \( X_n \) adapted, then is \( X_T \in F_T \)? Not necessarily, this is only guaranteed when \( P(T < \infty) = 1 \).

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1. Define: Conditional Expectation

   \( (\Omega, \mathcal{F}, P) \) w/\( X \in L^1, G \subseteq \mathcal{F}, Y := \mathbb{E}[X | G] \) is unique s.t.

   \( Y \) is \( G \)-measurable and \( \mathbb{E}[Y] < \infty \).

   \[
   \mathbb{E}[\mathbb{E}[X | G] 1_A] = \mathbb{E}[Y 1_A] = \mathbb{E}[X 1_A], \quad A \in G
   \]
2. **Name a Conditional Expectation Theorem**
   - **Conditional MCT**: Let $G \subseteq \mathcal{F}$.
     Let $X_n \geq 0$ be integrable r.v.s and $X_n \uparrow X$.
     Then $\mathbb{E}[X_n \mid G] \uparrow \mathbb{E}[X \mid G]$ a.s.
   - **Conditional DCT**: Let $G \subseteq \mathcal{F}$.
     If $X_n \to X$ a.s. and $|X_n| \leq Y$ for some integrable r.v. $Y$.
     Then $\mathbb{E}[X_n \mid G] \to \mathbb{E}[X \mid G]$ a.s.
   - **Conditional Jensen’s**: Let $G \subseteq \mathcal{F}$.
     If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, $\mathbb{E}|X| < \infty$ and $\mathbb{E} \varphi(X) < \infty$.
     Then $\mathbb{E}[\varphi(X) \mid G] \geq \varphi(\mathbb{E}[X \mid G])$ a.s.

3. **Does (a version of 1) always have ______ property (related to 2)**

4. **Question that leads to a Counterexample/Example.**
   - If $X,Y$ are two random variables and $\mathbb{E}(X|Y)=\mathbb{E}(X)$, are $X$ and $Y$ independent?
     Not necessarily. Let $X \in \{-1,0,1\}$, each with probability $\frac{1}{3}$. Let $Y=X^2$. Note that $X$ and $Y$ are not independent. However, observe that $\mathbb{E}(X)Y=0=0$ and $\mathbb{E}(X|Y=1) = \frac{1}{3} \cdot (-1) + \frac{1}{3}(1) = 0$, so $\mathbb{E}(X|Y)=0=\mathbb{E}(X)$ with probability $1$.

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1. **Define: Uniform Integrability**
   Family of r.v.s $(X_{a})_{a \in A}$ is uniformly integrable (UI) if
   $$\sup_{a \in A} \mathbb{E}[|X_a| \mid \{X_a \mid \mathcal{P} \}] \to 0 \text{ as } M \to \infty.$$  
   Remark: Since $\mathbb{E}|X_a| \leq M + \mathbb{E}[|X_a| \mid \{X_a \mid \mathcal{P} \}]$, then UI $\Rightarrow L^1$-bounded uniformly for $(X_a)_{a \in A}$.

2. **Name a UI Theorem**
   - **Sub σ-field UI Lemma**: Let $X \in L^1(\Omega,F,P)$. Then, $\{\mathbb{E}(X \mid G) : G \sigma\text{-field} \subseteq \mathcal{F}\}$ is UI. Used in Levy’s Fwd Law.
   - If $X_n \to X$ in probability, then TFAE:
     - $(X_n)$ is UI.
     - $X_n \to X$ in $L^1$.
     - $\mathbb{E}[X_n - X] \to 0$.
     - $\mathbb{E}[X_n \to \mathbb{E}[X]<\infty$.
   - **Convergence in Prob Corollary**:
     - If $X_n \to X$ in prob. and $(X_n)$ is UI $\iff X_n \to X$ in $L^1$.
     - If $X_n \to X$ in prob and $|X_n| \leq Y$ for some $Y \in L^1$ ($L^1$ bounded), then $X_n \to X$ in $L^1$.
   - **Submartingale Equivalencies Thm**: For a submart $X_n$, TFAE:
     - $(X_n)$ is UI.
     - $X_n$ converges a.s. and in $L^1$.
     - $X_n$ converges in $L^1$.
     - If $X_n$ is a martingale, then $\exists$ integrable r.v. $X$ so that $X_n = \mathbb{E}[X \mid F_n]$.

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3. Does (a version of 1) always have __________ property (related to 2)?
   ● Do UI sub martingales converge almost surely? Yes.

4. Question that leads to a Counterexample/Example.
   ● For a reverse martingale $(X_n,\mathbb{F}_n)$, clearly, $E[X_0]=X_n$, for each $n\in\{1,2,\ldots\}$. Is $E[X_0 | \mathbb{F}_n]$ UI?
     Yes. Proof: Since $(X_n)$ is a martingale, we have: $E[X_0] < \infty$. So by the Subsigma Field UI Lemma, we have $E[X_0 | \mathbb{F}_n]$ is UI.

   Durrett Example 5.5.1. Suppose $X_1,X_2,\ldots$ are UI and $X_n \rightarrow X$ a.s. Need $E(X_n|F)$ converge a.s.?
   No. Let $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ be independent r.v.’s with $P(Y_1 = 1) = 1/n$, $P(Y_n = 0) = 1 - 1/n$, $P(Z_n = n) = 1/n$, $P(Z_n = 0) = 1 - 1/n$. So our counterexample uses $X_n := Y_n Z_n$. Observe that $E(X_n : |X_n| \geq 1) = n/n^2$, so $X_n$ is UI. Also, $P(X_n > 0) = 1/n^2$ so $\Sigma P(X_n > 0) < \infty$, $P((X_n > 0) \ i.o.)=0$, and the Borel-Cantelli lemma implies $X_n \rightarrow 0$ a.s. Let $F = (Y_1, Y_2, \ldots)$. Then, $E(X_n|F) = Y_n E(Z_n|F) = Y_n E(Z_n) = Y_n$. Since $Y_n \rightarrow 0$ in $L^1$ but not a.s., the same is true for $E(X_n|F)$. Since $\Sigma P(Y_n > 1/2) = \Sigma 1/n = \infty$, then, apply Borel-Cantelli.

   ● Does every sequence $X_n$ which converges almost surely, also converge in $L^1$?
     No, take the sequence $n \cdot 1_{[0,1/n]}$, and note that it converges almost surely to zero. Also note that $E[n \cdot 1_{[0,1/n]}] = 1$ for all $n$. So, $\lim E[n \cdot 1_{[0,1/n]} X] = \lim E[n \cdot 1_{[0,1/n]}] = 1 \neq 0$.

   ● For a martingale $X_n$, does UI imply integrability of sup$|X_n|$?
     No, but the counterexamples are not trivial.

   ● Non-trivial martingale which converges almost surely to 0
     Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $E(Y_n)=1$ and $P(Y_n=1)<1$.
     (i) Show that $X_n = \prod_{m\geq n} Y_m$ defines a martingale. (ii) Use an argument by contradiction to show $X_n \rightarrow 0$ a.s.
     (i) is easy to check.
     (ii) Let $X = \lim X_n$. The Hewitt-Savage zero one law says (since $X \in \mathcal{E}$ exchangeable sigma field) that $X$ is almost surely a constant. Also, $X=Y_1 \prod_{m=2} Y_m$ has the same distribution as $Y_1 X$. Since $Y_1$ is not constant a.s., this forces $X \in \{0,\infty\}$, but $X \neq \infty$ since by Fatou and $Y_n$ independence we have: $E(X) = E(\lim X_n) = E(\lim \prod_{m\geq n} Y_m) = \lim E(\prod_{m\geq n} Y_m) = \lim E(Y_n) = 1$. So $X=0$, and $X_n \rightarrow 0$ a.s.

Chapter 6

1. Define: Markov Chain
   An $(\mathbb{F}_n)$-adapted stochastic process $X_n$ taking values in $(S,S)$ is called a Markov chain if it has the Markov Property: $P(X_{n+1} \in B|X_n) = P(X_{n+1} \in B|X_n)$ a.s. for each $B \in S$, $n \geq 0$.

2. Name a Markov Chain Theorem
   ● Decomposition Theorem: Let $R=\{x: P_{xx}=1\}$ be the recurrent states of a Markov chain. $R$ can be written as $\cup_i R_i$, where each $R_i$ is closed and irreducible. [This results shows that for the study of recurrent states we can, without loss of generality, consider a single irreducible closed set.]
   ● For an irreducible and recurrent chain (Corollary 6.46):
     ○ The stat/inv measures are unique up to constant multiples.
     ○ If $\mu$ is a stat/inv measure, then $\mu(x)>0$ for all $x$.
   ● If $p$ is irreducible and has a stationary distribution $\pi$.
     ○ Calculating Stat/Inv Distribution: $\pi(x)=1/E_x[T_x]$.
     ○ Theorem D6.5.7: Any other stationary measure is a multiple of $\pi$.
Theorem 6.70 (Markov Chain Convergence Theorem): Consider an irreducible, aperiodic Markov chain with stationary distribution \( \pi \). Then, \( p^n(x,y) \to \pi(y) \) as \( n \to \infty \), for all \( x,y \in S \).

Theorem 6.62 (Asymptotic Density of Returns): Let \( y \in S \) be recurrent, and \( N_n(y) = \sum_{i=1}^{n} 1_{\{X_i = y\}} \), then \( \lim N_n(y)/n = (1/E_x[T_x])1_{\{T_x = \infty\}} P_x - a.s. \)

3. Does (a version of 1) always have _________ property (related to 2)?

4. Question that leads to a Counterexample/Example.
   - Multivalued Markov Chain: If \( \xi_0, \xi_1, \ldots \) are iid \( H,T \), each with \( p = 1/2 \), then \( X_n := \{\xi_n, \xi_{n+1}\} \) is a Markov chain.
    - (HW 3): If \( \xi_0, \xi_1, \ldots \) are iid \( -1,1 \) with \( p = 1/2 \), and \( S_0 = 0 \), \( S_n := \xi_1 + \xi_2 + \ldots + \xi_n \), and \( X_n = \max \{S_m : 0 \leq m \leq n\} \). Then is \( X_n \) is a Markov chain?
    - No. Observe the sequence \( (X_1, X_2, X_3) = (1,1,1) \). This can occur if \( (S_1, S_2, S_3) = (1,0,1) \), or if \( (S_1, S_2, S_3) = (1,0,-1) \). Therefore, we have: \( P(X_1 = 2 | X_1 = 1, X_2 = 1, X_3 = 1) = (1/2)^3 = 1/8 \). Alternatively, take the sequence \( (X_1, X_2, X_3) = (0,0,1) \), and observe that this only occurs in one way, namely if \( (S_1, S_2, S_3) = (-1,0,1) \). Therefore, \( P(X_1 = 2 | X_1 = 0, X_2 = 0, X_3 = 1) = 1 \). Since the dependence includes more than just the previous value, \( X_n \) is not a Markov chain.

1. Define: Stationary Distribution
   It’s a stationary/invariant measure that is also a probability measure: \( \pi p = \pi \) such that \( \pi(y) = \sum_{x \in S} \pi(x)p(x,y) \), and \( \sum_{x \in S} \pi(x) = 1 \). It represents a possible equilibrium for the chain.

2. Name a Stationary Distribution Theorem
   - If \( p \) is irreducible and has a stationary distribution \( \pi \).
     - Calculating Stat/Inv Distribution: \( \pi(x) = 1/\mathbb{E}_x[T_x] \).
     - Theorem D6.5.7: Any other stationary measure is a multiple of \( \pi \).
   - Recurrence from Positive Stat/Inv Distributions: If \( \pi \) is a stationary/invariant distribution of a Markov chain and \( \pi(x) > 0 \) for some \( x \), then that \( x \) is recurrent.
   - Theorem 6.70 (Markov Chain Convergence Theorem): Consider an irreducible, aperiodic Markov chain with stationary distribution \( \pi \). Then, \( p^n(x,y) \to \pi(y) \) as \( n \to \infty \), for all \( x,y \in S \).

3. Does (a version of 1) always have _________ property (related to 2)?
   - What are sufficient conditions for a Markov chain’s stat/inv measures to be unique up to constant multiples? That it be irreducible and recurrent.
   - What are sufficient conditions for a Markov chain’s stat/inv measure, if it exists, to have the property \( \mu(x) = 0 \) for all \( x \)? That it be irreducible and recurrent.
   - What are sufficient conditions for a Markov chain’s stat/inv distribution, if it exists, to be unique? That it be irreducible and recurrent.
   - Assume a Markov chain is irreducible and recurrent, what are sufficient conditions to allow us to conclude that the stat/inv distribution cannot exist? The stat/inv measure has infinite mass.
   - If \( \pi \) is a stat/inv distribution and \( \pi(x) > 0 \), what do we know about \( x \)? It is recurrent.
   - If you have an irreducible Markov chain, and there is a positive recurrent value, does this imply the existence of a stationary distribution? Yes.
   - If you have an irreducible Markov chain, and every state is positive recurrent, does this imply the existence of a stationary distribution? Yes.
   - If you have an irreducible Markov chain that has a stationary distribution, does this imply the existence of a positive recurrent value? Yes.

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4. Question that leads to a Counterexample/Example.
   - Let $X_n$ be a Markov chain, where $S$ is the state space and $P$ is the transition matrix. Is every closed class recurrent? No, for example a biased random walk on the integers is transient. Finite closed classes, on the other hand, are always recurrent.

1. Define: Markov Chain Recurrence
   A state $y \in S$ is called recurrent if $\rho_{yy}=1$, and is called transient if $\rho_{yy}<1$.

2. Name a Recurrence Theorem
   - Decomposition Theorem: Let $R = \{ x : \rho_{xx}=1 \}$ be the recurrent states of a Markov chain. $R$ can be written as $\bigcup R_i$, where each $R_i$ is closed and irreducible. [This result shows that for the study of recurrent states we can, without loss of generality, consider a single irreducible closed set.]
   - Theorem 6.62 (Asymptotic Density of Returns): Let $y \in S$ be recurrent. Then $\lim_{n \to \infty} N_n(y)/n = (1/E_x[T_y])1_{\{T_y<\infty\}}$, $P_x$-a.s.

3. Does (a version of 1) always have _________ property (related to 2)?

4. Question that leads to an Counterexample/Example.

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1. Define: Markov Chain Irreducibility
   Markov chain is irreducible if it is possible to get to any state from any state. Formally, if its state space is a single communicating class, i.e., $x \leftrightarrow y$ for all $x, y \in S$.

2. Name an Irreducibility Theorem
   - Decomposition Theorem: Let $R = \{ x : \rho_{xx}=1 \}$ be the recurrent states of a Markov chain. $R$ can be written as $\bigcup R_i$, where each $R_i$ is closed and irreducible. [This result shows that for the study of recurrent states we can, without loss of generality, consider a single irreducible closed set.]
   - For an irreducible and recurrent chain (Corollary 6.46):
     - The stat/inv measures are unique up to constant multiples.
     - If $\mu$ is a stat/inv measure, then $\mu(x) > 0$ for all $x$.
   - If $p$ is irreducible and has a stationary distribution $\pi$.
     - Calculating Stat/Inv Distribution: $\pi(x) = 1/E_x[T_y]$.
     - Theorem D6.5.7: Any other stationary measure is a multiple of $\pi$.
   - Theorem 6.70 (Markov Chain Convergence Theorem): Consider an irreducible, aperiodic Markov chain with stationary distribution $\pi$. Then, $p^n(x,y) \to \pi(y)$ as $n \to \infty$, for all $x, y \in S$.

3. Does (a version of 1) always have _________ property (related to 2)?

4. Question that leads to a Counterexample/Example.
   - If an irreducible Markov chain has period 2, then for every state $i \in S$ do we have $(P_i)^2 > 0$? No, consider $P$:
     
     \[
     \begin{bmatrix}
     0 & 1 \\
     1 & 0
     \end{bmatrix}
     \]
   
   Note that $P^2 = I_d$, so period=2 and $x \leftrightarrow y$. So it is irreducible. But, $P_i = 0$, so $(P_i)^2 = 0$. 

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Other Counterexamples/Examples

- **Are Martingales always Markov processes?**
  No, assume that \( Z_i > 2 \) are independent, integrable, nonconstant (say, standard normal distributions), \( \mu=0, \) and \( Z_i \) independent of some \( X_0, \) where \( X_0 := X_i = 1 \) and \( X_i = X_{i-1} + Z_i \) for every \( t \geq 2. \) \( F_n = \sigma\{X_n, \ldots, X_1\}. \)
  Then \( E[X_i \mid F_{i-1}] = E[X_{i-1} + Z_{i} \mid F_{i-1}] = X_{i-1} + X_i \) (the martingale) for every \( t \geq 1 \) (hence, if \( X_0 \) is integrable, \( (X_i)_{i \geq 0} \) is a martingale) but \( (X_i)_{i \geq 0} \) is not a Markov process since the conditional distribution of \( X_i \) on \( F_{i-1} \) does not depend on \( X_{i-1} \) only, but on \( (X_{i-1}, X_{i-2}). \)

- **If \( X_n \) is a homogeneous Markov chain, is it true that \( X_{n+2} \) is also a homogeneous Markov chain?**
  No. Consider the random walk on \( \{ -1, 0, 1, \ldots, 6 \} \) that with probability \( \frac{1}{6} \) each either: stays at its position, goes to the right, or to the left. We consider the particular transition probability:
  \[
p'(0,2) = P(X_{n+2} = 2 \mid X_{n+1} = 0, X_n = 0) = 0, \text{ while } p'(0,2) = P(X_4 = 2 \mid X_3 = 0) > 0.
  \]

- **If \( X_n \in \{-1, 1\}, S_0 = 0, \text{ and } S_n := X_1 + \cdots + X_n. \) Then is \( (\{S_n\}_{n \geq 0}) \) a Markov-chain?**
  Not necessarily. Let \( F_n = \sigma\{X_1, \ldots, X_n\}. \) It is not a markov chain unless \( p = \frac{1}{2} \) (probability of a step to the left), and a counterexample is to take \( n = 1; \) then \( |S_1| = 1 \) but \( P(|S_2| = 1) = p \neq \frac{1}{2} \) if the first step was to \( S_1 = -1, \) but is \( P(|S_2| = 2) = 1 - p \neq \frac{1}{2} \) if the first step was to \( S_1 = 1. \) So, \( P(|S_2| = 2 : F_n) \in \{p, 1-p\} \) is not equal to \( P(|S_2| = 2 : |S_1| = \frac{1}{2}(1-p) + \frac{1}{2}p = \frac{1}{2} \not\in \{p, 1-p\}, \) and \( (\{S_n\}_{n \geq 0}) \) is not a Markov-chain.

- **Does every chain that has a stationary distribution have a limiting distribution? No.**
  Recall that a Markov Chain has a limiting distribution if \( \pi = \lim_{n \to \infty} \pi_n, \forall n \in S, \) exists. In particular, if the limit does not depend on the starting state (and hence distribution) of the chain.
  We know a Markov Chain \( \{X_n\} \) with a stat. distr. \( \mu \) as its initial distribution is a stationary process, because if \( X_0 \sim \mu \) is a stationary distribution, then for each \( n, \ X_n \sim \mu_{X_{n-1}} \sim \mu. \) So, \( (X_0, X_1, \ldots, X_n) \sim (X_0, X_1, \ldots, X_n) \) and \( \mu(0) = \mu(1) = \frac{1}{2}. \) Durrett said a special case to keep in mind for counterexamples is the Markov chain: \( X_t : \Omega \to S = \{0, 1\} \) with transition probability \( p(0,1) = p(1,0) = 1, \) and stationary distribution \( \mu(0) = \mu(1) = \frac{1}{2}. \) Now let \( X_t \in \{0, 1\} \)
  w/probability \( \frac{1}{2} \) (so not starting with the stat. dist.), so \( (X_0, X_1, \ldots) \sim (0, 1, 0, \ldots) \) or \( (1, 0, 1, \ldots) \) with probability \( \frac{1}{2}. \) Note that it does not have a limiting distribution. Durrett is demonstrating that this chain satisfies stationarity, and that it is useful to keep this Markov chain in mind when * picturing what stationarity means. In particular this is a commonly used counterexample to distinguish between stationary distributions, and limiting distributions.
  Regarding the limiting distribution, note that in this case \( \lim_{n \to \infty} p^n_{0,1} = 1 \) and \( \lim_{n \to \infty} p^n_{1,0} = 0, \) so the limit does not exist.
  Any chain that has a limiting distribution necessarily is stationary (since \( \pi \) can be shown to satisfy the stationarity property). The converse however is not true: and this is what the counterexample shows, since the limit above only exists if the chain is started from \( \mu(0) = \mu(1) = 1/2, \) and not from an arbitrary distribution. In general for finite, irreducible Markov chains
  - A stationary distribution always exists.
  - Existence of a limiting distribution implies stationarity.
  - If, in addition to being finite and irreducible, the chain is also aperiodic, then a limiting distribution is guaranteed to exist.