# Theory of Probability Flashcards


## Random Walks

<table>
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<th>Theory</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>Random Walk</td>
<td>Let $X_1, X_2, \ldots$ be iid taking values in $\mathbb{R}^d$ and let $S_n = X_1 + \ldots + X_n$. $S_n$ is a random walk.</td>
</tr>
</tbody>
</table>

| Stopping Time | $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ a filtered prob space. Stopping time $T : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ is r.v. s.t. $\{T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$, or equivalently, $\{T = n\} \in \mathcal{F}_n$ for all $n \geq 0$. |

| Stopping Time Examples | Constant times (e.g., $T = 10$) are always stopping times. $X_n$ an adapted process. Fix $A \in \mathcal{B}_\mathbb{R}$. Then first entry time into $A$, $T_A := \inf\{n \geq 0 : X_n \in A\}$, w/inf $\emptyset := +\infty$ is stopping time |

| Stopping Times Closure Lemma | If $S, T, T_n$ are stopping times on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$. Then so are: $S + T$, $S \land T := \min(S, T)$, $S \lor T := \max(S, T)$, $\liminf_n T_n$ and $\inf_n T_n$, $\limsup_n T_n$ and $\sup_n T_n$ |

| Permutable Event | Given random seq. $S$ and state space $\Omega := \{(\omega_1, \omega_2, \ldots) : \omega_i \in S\}$ Event $A \in \mathcal{F}$ is permutable if $\pi^{-1}A = \{\omega : \pi \omega \in A\} = A$, for any finite permutation $\pi$. $\varepsilon := \{A : A$ is permutable$\}$ |

| Symmetric Function | $f : \mathbb{R}^n \to \mathbb{R}$ is symmetric if $f(x_1, x_2, \ldots, x_n) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for each $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and for each permutation $\pi \in \{1, 2, \ldots, n\}$ |

| Exchangeable $\sigma$-field | $X_1, X_2, \ldots$ r.v.s on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $F_n := \{f : \mathbb{R}^n \to \mathbb{R}$ symmetric m’ble$\}$. Let $\varepsilon_n := \sigma(F_n, X_{n+1}, X_{n+2}, \ldots)$. Exchangeable $\sigma$-field $\varepsilon := \cap_{n=1}^\infty \varepsilon_n$. |

| Hewitt Savage 0-1 Law | $\varepsilon$ exchngble $\sigma$-field of iid $X_1, X_2, \ldots$, $\mathcal{F} = \sigma(X_1, X_2 \ldots)$, then $\mathbb{P}(A) \in \{0, 1\}$, $\forall A \in \varepsilon$ |

| Random Walk Possibilities on $\mathbb{R}$ | RWs on $\mathbb{R}$, 4 possibilities, one w/prob = 1. $S_n = 0 \forall n$, $S_n \to \pm\infty$, or $-\infty = \liminf S_n < \limsup S_n = \infty$ |

4/22/2020 Jodin Morey
| RW Conv/Transients Thm | Convergence (divergence) of $\sum_n \mathbb{P}(|S_n| < \varepsilon) \forall \varepsilon > 0$ is sufficient to determine transience (recurrence) of $S_n$ |
|------------------------|----------------------------------------------------------------------------------|
| RW Recurrence on $\mathbb{R}^d$ | $S_n$ recurrent in $d = 1$ if $S_n/n \xrightarrow{p} 0$. (or SSRW)  
$S_n$ recurrent in $d = 2$ if $S_n/\sqrt{n} \Rightarrow$ non-deg. norm. dist. (or SSRW)  
$S_n$ transient in $d \geq 3$ if it is “truly three-dimensional” |
| Recurrence Thm for RWs | $\{\text{recurrent values}\} = \emptyset$ or is closed subgroup of $\mathbb{R}^d$.  
If closed subgroup, then $\{\text{recurrent values}\} = \{\text{possible values}\}$ |
| RW Equivalencies Thm (Hint: Recurrence) | Let $\tau_0 = 0$ and $\tau_n = \inf\{m > \tau_{n-1} : S_m = 0\}$ be time of $n$th return to 0  
$\mathbb{P}(\tau_1 < \infty) = 1 \iff \mathbb{P}(S_m = 0 \text{ i.o.}) = 1 \iff \sum_{m=0}^{\infty} \mathbb{P}(S_m = 0) = \infty$ |
| Wald’s Identity | $\xi_1, \xi_2, \ldots$ be iid w/ $\mu := \mathbb{E}[\xi_n] < \infty$. Set $\xi_0$ and let $S_n = \xi_1 + \ldots + \xi_n$  
Let $T$ be stopping time w/ $\mathbb{E}[T] < \infty$. Then, $\mathbb{E}[S_T] = \mu \mathbb{E}[T]$ |
| Recurrent Value | $x \in S$ is recurrent if, $\forall \varepsilon > 0$, we have $\mathbb{P}(|S_n - x| < \varepsilon \text{ i.o.}) = 1$ |
| Possible Value (of RW) | $S := \{\text{possible values}\}$.  
$x \in S$ if for $\forall \varepsilon > 0$, $\exists n$ such that $\mathbb{P}(|S_n - x| < \varepsilon) > 0$. |
| Transient/Recurrent (RW) | If $\{\text{recurrent values}\} = \emptyset$, RW is transient, otherwise it is recurrent |
# Martingales

| Conditional Expectation | \((\Omega, \mathcal{F}, P) \) w/ \(X \in L^1, \ G \subseteq \mathcal{F}, \ Y:= \mathbb{E}[X|G]\) is unique s.t. 
| | \(Y \) is \(G\)-measurable and \(\mathbb{E}|Y| < \infty.\) 
| | \(\mathbb{E}[\mathbb{E}[X|G]1_A] = \mathbb{E}[Y1_A] = \mathbb{E}[X1_A], \ A \in G\) |
| \(E[X|A], \ where \ A \ is \ an \ event \ is: \) | Expected value of \(X\) given that \(A\) occurs |
| \(E[X|Y], \ where \ Y \ is \ a \ r.v. \ is: \) | \(r.v\) whose value at \(\omega \in \Omega\) is \(\mathbb{E}[X|A]\) where \(A\) is the event \(\{Y = Y(\omega)\}\) |
| \(\mathbb{E}[X1_A] \) is: | The case of \(\mathbb{E}[X|Y]\), for \(r.v. \ Y = 1_A,\) and \(1_A(\omega)\) is 1 if \(\omega \in A\) and 0 otherwise. 
\(\) It’s a \(r.v\) that returns \(\mathbb{E}[X|A]\) if \(\omega \in A\) and \(\mathbb{E}[X|A^c]\) if \(\omega \notin A\) |

## Absolute Continuity

Let \(\nu\) and \(\mu\) be \(\sigma\)-finite measures on \((\Omega, \mathcal{F})\). 
\(\nu << \mu, \) means that \(\mu(A) = 0 \Rightarrow \nu(A) = 0, \) for each \(A \in \mathcal{F}\)

## Radon-Nikodym Lemma

Let \(\nu\) and \(\mu\) be two \(\sigma\)-finite measures on \((\Omega, \mathcal{F})\). \(\nu << \mu \Leftrightarrow \exists \mathcal{F}\)-measurable \(f : \Omega \rightarrow [0, \infty)\) s.t. \(\nu(B) = \int_B f d\mu, \ \forall B \in \mathcal{F}\)

| If \(X \in G, \ then \ E[X|G] = \) | \(X\) a.s. |
| If \(G = \{\emptyset, \Omega\}, \ then \ E[X|G] = \) | \(\mathbb{E}[X]\) |
| If \(X\) independent of \(G, \ then \ E[X|G] = \) | \(\mathbb{E}[X]\) a.s. To prove this, observe that \(\mathbb{E}[X]\) is \(G\)-measurable and for any \(A \in G\) we have: 
| | \(\mathbb{E}[X1_A] = \mathbb{E}[X]\mathbb{E}[1_A] = \mathbb{E}[\mathbb{E}[X]1_A].\) |
| Pre-Tower Property | If \(\mathcal{F} \subset \mathcal{G}\) and \(\mathbb{E}[X|\mathcal{G}] \in \mathcal{F}, \) then 
| | \(\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X|\mathcal{G}]\) |
| Tower Property | Let $H \subseteq G$ be sub-$\sigma$-fields of $\mathcal{F}$. Then: $\mathbb{E}[\mathbb{E}[X|G]|H] = \mathbb{E}[X|H]$ a.s. |
|----------------|---------------------------------------------------------------|
| Take out what is known | If $X$ is $G$-measurable, then for any r.v. $Y$ s.t. $\mathbb{E}|Y| < \infty$ and $\mathbb{E}|XY| < \infty$, we have: $\mathbb{E}[XY|G] = X\mathbb{E}[Y|G]$ a.s. |
| Conditional MCT | Let $X, X_n \geq 0$ be integrable r.v.s and $X_n \uparrow X$. Then $\mathbb{E}[X_n|G] \uparrow \mathbb{E}[X|G]$ a.s. |
| Conditional Jensen’s Inequality | If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, $\mathbb{E}|X| < \infty$ and $\mathbb{E}|\varphi(X)| < \infty$, then $\mathbb{E}[\varphi(X)|G] \geq \varphi(\mathbb{E}[X|G])$ a.s. |
| \(L^p\) Contraction of Cond. Expectation | For $p \geq 1$, and $G \in \mathcal{F}$, $\mathbb{E}[\mathbb{E}[X|G]|^p] \leq \mathbb{E}[|X|^p]$.
**Proof:** Jensen’s $\Rightarrow \mathbb{E}[|X|G]^p \leq \mathbb{E}[|X|^p : G]$. Now take the expectation of both sides. |
<p>| Conditional Fatou’s Lemma | Let $X_n \geq 0$ be integrable r.v.s. and $\lim \inf_n X_n$ be integrable. Then $\mathbb{E}[\lim \inf_n X_n|G] \leq \lim \inf_n \mathbb{E}[X_n|G]$ a.s. |
| Conditional DCT | If $X_n \to X$ a.s. and $|X_n| \leq Y$ for some integrable r.v. $Y$. Then $\mathbb{E}[X_n|G] \to \mathbb{E}[X|G]$ a.s. |
| Chebyshev’s Conditional Inequality | If $a &gt; 0$, then $\mathbb{P}(|X| \geq a|\mathcal{F}) \leq a^{-2}\mathbb{E}[X^2|\mathcal{F}]$ |
| Martingale | $X_n$ on $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_n)$, s.t. $X_n$ is adapted to $\mathcal{F}<em>n$. $\mathbb{E}|X_n| &lt; \infty$ for each $n$. and, $\mathbb{E}[X</em>{n+1}|\mathcal{F}_n] = X_n$ a.s. $\forall n$. (or $\geq$, or $\leq$ resp.) |
| If $X_n$ is a martingale, then for $n &gt; m$, $\mathbb{E}[X_n|\mathcal{F}_m] =$ |
| If $X_n$ is a martingale wrt $\mathcal{F}_n$ and $\varphi$ is convex, then: |
| (or sub) | If $\mathbb{E}|\varphi(X_n)| &lt; \infty \forall n$, then $\varphi(X_n)$ is a sub-martingale wrt $\mathcal{F}_n$. Consequently, if $p \geq 1$ and $\mathbb{E}|X_n|^p &lt; \infty \forall n$, then $|X_n|^p$ is a sub-martingale wrt $\mathcal{F}_n$. |</p>
<table>
<thead>
<tr>
<th>Predictable Sequence</th>
<th>R.v.s $H_n$ are predictable wrt $\mathcal{F}<em>n$ if it is $\mathcal{F}</em>{n-1}$ measurable for each $n \geq 1$.</th>
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</thead>
<tbody>
<tr>
<td>Doob's Martingale Transform</td>
<td>Let $(X_n)_{n \geq 0}$ be a $(\mathcal{F}<em>n)</em>{n \geq 0}$–martingale, and $H_n$ predictable.\nTransform is: $(H \cdot X)<em>0 = 0$, $(H \cdot X)<em>n = \sum</em>{k=1}^{n} H_k(X_k - X</em>{k-1})$.\nIf $(H \cdot X)_n$ integrable, then $(H \cdot X)_n$ is a martingale.</td>
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<tr>
<td>Doob's Mart Transform Lemma</td>
<td>Assume that $X_n$ is a martingale and $(H \cdot X)_n \in L^1$, $\forall n$.\nThen, $H \cdot X$ is a $(\mathcal{F}<em>n)</em>{n \geq 0}$-martingale.</td>
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<tr>
<td>Doob's Decomp</td>
<td>Submart $X_n$ wrt $\mathcal{F}_n$ can be uniquely written as sum of mart $M_n$ and increasing predictable process $A_n$. Let $D_0 = X_0$, $D_i = X_i - E[X_i</td>
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<tr>
<td>Stopping Time SuperMartingale Prop</td>
<td>If $T$ is a stopping time and $(X_n)<em>{n \geq 0}$ is a supermart\nthen $(X</em>{T/n})_{n \geq 0}$ is a supermart</td>
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<tr>
<td>Stopped Martingale Corollary</td>
<td>If $T$ is a stopping time and $(X_n)<em>{n \geq 0}$ is a martingale\nthen $(X</em>{T/n})_{n \geq 0}$ is a martingal</td>
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<tr>
<td>Let $T$ be a stopping time w/ $E[T] &lt; \infty$, then $E[T] =$</td>
<td>$\sum_{i=1}^{\infty} P(T \geq i)$.</td>
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<tr>
<td>Doob’s Upcrosing Inequality</td>
<td>Let $a &lt; b$, and $U_n[a,b]$ the # of upcrossings from $a \to b$ by $n$.\nIf $X_n$ is submart, then $\mathbb{E}[U_n[a,b]] \leq \frac{\mathbb{E}[(X_n-a^+)] - \mathbb{E}[(X_n-a^+)]}{b-a}$</td>
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<tr>
<td>Martingale Convergence</td>
<td>Suppose that $(X_n)_{n \geq 0}$ is a sub-martingale with $\sup_n \mathbb{E}[X_n^+] &lt; \infty$\nThen for some $X$, we have $X_n \to X$ a.s., where $\mathbb{E}</td>
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<tr>
<td>$L^1$-Bounded Martingale Convergence</td>
<td>If $(X_n)_{n \geq 0}$ is a martingale with $\sup_n \mathbb{E}</td>
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<td>Non-negative Super-Mart Convergence</td>
<td>If $(X_n)_{n \geq 0}$ is a super-martingale with $X_n \geq 0$,\nthen $X_n \to X$ a.s. and $\mathbb{E}[X] \leq \mathbb{E}[X_0]$.</td>
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</tbody>
</table>
Let $\mathcal{F}_n$ be filtration w/ $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $A_n$ events w/ $A_n \in \mathcal{F}_n$. 
Then, $\{A_n \ i.o.\} = \left\{ \sum_{n=1}^{\infty} P(A_n | \mathcal{F}_{n-1}) = \infty \right\}$.
If $A_n = X_n \leq \varepsilon \implies A_n \Rightarrow 0$. If $A_n = X_n > \varepsilon \implies X_n \Rightarrow 0$.

Let $\mu$ be finite, $\nu$ a prob. measure, $\mathcal{F}_n \uparrow \mathcal{F}$ be $\sigma$-fields, 
and $\mu_n, \nu_n$ be restrictions of $\mu, \nu$ to $\mathcal{F}_n$. If $\mu_n \ll \nu_n, \forall n$, 
and we let $X_n = d\mu_n/d\nu_n$. Then, $X_n$ is a martingale wrt $\mathcal{F}_n$.

If $\xi^n_i$ iid nonnegative integer r.v.s w/ $\mu := \mathbb{E}[\xi^n_1] \in (0, \infty)$.
Let $Z_0 \deq 1$ and $Z_{n+1} \deq \sum_{i=1}^{\xi^n_1} \xi^n_i$, if $Z_n > 0$; or 0 otherwise.
Then, $\xi^n_1 \mathcal{F}_{n-1}$ is a mart wrt $\mathcal{F}_n = \sigma(\xi^n_i : i \geq 1, 0 \leq m < n)$.

If $\mu < 1$, then $Z_n = 0 \forall n$ sufficiently large, so $Z_n/\mu^n \to 0$
If $\mu = 1$ and $\mathbb{P}(\xi^n_1 = 1) < 1$, then $Z_n = 0, \forall n$ sufficiently large.
If $\mu > 1$, then $\rho < 1$, that is, $\mathbb{P}(Z_n > 0 \text{ for all } n) > 0$.

If $X_m$ is submart & $T$ is stopping time w/ 
$\mathbb{P}(T \leq k) = 1$, for some $k \in \mathbb{Z}_+$, then $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] \leq \mathbb{E}[X_k]$.
(or $\mathbb{E}[X_0] = \mathbb{E}[X_T] = \mathbb{E}[X_k]$ for mart)

Let $X_m$ be nonnegative submart, $X^*_n = \max_{0 \leq m \leq n} X_m$, $\lambda > 0$,
and $A = \{X^*_n \geq \lambda\}$. Then, $\mathbb{P}(A) \leq \frac{1}{\lambda} \mathbb{E}[X_n \mathbb{1}_A] \leq \frac{1}{\lambda} \mathbb{E}[X_n]$.

Suppose $X_n$ is mart w/ sup $\mathbb{E}[|X_n|^p] < \infty$ for some $p > 1$.
Then, $X_n \Rightarrow X$ a.s. and in $L^p$.

Family of r.v.s $(X_n)_{n \in \Lambda}$ is uniformly integrable (UI) if 
$\sup_{n \in \Lambda} \mathbb{E}[|X_n| 1_{\{|X_n| > M\}}] \to 0$ as $M \to \infty$. Remrk: Since 
$\mathbb{E}|X_n| \leq M + \mathbb{E}[|X_n| 1_{\{|X_n| > M\}}]$, then UI $\Rightarrow L^1$-bounded

Let $X \in L^1(\Omega, \mathcal{G}, \mathbb{P})$.
Then, $\left\{ \mathbb{E}[X|\mathcal{G}] : \mathcal{G} \text{ a } \sigma \text{-field } \subset \mathcal{F} \right\}$ is uniformly integrable.

If $X_n \Rightarrow X$ in probability, then TFAE: 
$\Rightarrow \{X_n : n \geq 0\}$ is uniformly integrable 
$\Rightarrow X_n \Rightarrow X$ in $L^1$ ($\mathbb{E}|X_n - X| \to 0$) 
$\Rightarrow \mathbb{E}|X_n| \Rightarrow \mathbb{E}|X| < \infty$.
[Note: $L^1$ convergence $\Rightarrow$ convergent in probability and UI]
### Martingale Convergence in Probability Corollary

If $X_n \xrightarrow{L^1} X$,

$$(X_n)_{n \geq 0} \text{ is UI } \iff X_n \xrightarrow{L^1} X.$$  

$|X_n| \leq Y$ for some $Y \in L^1$, then $X_n \xrightarrow{L^1} X$.

### Sub-martingale Equivalencies Thm

For a submart $X_n$, TFAE:

- ♦ $(X_n)_{n \geq 0}$ is UI. ♦ $X_n$ converges a.s. and in $L^1$.

- ♦ $X_n$ converges in $L^1$. Also, if $(X_n)_{n \geq 0}$ is a martingale, then

- ♦ $\exists$ integrable r.v. $X$ so that $X_n = \mathbb{E}[X|\mathcal{F}_n]$.

### Levy’s 0-1 Law

Suppose that $\mathcal{F}_n \uparrow \mathcal{F}_\infty := \sigma(\bigcup_n \mathcal{F}_n)$.

and $A \in \mathcal{F}_\infty$, then $\mathbb{E}[1_A|\mathcal{F}_n] \to 1_A$ a.s.

From which you can conclude Kolmogorov’s 0-1.

### Levy’s Forward Law

Suppose that $\mathcal{F}_n \uparrow \mathcal{F}_\infty := \sigma(\bigcup_n \mathcal{F}_n)$.

If $X \in L^1$, then $\mathbb{E}[X|\mathcal{F}_n] \to \mathbb{E}[X|\mathcal{F}_\infty]$ a.s. and in $L^1$.

### Kolmogorov’s 0-1 Law

Let $\xi_1, \xi_2, \ldots$ be independent r.v.s and $\mathcal{F}_n = \sigma(\xi_1, \xi_2, \ldots, \xi_n)$, $\forall n$.

Let $\mathcal{T} = \bigcap_{k=1}^\infty \sigma(\xi_k, \xi_{k+1}, \ldots)$ be tail σ-field.

Then $\forall A \in \mathcal{T}$, $\mathbb{P}(A) \in \{0, 1\}$.

### DCT for Filtered Conditional Expectation

Suppose $Y_n \to Y$ a.s. and $|Y_n| \leq Z$, $\forall n$ where $\mathbb{E}[Z] < \infty$.

If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, then $\mathbb{E}[Y_n|\mathcal{F}_n] \to \mathbb{E}[Y|\mathcal{F}_\infty]$ a.s.

$\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F})$

### Backward Martingale

Let $(\mathcal{F}_{-n})_{n \geq 0}$ be sub-σ-fields, w/ $\ldots \subseteq \mathcal{F}_{-2} \subseteq \mathcal{F}_{-1} \subseteq \mathcal{F}_0$.

- ♦ $X_{-n} \in \mathcal{F}_{-n}$ for each $n \in \mathbb{Z}_+$.

- ♦ $X_{-n} \in L^1$ for each $n \in \mathbb{Z}_+$.

- ♦ $\mathbb{E}[X_{-n}|\mathcal{F}_{-(n+1)}] = X_{-(n+1)}$ for each $n \in \mathbb{Z}_+$.

### Example of UI Martingale

For reverse martingale: clearly, $\mathbb{E}[X_0|\mathcal{F}_{-n}] = X_{-n}$ for each $n \in \mathbb{Z}_+$.

Hence, if $(X_{-n})_{n \in \mathbb{Z}_+}$ is a reverse martingale, then it is UI.

Proof: $\mathbb{E}[|X_0|] < \infty$, so by Sub σ-field UI Lemma, $\mathbb{E}[X_0|\mathcal{F}_{-n}]$ is UI.

### Convergence of Reverse Mart Thm

Let $(X_n)_{n \geq 0}$ be reverse mart.

Then $X_{-n} \xrightarrow{n \to \infty} X_{-\infty}$ a.s. and in $L^1$.

Moreover, $\mathbb{E}[X_0|\mathcal{F}_{-\infty}] = X_{-\infty}$ where $\mathcal{F}_{-\infty} = \cap_{n \in \mathbb{Z}_+} \mathcal{F}_{-n}$.

### Levy’s Backward Law

Let $Y \in L^1$. Suppose decreasing σ-fields $\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \ldots$

and $\mathcal{G}_\infty = \cap_{n=0}^\infty \mathcal{G}_n$. Then, $\mathbb{E}[Y|\mathcal{G}_n] \to \mathbb{E}[Y|\mathcal{G}_\infty]$ a.s. and in $L^1$.
<table>
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<th>Concept</th>
<th>Description</th>
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<tbody>
<tr>
<td>Exchangeable Sequence</td>
<td>( X_n ), where for each ( n ), ((X_1, X_2, \ldots, X_n) \overset{d}{=} (X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}) ), ( \forall ) permutations ( \pi ).</td>
</tr>
<tr>
<td>de Finetti’s Thm</td>
<td>If ( X_n ) are exchangeable, then, conditional on ( \epsilon ), we have ( X_1, X_2, \ldots ) are iid.</td>
</tr>
<tr>
<td>Optional Stopping ( \sigma )-field ( \mathcal{F}_T )</td>
<td>Let ((\Omega, \mathcal{F}, (\mathcal{F}<em>n)</em>{n \geq 0}, \mathbb{P})) and ( T ) be stopping time. Denote by ( \mathcal{F}_T ), the ( \sigma )-field of &quot;events which occur prior to time ( T ).&quot; In symbols: ( \mathcal{F}_T := { A \in \mathcal{F} : A \cap { T \leq n } \in \mathcal{F}_n, \ \forall n \geq 0 } ).</td>
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<tr>
<td>Optional Stopping Proposition</td>
<td>If ( T ) is stopping time, then ( \mathcal{F}_T ) is ( \sigma )-field &amp; ( T ) is ( \mathcal{F}_T )-measurable. If ( S \leq T ) is stopping time, then ( \mathcal{F}_S \subseteq \mathcal{F}_T ). Let ( T ) be stopping time ( \mathbb{P}(T &lt; \infty) = 1 ) &amp; ( X_n ) be adapted, then ( X_T \in \mathcal{F}_T ).</td>
</tr>
<tr>
<td>UI SubMart Stopping Time Closure</td>
<td>If ((X_n)<em>{n \geq 0}) is UI sub-mart, then for any stopping time ( T ), ((X</em>{T\wedge n})_{n \geq 0}) is UI.</td>
</tr>
<tr>
<td>UI SubMart Stopping Time Ineq.</td>
<td>If ( X_n ) is UI submart, then ( \forall ) stopping time ( T \leq \infty ), we have: ( \mathbb{E}[X_0] \leq \mathbb{E}[X_T] \leq \mathbb{E}[X_\infty] ), where ( X_\infty = \lim X_n ).</td>
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<tr>
<td>Optional Stopping Thm for SubMarts (or mart)</td>
<td>If ( S, T ) are stopping times ( \mathbb{P}(S \leq T &lt; \infty) = 1 ), and ((X_{T\wedge n})_{n \geq 0}) is UI submart, then ( \mathbb{E}[X_T</td>
</tr>
<tr>
<td>Finite Differences Submartingale w/Stopping Times</td>
<td>Suppose ( X_n ) is a submart and ( \mathbb{E}[</td>
</tr>
<tr>
<td>Nonneg SuperMart Stopping Time Thm</td>
<td>( X_n ) is nonnegative supermart and ( T \leq \infty ) is stopping time, then ( \mathbb{E}[X_0] \geq \mathbb{E}[X_T] ) where ( X_\infty = \lim X_n ).</td>
</tr>
<tr>
<td>Asymmetric Simple RW w/generating fnct ( \varphi(x) := \sum_{k \geq 0} p_k x^k ) w/( p_k := \mathbb{P}(\xi_i = k) )</td>
<td>( \xi_1, \xi_2, \ldots ) iid, ( S_n := \xi_1 + \ldots + \xi_n ), ( \mathbb{P}(\xi_i = 1) = p, \mathbb{P}(\xi_i = -1) = q = 1 - p ), with ( \frac{1}{2} &gt; p &lt; 1 ). ( \varphi(x) := \left( \frac{q}{p} \right)^x \Rightarrow \varphi(S_n) ) is mart. ( T_a := \inf { n : S_n = a } ), ( a &lt; 0 &lt; b ) ( \Rightarrow \mathbb{P}(T_a &lt; T_b) = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)} ). ( a &lt; 0 ) ( \Rightarrow \mathbb{P}(\min_n S_n \leq a) = \mathbb{P}(T_a &lt; \infty) = \left( \frac{1 - p}{p} \right)^-a ). ( b &gt; 0 ) ( \Rightarrow \mathbb{P}(T_b &lt; \infty) = 1 ) &amp; ( \mathbb{E}[T_b] = \frac{b}{2p-1} ).</td>
</tr>
</tbody>
</table>
Let $X_1, X_2, \ldots$ be a martingale with $|X_{n+1} - X_n| \leq M < \infty$.

Let $C := \{ \lim X_n \text{ exists and finite} \}$,

and $D := \{ \lim \sup X_n = +\infty \text{ and } \lim \inf X_n = -\infty \}$. Then, $P(C \cup D) = 1$.
Markov Chains

**Example such that sup\(_{n \geq 1} |E[X_n]| < \infty\) but \((X_n)_{n \geq 1}\) are not uniformly integrable**

Let \(\Omega = [0, 1]\) with Lebesgue measure, and \(X_n = n \cdot 1_{[0,\frac{1}{n}]}\). Then the \(X_n\) are bounded in \(L^1\), but not uniformly integrable.

**Convergence in Probability**

A sequence \(\{X_n\}\) of random variables converges in probability towards the random variable \(X\) if for all \(\varepsilon > 0\), we have:
\[
\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0.
\]

**Convergence in Distribution**

*(Weak Convergence):*

Let \(X_n, X\) be r.v.s w/CDFs \(F_n\) & \(F\) resp. We say that \(X_n \xrightarrow{d} X\) or \(X_n \Rightarrow X\) if \(F_n(x) \to F(x)\ \forall x\) where \(F\) continuous at \(x\) \((C_F)\). If above holds, then \(\pi_n \xrightarrow{d} \pi\), where \(\pi_n\) and \(\pi\) are distributions of \(X_n/X\) resp.

**Convergence Almost Surely**

To say that the sequence \(X_n\) converges a.s., almost everywhere, with probability 1, or strongly towards \(X\) means that
\[
P\left(\lim_{n \to \infty} X_n = X\right) = 1.
\]

**Markov Chain**

An \((\mathcal{F}_n)_{n \geq 0}\)-adapted stochastic process \((X_n)_{n \geq 0}\) taking values in \((\mathcal{S}, \mathcal{S})\) is called a Markov chain if it has the **Markov Property**:
\[
P(X_{n+1} \in B | \mathcal{F}_n) = P(X_{n+1} \in B | X_n) \text{ a.s. for each } B \in \mathcal{S}, n \geq 0.
\]

**Markov Chain Transition Probability**

We define a Markov chain’s \((X_n)_{n \geq 0}\) transition probabilities \((p_n)_{n \geq 0}\) as
\[
P(X_{n+1} \in B | \mathcal{F}_n) =: p_n(X_n, B) \text{ almost surely for each } n \geq 0 \text{ and } B \in \mathcal{S}.
\]

**Transition Matrix**

If for all \(\varepsilon > 0\), we have:
\[
\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0.
\]

**Time Homogeneous Markov Chain**

*(finite dimensional, continuous state space)*

A Markov chain in which the transition probabilities are all the same \(p_n = p\) for all time \(n \geq 0\).

**Markov Chain Distributions**

\(X_n\) is Markov w/trans. prob. \((p_n)_{n \geq 0}\) & init. dist. \(\mu\), then finite dimensional dist. are given by
\[
P(\{X_0 \in A_0, X_1 \in A_1, \ldots, X_k \in A_k\}) = \int_{A_0} \mu(dx_0) \int_{A_1} p_0(x_0, dx_1) \ldots \int_{A_k} p_k(x_{k-1}, dx_k)
\]

\(\mathcal{F}_n := \sigma(X_0, \ldots, X_n)\). \(\theta : \mathbb{S}^{\mathbb{Z}_+} \to \mathbb{S}^{\mathbb{Z}_+}\) where \(\theta(x_0, x_1, \ldots) = (x_1, x_2, \ldots)\)

**Strengthened Markov Prop.**

Let \(X_n\) be Markov w/init dist \(\mu\).
\(X_n\) coordinate maps on \((\mathbb{S}^{\mathbb{Z}_+}, \mathbb{S}^{\mathbb{Z}_+}, P_\mu)\)

For any bounded measurable function \(f : \mathbb{S}^{\mathbb{Z}_+} \to \mathbb{R}\), and any \(k \geq 0\),
\[
E_\mu[f \circ \theta^k | \mathcal{F}_k] = E_{X_k}[f] \text{ } P_\mu\text{ a.s.}
\]
<table>
<thead>
<tr>
<th>Chapman-Kolmogorov Equation</th>
<th>( \mathbb{P}<em>x(X</em>{m+n} = z) = \sum_y \mathbb{P}_x(X_m = y) \mathbb{P}_y(X_n = z) ) for each ( m,n \in \mathbb{Z}^+ ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absorbing</td>
<td>A state ( a ) is called absorbing if ( \mathbb{P}_a(X_1 = a) = 1 ).</td>
</tr>
<tr>
<td>Strong Markov Property</td>
<td>For any bounded measurable function ( f : S^{\mathbb{Z}^+} \to \mathbb{R} ) and for any stopping time ( T ), ( \mathbb{E}<em>\mu[f \circ \theta^T \mathcal{F}<em>T] = \mathbb{E}</em>{X_T}[f] ) on ( { T &lt; \infty } ) ( \mathbb{P}</em>\mu )-a.s.</td>
</tr>
<tr>
<td>Reflection Principle</td>
<td>Let ( \xi_1, \xi_2, \ldots ) be iid w/distribution symmetric about 0. Let ( S_n = \xi_1 + \ldots + \xi_n ). If ( a &gt; 0 ), then ( \mathbb{P}(\sup_{m \leq n} S_m &gt; a) \leq 2\mathbb{P}(S_n &gt; a) ).</td>
</tr>
<tr>
<td>( k )th Return to ( y )</td>
<td>Let ( T^y_0 := 0 ), and for ( k \geq 1 ), let ( T^y_k := \inf{ n &gt; T^y_{k-1} : X_n = y } ), the time of the ( k )th return to ( y ).</td>
</tr>
<tr>
<td>( \rho_{yz} )</td>
<td>( \mathbb{P}_y(T_z &lt; \infty) )</td>
</tr>
<tr>
<td>Finite ( k )th Return Prob. to ( z ) starting at ( y ) :</td>
<td>For ( k \geq 1 ), ( \mathbb{P}<em>y(T^y_z &lt; \infty) = \rho</em>{yz}\rho_{zz}^{k-1} ).</td>
</tr>
<tr>
<td>Recurrent</td>
<td>A state ( y \in S ) is called recurrent if ( \rho_{yy} = 1 ) and is called transient if ( \rho_{yy} &lt; 1 ).</td>
</tr>
<tr>
<td>If ( y ) is recurrent, then</td>
<td>( \lim_{k \to \infty} \mathbb{P}<em>y(T^y_z &lt; \infty) = \lim</em>{k \to \infty} \rho_{zy}^k = 1 ).</td>
</tr>
<tr>
<td>If ( y ) is transient, then ( P_y(X_n = y \text{ i.o.}) ) =</td>
<td>( = \lim_{k \to \infty} \rho_{zy}^k = 0 ).</td>
</tr>
<tr>
<td>Total number of visits to ( y ) by the Markov chain ( X_n ) is notated as ( N(y) := )</td>
<td>( \sum_{n=1}^{\infty} 1_{{X_n = y}} ).</td>
</tr>
</tbody>
</table>
A state \( x \) leads to, or is accessible from another state \( y \neq x \), denoted by \( x \rightarrow y \), if:

\[
\rho_{xy} > 0 \text{ (or equivalently, for some } n \geq 1, p^n(x,y) > 0). 
\]

Formally, \( x \rightarrow y \) if \( \exists n \geq 0 \) such that \( \mathbb{P}(X_{n+} = y | X_0 = x) = p_{xy}^{(n+)} > 0 \).

**Communicating Class**

"\( \leftrightarrow \)" is an equivalence relation.

Therefore, there is a partition \( C_1, C_2 \) of \( S \), with each block \( C_i \) being referred to as a communicating class.

**Irreducible Subset**

A closed subset \( A \subseteq S \) is called irreducible if \( x \leftrightarrow y \) for all \( x, y \in A \).

By definition, each class is irreducible.

**Irreducible Markov Chain**

Markov chain is irreducible if it is possible to get to any state from any state. Formally, Markov chain is irreducible if its state space is a single communicating class, i.e., \( x \leftrightarrow y, \forall x, y \in S \).

**Properties when \( x \) is recurrent and \( \rho_{xy} > 0 \)**

i) \( \rho_{yx} = 1 \), ii) \( y \) is recurrent, iii) \( \rho_{yx} = 1 \).

**Closed Subset of States**

We call a subset of states \( A \subseteq S \) closed if

\[
\rho_{xy} = 0 \text{ for all } x \in A \text{ and } y \notin A \]

**Is a recurrent class \( C \) closed, open, neither?**

Closed.

**:-(**

In a finite state Markov chain, a class is recurrent (respectively transient) if and only if:

**Birth & Death Chains \( X_n \) on \( \{0, 1, 2, \ldots \} \).**

\[
p_i := p(i, i + 1), \quad q_i := p(i, i - 1), \quad r_i := p(i, i) 
\]

Let: \( \varphi(0) := 0, \varphi(1) := 1, \) and \( \varphi(k + 1) = ? \)

**Birth Death Chain:** the state 0 is recurrent if and only if

For irreducible: \( \varphi(m + 1) = \varphi(m) + \prod_{j=1}^{m} \frac{q_j}{p_j} \) for \( m \geq 1 \),

and \( \varphi(n) = \sum_{m=0}^{n-1} \prod_{j=1}^{m} \frac{q_j}{p_j} \) for \( n \geq 1 \).

**Birth Death Chain:**

the state 0 is recurrent if and only if

\[
\varphi(M) \rightarrow \infty \text{ as } M \rightarrow \infty, \text{ that is:} \\
\varphi(\infty) = \sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{q_j}{p_j} = \infty. \\
\text{If } \varphi(\infty) < \infty, \text{ then } \mathbb{P}(T_0 = \infty) = \frac{\varphi(x)}{\varphi(\infty)}.
\]
| Stationary/Invariant Measure | \( \mu \) is a stationary/invariant distribution that is a probability measure. 
|\( \mu P = \mu(y) = \sum_{x \in S} \mu(x)p(x,y) \) (\( \mu \) is left eigenvector of \( p \)). The last equation says \( \mathbb{P}_\mu(X_1 = y) = \mu(y) \). Using the Markov property and induction, we have \( \mathbb{P}_\mu(X_n = y) = \mu(y) \forall n \geq 1 \). |
| Stationary/Invariant Distribution | \( \pi \) is a stationary/invariant measure that is a probability measure. 
|\( \pi P = \pi(y) = \sum_{x \in S} \pi(x)p(x,y) \), and \( \sum_{x \in S} \pi(x) = 1 \). It represents a possible equilibrium for the chain. |
| Suppose \( p \) is irreducible. A necessary and sufficient condition for the existence of a reversible measure is | i) \( p(x,y) > 0 \) implies \( p(y,x) > 0 \), and 
ii) for any loop \( x_0, \ldots, x_n = x_0 \) 
with \( \prod_{1 \leq i \leq n} p(x_i, x_{i-1}) > 0 \), \( \prod_{i=1}^n \frac{p(x_{i+1}, x_i)}{p(x_i, x_{i+1})} = 1 \). |
| Recurrent Time in \( y \) | \( \mu_s(y) := \) Define \( \mu_s(y) \) as the expected time spent in \( y \) between visits to \( x \). |
| Positive Recurrent | \( \mathbb{E}_x[T_x] = \sum_{n=1}^{\infty} n \mathbb{P}(T_x = n) = \sum_{y \in S} \mu_s(y) < \infty \), and \( \mathbb{P}_x(T_x < \infty) = 1 \).  
Positive Recurrent \( \Rightarrow \) Recurrent |
| Null-Recurrent | \( x \in S \) is said to be null recurrent if \( \mathbb{P}_x(T_x < \infty) = 1 \), but \( \mathbb{E}_x[T_x] = \infty \).  
If \( \{X_n\} \) is recurrent but not null recurrent then it is called positive recurrent. \( X_n \) is null recurrent if all \( X_i \) are null recurrent. |
| If a chain is finite and irreducible, then there exists: | A unique stationary/invariant distribution \( \pi \), and it is positive recurrent. |
| If \( \{X_n\} \) is positive recurrent, then for every \( x, y \in S \) : | \( \lim_{n \to \infty} p^n(x,y) = \pi(y) > 0 \) where \( \pi : S \to [0,1] \) is the stationary/invariant distribution.  
\( p^n(x,y) := \frac{1}{\pi} \sum_{i=S} \pi(X_n = y|X_0 = x) \)  
It’s unique! |
| For an irreducible, positive recurrent Markov chain, what quality does the stat/invariant distribution \( \pi \) have? | \( \ast \) Stat. measures are unique up to constant multiples.  
\( \ast \) \( \mu \) a stat. measure \( \Rightarrow \mu(x) > 0, \forall x \). \( \ast \) Stat. dist. \( \pi \), if exists, is unique  
\( \ast \) Stat. measure has infinite mass \( \Rightarrow \) Stat. dist. \( \pi \) cannot exist. |
| For an irreducible and recurrent chain, the following are true. | If \( \pi \) is a stat/invariant distribution of a Markov chain and \( \pi(x) > 0 \), then \( x \) is recurrent. |
For an irreducible Markov chain, the following are equivalent.

i) There exists \( x \in S \) that is positive recurrent.

ii) There exists a stationary distribution \( \pi \).

iii) Every state is positive recurrent.

If \( p \) irreducible and has stat. dist. \( \pi \), then any other stationary measure is a multiple of \( \pi \).

**Doubly Stochastic**

Prob. transition matrix \( p_{ij} = P(X_{n+1} = j | X_n = i) \)

is doubly stochastic if \( \Sigma_j p_{ij} = 1 \ \forall j \) and \( \Sigma_i p_{ij} = 1 \ \forall i \).

Uniform distribution is stat. dist. of \( p \) \( \iff \) \( p \) is doubly stochastic.

**Stationary Sequence**

\((X_n)_{n \geq 0}\) is stationary if \((X_n, X_{n+1}, \ldots) \sim (X_0, X_1, \ldots)\), \( \forall n \geq 0 \)

or equivalently, \((X_n, X_{n+1}, \ldots, X_{n+m}) \sim (X_0, X_1, \ldots, X_m)\), \( \forall n, m \geq 0 \)

Exchangeable sequences are stationary.

**Reversible Measure**

measure \( \mu \) such that \( \mu(x)p(x,y) = \mu(y)p(y,x) \).

Is always stationary since \( \Sigma_{x \in S} \mu(x)p(x,y) = \Sigma_{x \in S} \mu(y)p(y,x) = \mu(y) \),

i.e., it is invariant under multiplication by \( p \).

**Aperiodic Markov Chain**

For \( x \), \( I_x := \{n \geq 1 : p_n(x,x) > 0\} \). Let \( d_x \) be the GCD of \( I_x \)

\( x \) has period \( d_x \). If every state of a Markov chain has period 1, then we call the chain aperiodic.

**What could cause \( d_x = d_y \)?**

If \( x \leftrightarrow y \).

In other words, if \( \rho_{xy} > 0 \) and \( \rho_{yx} > 0 \).

**If \( d_x = 1 \), then \( \exists n_0 \geq 1 \) such that:**

\( p^n(x,x) > 0 \) for all \( n \geq n_0 \).

e.g., if \( I_x = \{5,7\} \).

**An irreducible aperiodic Markov chain has the following property: for each \( x, y \in S \), there exists:**

\( n_0 = n_0(x,y) \geq 1 \) such that \( p^n(x,y) > 0 \) for all \( n \geq n_0 \).

**Irreducible Aperiodic Markov \( X_n \)**

is Null Recurrent if:

\( \langle X_n \rangle \) is recurrent and \( \lim_{n \to \infty} p_n(x,y) = 0 \) for all \( x, y \in S \).

**Markov Chain Convergence Theorem**

Consider irreducible, aperiodic Markov with stat. dist. \( \pi \)

Then, \( p^n(x,y) \to \pi(y) \) as \( n \to \infty \), for all \( x, y \in S \).
Total Variation Distance

Chain is coupled if:

i) marginals \( X_n \) and \( Y_n \) are Markov w/same \( p \) & init. dist. \( \mu, \nu \) resp.

ii) \( X_n = Y_n \) for \( n \geq T \), where \( T := \inf \{ n \geq 0 : X_n = Y_n \} \).

For two probability measures \( \mu, \nu \) on \( S \), their total variation distance is given by:

\[
d_{TV}(\mu, \nu) := \frac{1}{2} \sum_{x \in S} |\mu(x) - \nu(x)| = \sup_{A \subseteq S} |\mu(A) - \nu(A)|
\]

Coupled Markov Chain.

Let \( \mu, \nu \) be prob. measures on countable \( S \), & \( (X_n, Y_n)_{n \geq 0} \) on product space \( S \times S \).

Markov Recurrent Corollary

A state \( x \in S \) is recurrent \( \iff \)

Asymptotic Density of Returns

where \( N_n(y) := \sum_{m=1}^{n} 1 \{ x_m = y \} \), is # visits to \( y \) by \( n \). Let \( y \in S \) recurrent. Then \( \lim \limits_{n \to \infty} \frac{N_n(y)}{n} = \frac{1}{E_y[T_y]} 1_{T_y < \infty} \P - a.s. \)

For a Markov chain and any \( x, y \in S \),

if \( N(y) := \sum_{n=1}^{\infty} 1 \{ x_n = y \} \) is total # visits to \( y \), then we have \( E_x[N(y)] = \frac{p_{xy}}{1 - \rho_{yx}} = \sum_{n=1}^{\infty} p^n(x,y) \)

(where we interpret \( \frac{0}{0} = 0 \), \( \frac{c}{0} = +\infty \) for \( c > 0 \))

For a Markov chain and \( x, y \in S \),

if \( N(y) := \sum_{n=1}^{\infty} 1 \{ x_n = y \} \) is total # visits to \( y \), then we have \( P_x(N(y) = k) = \rho_{xy} \rho_{yy}^{k-1} (1 - \rho_{yy}^k) \)

Consider Markov \( X_n \) started from stat. dist. \( \pi \) & trans. matrix \( p \).

Fix \( N \geq 1 \) & \( Y_n := X_{N-n} \) for \( n = 0, 1, \ldots, N \). Then:

Birth Death Chain:

For any \( c \in R \), let \( T_c = \inf \{ n \geq 1 : X_n = c \} \),

If \( a < x < b \), then: \( \P_x(T_a < T_b) = \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)} \)

\((Y_n)_{0 \leq n \leq N}\) is a time-homogeneous Markov chain with initial distribution \( \pi \) and transition matrix \( q \) given by \( q(x,y) = \frac{\pi(y)q(y,x)}{\pi(x)} \)

\( \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)} \) and

\( \P_x(T_b < T_a) = \frac{\phi(x) - \phi(a)}{\phi(b) - \phi(a)} \)
Stationary/Invariant Measure Theorem
Let $x$ be a recurrent state. Then: $\mu_x(y) := \mathbb{E}_x \left[ \sum_{n=0}^{T_x-1} 1_{\{X_n = y\}} \right] = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y, T_x > n)$, is a stationary measure.

Pairs of states $x, y$ communicate, denoted by $x \leftrightarrow y$, if:
$x \rightarrow y \text{ and } y \rightarrow x$.
In other words, if $\rho_{xy} > 0$ and $\rho_{yx} > 0$.

Suppose Markov irreducible & recurrent.
Let $\mu$ be stat. measure $w/\mu(y) > 0, \forall y \in S$.
If $\nu$ is another stat. measure, then
$\mu = c\nu$ for some $c > 0$.

Stat./Invariant Distribution $\pi$:
Suppose that $S$ is finite and $p$ is irreducible.
Then:
there exists a unique solution to $\pi p = \pi$
with $\sum_{i \in S} \pi(i) = 1$ and $\pi(i) > 0$ for all $i \in S$.

On a Markov chain, if $C$ is a finite closed set, then it contains...
at least one recurrent state.
In particular, a finite closed class $C$ is recurrent.

Calculating Stat./Invariant Distribution
If $p$ is irreducible and has stat. distribution $\pi$,
then $\pi(x) = \frac{1}{\mathbb{E}_x[T_x]}$.
The Markov chain $X_n$ is recurrent.

Birth Death Chain: If $S$ irreducible, $\varphi \geq 0$
$w/E_x[\varphi(X_1)] \leq \varphi(x)$ for $x \notin F$ (finite set),
and $\lim_{x \to \infty} \varphi(x) \to \infty$ as $x \to \infty$, then: