1 Nuts and bolts

1. Read the following sections before workshop tomorrow:
   - 6395 The fundamental theorem of calculus
   - 6403 Volume by disks

2. Office hours this week: MW 11-12, and F 12-1.

2 What’s happening today

1. Heavy lifting – why the fundamental theorem of calculus is true – the mean value theorem gives the link between the derivative and the integral.

2. Application of definite integrals and the fundamental theorem – finding volumes using the “disk method”

3 The fundamental theorem of calculus

If \( F'(x) = f(x) \) (that is, if \( F(x) \) is an antiderivative for \( f(x) \)) on \([a, b]\), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

We used this on Monday to calculate some definite integrals – much easier than finding these values directly from the definition.

This theorem is true for continuous functions \( f(x) \). Why is it true?

4 Mean value theorem

If \( f(x) \) is continuous on \([a, b]\) and has a derivative on \((a, b)\), then there is a number \( c \) with \( a < c < b \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]
On Monday, we saw the geometric meaning of that statement, and we calculated those numbers $c$ explicitly in an example. Notice how essential the hypotheses are:

Example 1. Let $f(x) = |x|$ on $[-1, 1]$. Is the conclusion of the mean value theorem true for this function?

5 Warmup to the proof of the mean value theorem

Mean value theorem, Junior. (also known as Rolle’s theorem) If $f(x)$ is continuous on $[a, b]$ and has a derivative on $(a, b)$, and $f(a) = f(b) = 0$, then there is a number $c$ with $a < c < b$ such that $f'(c) = 0$.

6 Now we use junior to prove the big mean value theorem

The idea behind the proof: Apply Junior to “the function minus the line”.

![Graph of a function and a line representing the mean value theorem](image-url)
Write a function of \( x \) that tells you the difference between \( f(x) \) and the line shown. That function is zero at the endpoints, and we can apply Junior.

The line is described by

\[
y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a).
\]

Write

\[ L(x) = f(a) + \frac{f(b) - f(a)}{b - a} (x - a). \]

Now consider this function:

\[ h(x) = f(x) - L(x). \]

This is the difference between the function \( f(x) \) and the line.

\[
h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} (x - a).
\]

Notice that \( h(a) = 0 \), \( h(b) = 0 \), and \( h \) is differentiable since \( f(x) \) and \( L(x) \) are.

Then Junior says that there is a \( c \) with \( a < c < b \) where \( h'(c) = 0 \).

But \( h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \), so at that \( c \) we have

\[
0 = f'(c) - \frac{f(b) - f(a)}{b - a},
\]

which proves the big mean value theorem.

Now, how could that possibly have anything to do with the fundamental theorem of calculus?

Let’s remember the definition of the definite integral: for a continuous function \( f(x) \) on \([a, b]\), we have

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^n \frac{b - a}{n} f(x_k^*),
\]

where \( x_k^* \) is the midpoint of the \( k \)th subinterval of length \( \frac{b - a}{n} \) on \([a, b]\).

It turns out that this limit exists and equals the same number if we allow \( x_k^* \) to be ANY NUMBER in the \( k \)th subinterval.

We used the midpoint of each subinterval before, because that makes the (painful but direct) calculations work out nicely.

7 Proof of the fundamental theorem:

Write \( a = x_0, x_1, \ldots, x_{n-1}, x_n = b \) for the endpoints of the \( n \) subintervals of width \( \frac{b - a}{n} \).

Let’s use the mean value theorem on the function \( F(x) \) on each subinterval \([x_{k-1}, x_k]\).
The mean value theorem says that there is a number \( x^*_k \) between \( x_k \) and \( x_{k-1} \) such that

\[
F'(x^*_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = \frac{F(x_k) - F(x_{k-1})}{\frac{b-a}{n}}.
\]

Now go back to the definition of the definite integral, using all of these points that the mean value theorem provides:

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^n \frac{b-a}{n} \cdot f(x^*_k)
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^n \frac{b-a}{n} \cdot F'(x^*_k)
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^n \frac{b-a}{n} \cdot \frac{F(x_k) - F(x_{k-1})}{\frac{b-a}{n}}
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^n F(x_k) - F(x_{k-1})
\]

\[
= \lim_{n \to \infty} F(b) - F(a) = F(b) - F(a).
\]

8 Application: using definite integrals to calculate volumes

Let \( f(x) \) be a function that is continuous on \([a, b]\). By rotating the graph of \( f(x) \) around the \( x \)-axis, we describe a solid, whose volume is

\[
V = \lim_{n \to \infty} \sum_{k=1}^n \frac{b-a}{n} \pi f(x^*_k)^2 = \int_a^b \pi f(x)^2 \, dx.
\]

Example 2. Find the volume of a cone of base radius \( r \) and height \( h \).

Example 3. Find the volume of a sphere of radius \( r \).