Math 1371 – Lecture 21

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1 Nuts and bolts

1. Read the following sections before workshop tomorrow:
   - 6395 The fundamental theorem of calculus
   - 6403 Volume by disks
2. Office hours this week: MW 11-12, and F 12-1.

2 What's happening today

1. Heavy lifting – why the fundamental theorem of calculus is true – the mean value theorem gives the link between the derivative and the integral.
2. Application of definite integrals and the fundamental theorem – finding volumes using the "disk method"

3 The fundamental theorem of calculus

If $F'(x) = f(x)$ (that is, if $F(x)$ is an antiderivative for $f(x)$) on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We used this on Monday to calculate some definite integrals – much easier than finding these values directly from the definition.

This theorem is true for continuous functions $f(x)$. Why is it true?

4 Mean value theorem

If $f(x)$ is continuous on $[a, b]$ and has a derivative on (a, b), then there is a number c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

On Monday, we saw the geometric meaning of that statement, and we calculated those numbers c explicitly in an example.

Notice how essential the hypotheses are:

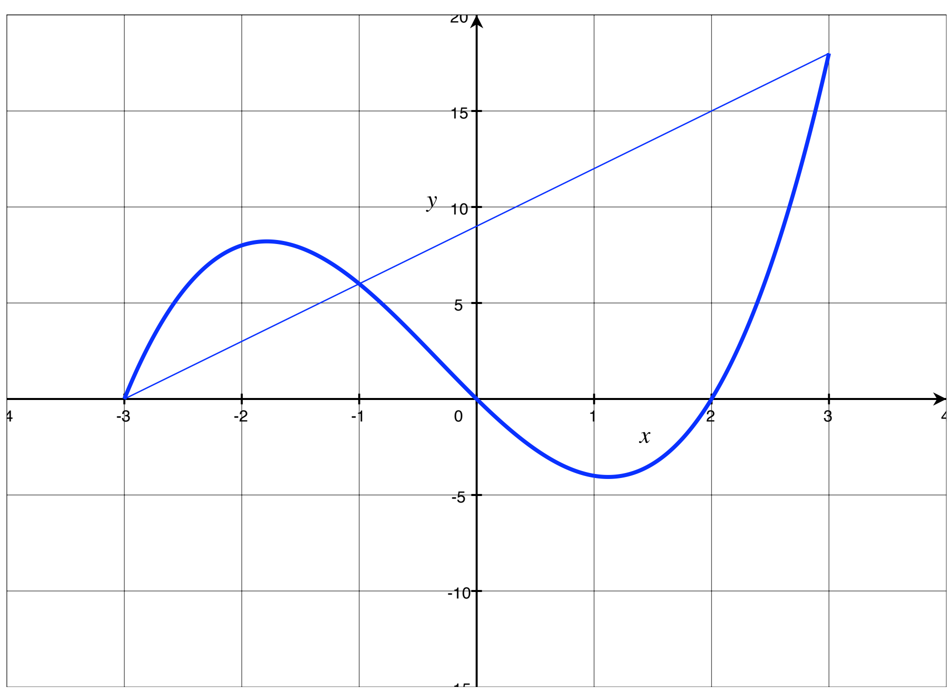
Example 1. Let $f(x) = |x|$ on $[-1, 1]$. Is the conclusion of the mean value theorem true for this function?

5 Warmup to the proof of the mean value theorem

Mean value theorem, Junior. (also known as Rolle's theorem) If $f(x)$ is continuous on $[a, b]$ and has a derivative on (a, b), and $f(a) = f(b) = 0$, then there is a number c with $a < c < b$ such that $f'(c) = 0$.

6 Now we use junior to prove the big mean value theorem

The idea behind the proof: Apply Junior to "the function minus the line".



Write a function of x that tells you the difference between $f(x)$ and the line shown. That function is zero at the endpoints, and we can apply Junior.

The line is described by

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a).$$

Write

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Now consider this function:

$$h(x) = f(x) - L(x).$$

This is the difference between the function $f(x)$ and the line.

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Notice that $h(a) = 0$, $h(b) = 0$, and h is differentiable since $f(x)$ and $L(x)$ are.

Then Junior says that there is a c with $a < c < b$ where $h'(c) = 0$.

But $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, so at that c we have

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which proves the big mean value theorem.

Now, how could that possibly have anything to do with the fundamental theorem of calculus?

Let's remember the definition of the definite integral: for a continuous function $f(x)$ on $[a, b]$, we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b - a}{n} f(x_k^*),$$

where x_k^* is the midpoint of the kth subinterval of length $\frac{b-a}{n}$ on $[a, b]$.

It turns out that this limit exists and equals the same number if we allow x_k^* to be **ANY NUMBER** in the kth subinterval.

We used the midpoint of each subinterval before, because that makes the (painful but direct) calculations work out nicely.

7 Proof of the fundamental theorem:

Write $a = x_0, x_1, \ldots, x_{n-1}, x_n = b$ for the endpoints of the n subintervals of width $\frac{b-a}{n}$.

Let's use the mean value theorem on the function $F(x)$ on each subinterval $[x_{k-1}, x_k]$.

The mean value theorem says that there is a number x_k^* between x_{k-1} and x_k such that

$$F'(x_k^*) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = \frac{F(x_k) - F(x_{k-1})}{\frac{b-a}{n}}.$$

Now go back to the definition of the definite integral, using all of these points that the mean value theorem provides:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} \cdot f(x_k^*)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} \cdot F'(x_k^*)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} \cdot \frac{F(x_k) - F(x_{k-1})}{\frac{b-a}{n}}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k) - F(x_{k-1})$$

$$= \lim_{n\to\infty} F(b) - F(a) = F(b) - F(a).$$

8 Application: using definite integrals to calculate volumes

Let $f(x)$ be a function that is continuous on $[a, b]$. By rotating the graph of $f(x)$ around the x-axis, we describe a solid, whose volume is

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} \pi f(x_k^*)^2 = \int_a^b \pi f(x)^2\,dx.$$

Example 2. Find the volume of a cone of base radius r and height h.

Example 3. Find the volume of a sphere of radius r.